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# MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 2001 SOLUTIONS

| | | | |
|---|---|---|---|
| 1. C | 8. E | 15. D | 22. C |
| 2. C | 9. D | 16. D | 23. B |
| 3. B | 10. A | 17. B | 24. E |
| 4. D | 11. C | 18. D | 25. D |
| 5. B | 12. D | 19. A | 26. C |
| 6. A | 13. B | 20. A | 27. B |
| 7. C | 14. E | 21. B | |

**Part I – Multiple-choice solutions**

**Question 1**

For the function $y = \frac{A}{x-b} + B$, where *A, b* and *B* are constants, the graph will have a vertical asymptote when $x - b = 0$, that is, when $x = b$. So we require that $b = -1$ so $x+1$ will be our denominator. The horizontal asymptote is given by $y = B$ so in our case $B = 2$.

The required equation could be $y = \frac{1}{x+1} + 2$.

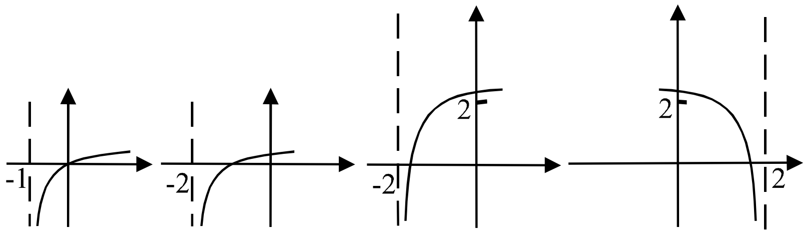
The answer is C.

**Question 2**

The function *f* intersects with the *x*-axis at
$x = -b$ and $x = c$ and so $x + b$ and $x - c$ are factors of $f$.
The function *f* touches the *x*-axis at $x = -a$ and so $x + a$ is a repeated factor of *f*. So the rule for *f* would be $y = (x+a)^2(x+b)(x-c)$.
The answer is C.

**Question 3**

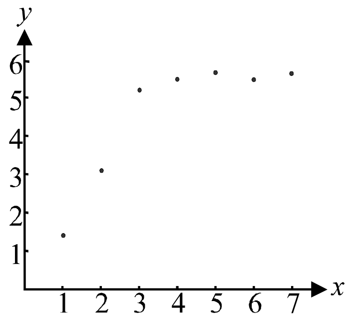
Sketch the graph of $y = \log_e(x+1)$



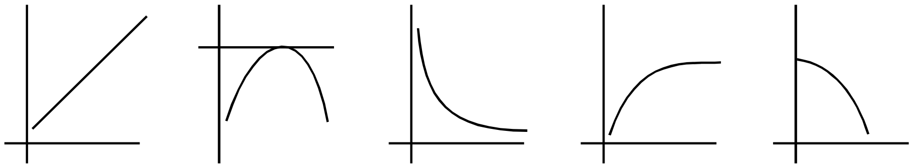

The original graph. | Translated 1 unit left. | Translated 2 units up. | Refected in the *y* axis.

The answer is B.

**Question 4**
Use your graphics calculator to plot the data on a scatterplot.



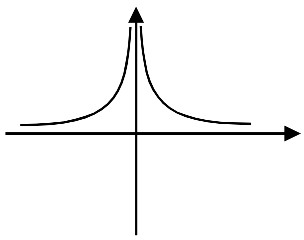

The functions given have the general shapes below:



$y = ax$ $\quad y = -(x-a)^2$ $\quad y = \frac{1}{ax}$ $\quad y = \log_e ax$ $\quad y = \cos(ax)$

The logarithmic function best models the data.
The answer is D.

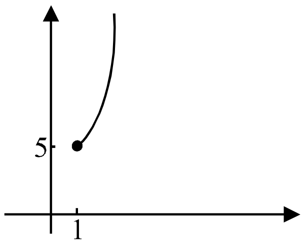
**Question 5**
Sketch the graph of $g$.



The domain is $(-\infty, 0) \cup (0, \infty)$.
The range is $(0, \infty)$.
The answer is B.

**Question 6**
Using addition of ordinates, choose some key points, for example the $y$ – intercepts. When $x = 0$, $f(x) = 1$, $g(x) = 5$ and $h(x) = -4$.
*So* $f(x) = g(x) + h(x)$.
$h(x) \neq g(x) + f(x)$ and so option B is eliminated.
$g(x) \neq f(x) + h(x)$ and so option C is eliminated.
$g(x) \neq 2f(x)$ and so option D is eliminated.
$h(x) \neq -(g(x+2))$ since $h(0) = -4$ and $-g(2) = -2$ and so option E is eliminated.
Choose some other key points for example when $x = 1$ and check that the corresponding value of the functions ie. $f(1)$, $g(1)$ and $h(1)$ satisfy the rule $f(x) = g(x) + h(x)$.
The answer is A.

**Question 7**
Sketch $g(x)$.




From the sketch, we have $d_g = [1,\infty)$, $\mathrm{r_g} = [5,\infty)$ so $d_{g-1} = [5,\infty)$, $r_{g-1} = [1,\infty)$

Now let $y = 3e^{x-1} + 2$

Swapping $x$ and $y$ gives us $x = 3e^{y-1} + 2$

Rearranging, $\frac{x-2}{3} = e^{y-1}$

So, $\log_e\left(\frac{x-2}{3}\right) = y - 1$

$$y = 1 + \log_e\left(\frac{x-2}{3}\right)$$

So we have $g^{-1} : [5,\infty) \to R$, $\quad g^{-1}(x) = 1 + \log_e\left(\frac{x-2}{3}\right)$

The answer is C.

**Question 8**

Use combinations:

$$^6C_3 = \frac{6!}{3!3!} = 20$$

Or use Pascal's triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

The $x^3$ term will be $20 \times (ax)^3 \times b^3 = 20(ab)^3 x^3$. The coefficient is $20(ab)^3$

The answer is E.

**Question 9**

$$\frac{1}{2}\log_2 x + 2\log_2 \sqrt{x} - 4\log_2 x = -5$$

$$\log_2 x^{\frac{1}{2}} + \log_2 x - \log_2 x^4 = -5 \qquad \text{(Note that } 2\log_2 \sqrt{x} = \log_2\left(x^{\frac{1}{2}}\right)^2$$

$$\log_2 \frac{x^{\frac{1}{2}} \times x}{x^4} = -5 \qquad = \log_2 x\text{ )}$$

$$\log_2 \frac{x^{\frac{3}{2}}}{x^4} = -5$$

$$\log_2 x^{-2.5} = -5$$

So, $\quad -2.5\log_2 x = -5$

So, $\quad \log_2 x = 2$

$$2^2 = x$$

$$x = 4$$

The answer is D.

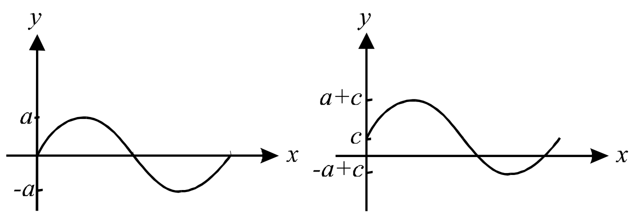
**Question 10**

The amplitude is 1. The period is $2\pi$ and the range is $[0,2]$.

The answer is A.

**Question 11**

The period of *f* is $2\pi$. The domain of *f* is $\left[-\frac{\pi}{6}, \frac{11\pi}{6}\right]$ and *f* shows one complete period of the graph of $y = a\sin(x-b)+c$. Now, *a, b* and *c* are positive constants, *a* is the amplitude, *c* is the vertical translation, and in this case *c* represents translation up $(c>0)$.




In the first diagram, we have the minimum value of the function is $-a$ and the maximum value is $a$. After the function has been translated *c* units up, the minimum value is $-a+c$ (or $c-a$) and the maximum value is $a+c$.

So, the minimum value will be $c-a$ and the maximum value will be $c+a$.

The answer is C.

**Question 12**

The period of the graph of $y = \tan\frac{x}{2}$ is $\frac{\pi}{1/2} = 2\pi$. The asymptotes of the graph will occur at $x = \cdots -3\pi, -\pi, \pi, 3\pi, 5\pi \ldots$ The point (b,0) is halfway between the asymptotes and so could occur at $x = \cdots -2\pi, 0, 2\pi, 4\pi \ldots$ So *a* could be $3\pi$ and *b* could be $4\pi$.

The answer is D.

**Question 13**

$$\sqrt{2}\cos\frac{x}{3} = -1$$

$$\cos\frac{x}{3} = \frac{-1}{\sqrt{2}}$$

$$\frac{x}{3} = \frac{3\pi}{4}, \frac{5\pi}{4} \ldots$$

$$x = \frac{9\pi}{4}, \frac{15\pi}{4} \ldots$$

Over the interval $[0, 3\pi]$ there is only 1 solution and that is $\frac{9\pi}{4}$.

The answer is B.

**Question 14**

For $x \in (3, \infty)$ the gradient of *f* is zero so we can eliminate option C.

For $x \in (-\infty, 0)$ the gradient of *f* is negative so we can eliminate A and B.

At $x = 0$ and $x = 3$ we cannot draw a tangent to the graph of *f* and so the gradient function does not exist for those values of *x*. So, we eliminate the graph of D.
The answer is E.

**Question 15**

$$y = \sqrt{\tan 2x}$$

$$= (\tan 2x)^{\frac{1}{2}}$$

$$\text{So, } \frac{dy}{dx} = \frac{1}{2}(\tan 2x)^{-\frac{1}{2}}.2\sec^2 2x$$

$$= \frac{\sec^2 2x}{\sqrt{\tan 2x}}$$

The answer is D.

**Question 16**

$$f(x) = 3x^2 \log_e(2x)$$

$$f'(x) = 6x\log_e(2x) + 3x^2 \cdot \frac{1}{x}$$

$$= 6x\log_e(2x) + 3x$$

The answer is D.

**Question 17**

$f(x) = e^{\cos(2x)}$

let $y = e^{\cos(2x)}$

Let $y = e^u$ where $u = \cos(2x)$

$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = -2\sin(2x)$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= e^u \cdot -2\sin(2x)$

$= -2\sin(2x)e^{\cos(2x)}$

Now, $f'(x) = -2\sin(2x)e^{\cos(2x)}$

So $f'(\pi) = -2\sin(2\pi)e^{\cos 2\pi}$

$= -2 \times 0 \times e^1$

$= 0$

Alternatively, use a graphics calculator to sketch the graph of $y = e^{\cos(2x)}$. Then calculate the value of $\frac{dy}{dx}$ when $x = \pi$. The answer is 0.

The answer is B.

**Question 18**

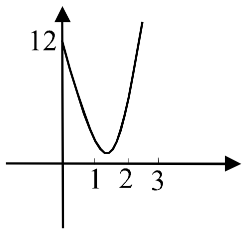
Do a quick sketch of the graph T(t) on your graphics calculator. There is a minimum turning point between t = 1 and t = 2.

Use your graphics calculator to locate this point.

It is (1.46, 1.88) to 2 places.

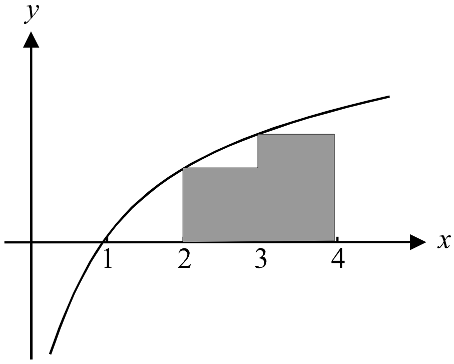
The minimum temperature is 1.9 (to 1 place).

The answer is D.




**Question 19**

We are looking for the shaded area shown.




Area required $= 1 \times f(1) + 1 \times f(2) + 1 \times f(3)$

$= 0 + 1 \times \log_e 2 + 1 \times \log_e 3$

$= \log_e 2 + \log_e 3$

The answer is A.

**Question 20**

$$\int\left(\sin(3x)+e^{-x}\right)dx=-\frac{1}{3}\cos(3x)+\frac{e^{-x}}{-1}+c$$
$$=-\frac{1}{3}\cos(3x)-e^{-x}+c$$

The answer is A.

**Question 21**

$$\int\frac{x^3+1}{\sqrt{x}}dx=\int\frac{x^3+1}{x^{\frac{1}{2}}}dx$$
$$=\int\left(x^{\frac{5}{2}}+x^{-\frac{1}{2}}\right)dx$$
$$=\frac{2x^{\frac{7}{2}}}{7}+2x^{\frac{1}{2}}+c$$

An antiderivative is $\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{1}{2}}$

The answer is B.

**Question 22**

The shaded area that falls below the $x$-axis will have a 'negative value'.

Hence the area required is given by $-\int_{-1}^{1}f(x)dx+\int_{1}^{4}f(x)dx-\int_{4}^{6}f(x)dx$

The answer is C.

**Question 23**

We have a probability distribution for a discrete random variable X and therefore

$\Pr(X=0)+\Pr(X=1)+\Pr(X=2)+\Pr(X=3)=1$

So $2k+k+3k+4k=1$

So $10k=1$

$k=\frac{1}{10}$

So $\Pr(x=1)=k$

$=0.1$

The answer is B.

**Question 24**

The mean of the distribution is the $x$ value where the bell-shaped curve peaks. For distribution $X_1$, this is twice that of distribution $X_2$.

This eliminated option A, C and D.

The variance of the distribution measures the spread of the distribution. The variance of distribution $X_1$ is half that of distribution $X_2$ and so the distribution of $X_1$ will not be as spread out as the distribution of $X_2$. This eliminates option B.

The answer is E.

**Question 25**

The tablets are randomly selected without replacement from the bottle and therefore we have a hypergeometric distribution.

The correct expression is D. Note that this answer could also have been written as

$$\frac{\binom{50}{5}\binom{100}{5}}{\binom{150}{10}}$$

The answer is D.

**Question 26**

The number of times in a week that Jack is late for his basketball commitments follows a binomial distribution where $n = 3$ and $p = 0.4$.

$$\begin{aligned}\text{Now, the mean } &= np \\ &= 1.2\end{aligned}$$

$$\begin{aligned}\text{And the variance } &= np(1-p) \\ &= 3 \times 0.4 \times 0.6 \\ &= 0.72\end{aligned}$$

The answer is C.

**Question 27**

The boys return what is in their net and so we have sampling with replacement. We have a binomial distribution with the probability of netting an eel being $\frac{1}{4}$.

So $\Pr(X = 2) = {}^5C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3$

The answer is B.

**PART II**

**Question 1**

**a.**

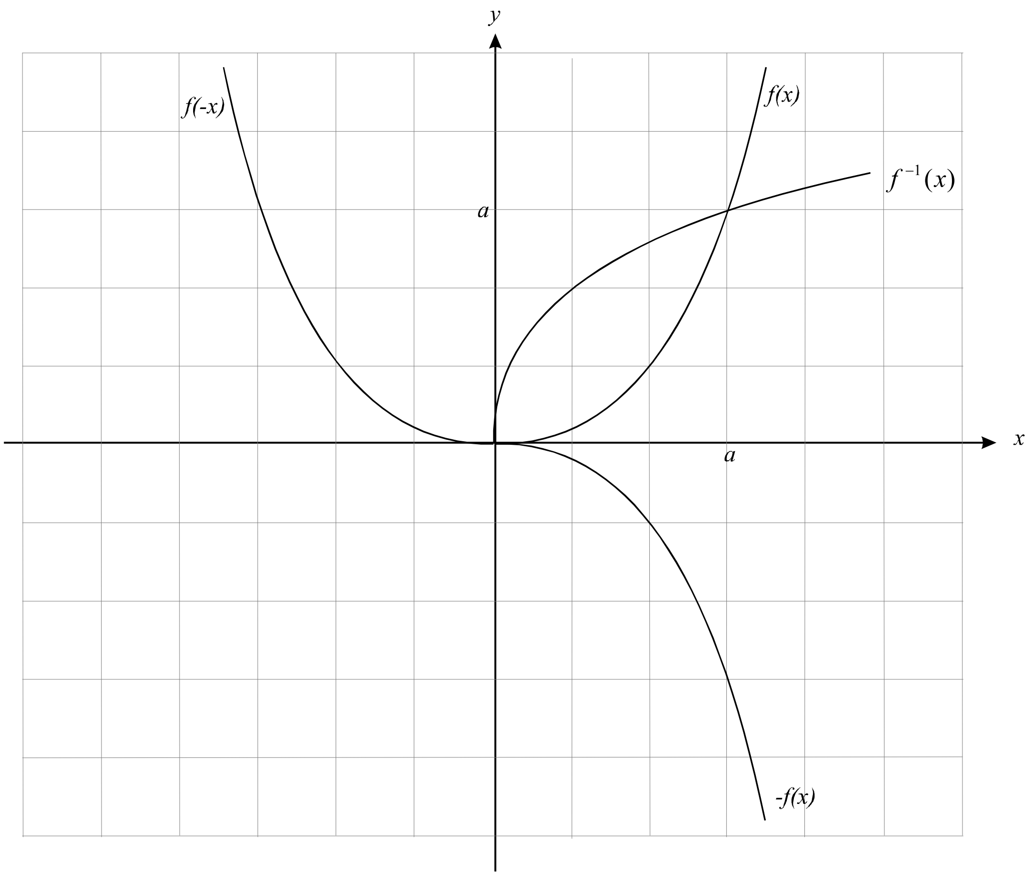
| Number of goals $(x)$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\Pr(X = x)$ | $\frac{2}{20}$ | $\frac{8}{20}$ | $\frac{4}{20}$ | $\frac{6}{20}$ |

**(1 mark)**

**b.** $$E(x) = 0 \times \frac{2}{20} + 1 \times \frac{8}{20} + 2 \times \frac{4}{20} + 3 \times \frac{6}{20}$$
$$= \frac{8}{20} + \frac{8}{20} + \frac{18}{20}$$
$$= \frac{34}{20}$$
$$= 1.7$$

**(1 mark)**

**Question 2**




**(3 marks)**

**Question 3**

**i.** The amplitude is 2, the period is $\pi$, there is no horizontal translation and there has been a vertical translation of 1 unit up. The general equation

$y = A\sin(a(x+b)) + B$

becomes $y = 2\sin(2(x+0)) + 1$

So the function required is $f(x) = 2\sin(2x) + 1$

Check this by graphing it on your graphics calculator.

**(1 mark)**

**ii.** The amplitude is 2, the period is $\pi$, there has been a horizontal translation $\frac{\pi}{4}$ units to the right and a vertical translation of 1 unit up. The general equation

$y = A\cos(a(x+b)) + B$

becomes $y = 2\cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$

So the function required is $f(x) = 2\cos\left(2x - \frac{\pi}{2}\right) + 1$

**(1 mark)** for horizontal translation
**(1 mark)** for the complete rule

**Question 4**

**a.** Average rate of change $= \dfrac{D(2) - D(1)}{2-1}$

$= -2.91\,\text{m/s}$ correct to 2 decimal places

**(1 mark)**

**b**. $D(t) = (t^2 - 5t)e^{0.1t}$

$D'(t) = (2t-5)e^{0.1t} + (t^2 - 5t) \times 0.1e^{0.1t}$

**(1 mark)**

$D'(2) = -1.95$ m/s correct to 2 decimal places

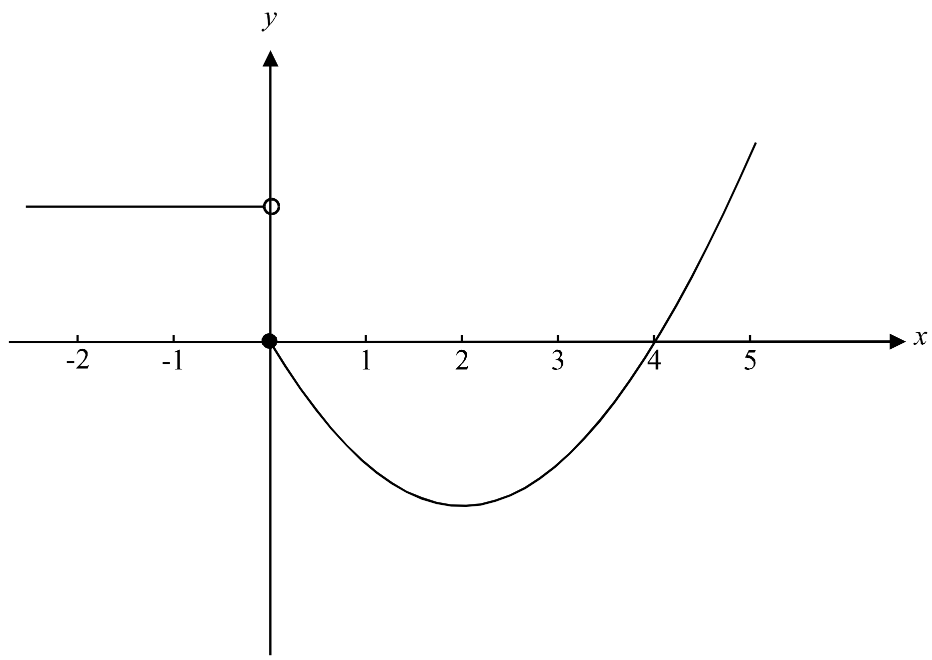
**(1 mark)**

**Question 5**

$f(0)=0$ tells us that the graph passes through the point $(0,0)$
$f(4)=0$ tells us that the graph passes through the point $(4,0)$
$f(-1)>0$ tells us that the point where $x=-1$ lies above the $x$-axis
$f'(2)=0$ tells us that at the point where $x=2$, we have a turning point or a stationary point of inflection.
$f'(x)=0, x\in(-\infty,0)$ tells us that the gradient of the curve for values of $x$ less than 0 is zero and hence we have a horizontal line. Combining this with the clue $f(-1)>0$ we now know that we have a horizontal line, which runs above the $x$-axis for $x<0$.

$f'(x)<0, x\in(0,2)$ tells us that the graph has a negative gradient for values of $x$ between 0 and 2.
$f'(x)>0, x\in(2,\infty)$ tells us that the graph has a positive gradient for values of $x$ greater than 2.
Putting these all together we obtain the following graph.

**(1 mark)** for linear branch
**(1 mark)** for (0,0), (4,0) and turning point
**(1 mark)** for shape of curve



**Question 6**

We know that $f(x+h) \approx f(x) + hf'(x)$.

Now, $f(x) = \log_e(x^2 + 1)$

so, $f(2) = \log_e 5$

Also $f'(x) = \dfrac{2x}{x^2+1}$ **(1 mark)**

so, $f'(2) = \dfrac{4}{5}$

Now, $h = 2.01 - 2 = 0.01$

So $f(x+h) = f(2.01)$

$$\approx \log_e 5 + 0.01 \times \frac{4}{5}$$

$$= \log_e 5 + 0.008$$

So, $Y = \log_e 5 + 0.008$ **(1 mark)**

**Question 7**

**a.** $\Pr(1582 < X < 1642)$

$= \Pr(-1.5 < Z < 3.5)$ **(1 mark)**

$= \Pr(X < 3.5) - \Pr(X < -1.5)$

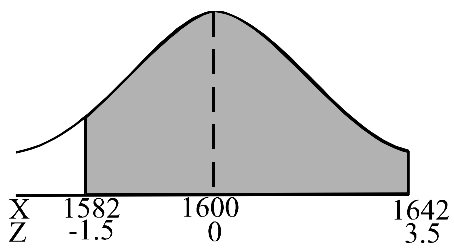
$= \Pr(X < 3.5) - (1 - \Pr(X < 1.5))$

$= 0.9998 - 1 + 0.9332$

$= 0.933$ to 3 decimal places **(1 mark)**

$$Z = \frac{X-\mu}{\sigma} = \frac{1642-1600}{12} = \frac{42}{12} = 3.5$$

$$Z = \frac{X-\mu}{\sigma} = \frac{1582-1600}{12} = \frac{-18}{12} = -1.5$$




**b.** Pr(X<1612) = Pr(Z<1)

$= 0.8413$

$$Z = \frac{X-\mu}{\sigma} = \frac{1612-1600}{12} = 1$$

**(1 mark)**

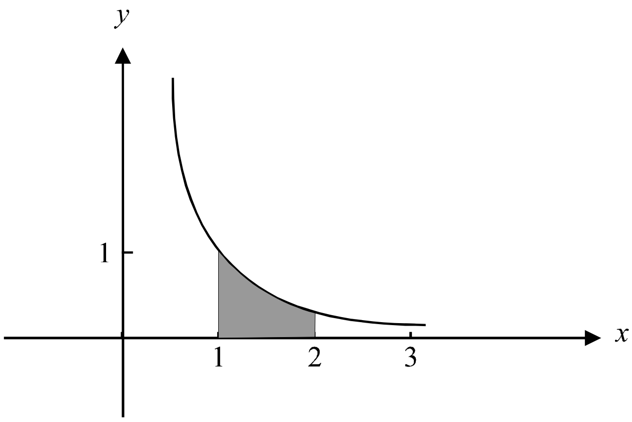
So $\Pr(x = 2) = {}^3C_2 (0.8413)^2 (0.1587)^1$

$= 0.3370$ correct to 4 decimal places

**(1 mark)**

**Question 8**

Sketch a graph first.




Area required $= \int_1^2 \frac{1}{(3x-2)^{\frac{3}{2}}}\,dx$ **(1 mark)**

$$= \int_1^2 (3x-2)^{-\frac{3}{2}}\,dx$$

$$= \left[\frac{(3x-2)^{-\frac{1}{2}}}{3\times -\frac{1}{2}}\right]_1^2$$

$$= -\frac{2}{3}\left[\frac{1}{\sqrt{3x-2}}\right]_1^2$$ **(1 mark)**

$$= -\frac{2}{3}\left\{\left(\frac{1}{\sqrt{4}}\right) - \left(\frac{1}{\sqrt{1}}\right)\right\}$$

$$= -\frac{2}{3}\left(\frac{1}{2} - 1\right)$$

$$= -\frac{2}{3} \times -\frac{1}{2}$$

$= \frac{1}{3}$ square units **(1 mark)**