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VCE Specialist Mathematics ½
Complex Numbers I [8.1]
Workbook

Outline:

Section A: Introduction to Complex Numbers

Sub-Section: Imaginary Numbers

What are imaginary numbers?

Imaginary Number

$$\sqrt{-1} = i$$

$$-1 = i^2$$

- Imaginary number is simply the square root of a negative number.

Question 1

Evaluate the following using imaginary numbers.

a. $\sqrt{-16}$

b. $\sqrt{-4} - \sqrt{-25}$

TIP: Simply substitute $-1 = i^2$.

History: Where did Imaginary Numbers come from?

- For many centuries, the general solution for quadratic equations existed. However, mathematicians struggled to discover the general solutions for cubic equations.
- In 1545, Gerolamo Cardano (1505-1576) came up with a general solution for cubic equations.
- In his general solution, he often encountered the square root of -1 which he deemed insignificant.

$$\sqrt[3]{2+\sqrt{-121}} = a + \_\_\_\_\_\_\_\_\_\_$$

$$\sqrt[3]{2-\sqrt{-121}} = a - \_\_\_\_\_\_\_\_\_\_$$

- However, Rafael Bombelli realises that the general solution for cubic equations still works despite encountering the square root of -1.
- From here, the concept of imaginary numbers was introduced.

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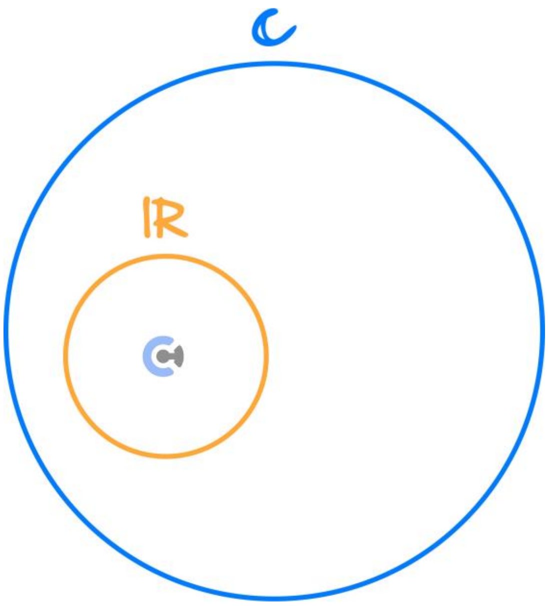
Sub-Section: Complex Numbers

What are complex numbers?

Complex Number

- The set of complex number is given by $\mathbb{C}$.
- It is a combination of real and imaginary numbers.



$$z = x + yi$$

Discussion: Are all imaginary numbers complex number? And is it true for vice versa?

Question 2

Simplify the following complex numbers.

a. $4 + 5$

b. $\sqrt{-9} + 3 - \sqrt{-25}$

c. $2 + 4i - \sqrt{-1}$

NOTE: Real numbers are subset of complex numbers.

<u>**Discussion:**</u> **What are complex numbers comprised of?**

Definition

Real and Imaginary Part of Complex Numbers

$$z = x + yi$$

- For complex number $x + yi$,

$$Re(z) = x$$

$$Im(z) = y$$

- Real part $= x$ and Imaginary part $= y$.

Question 3

Evaluate the following:

a. $\text{Im}(2 + 3i)$

b. $\text{Re}(2 + 3i + 4i - 5i - 6i + 10i - 100i)$

c. $\text{Im}(\sqrt{-25})$

d. $\text{Im}(\text{Im}(2 + 3i))$

Misconception

"Imaginary part of $2 + 3i$ ($\text{Im}(2 + 3i)$) is $3i$."

TRUTH:
Both the real part and the imaginary part of a complex number are real numbers.
So, $\text{Im}(2 + 3i) = 3$.

How do we get i on our technology?

Calculator Commands: Imaginary Numbers

Mathematica	TI-Nspire	Casio Classpad
Capital i.	Press the π button twice.	Under maths 2.

Question 4 Tech-Active.

Evaluate $2 + 3i - 4i$ on your technology.

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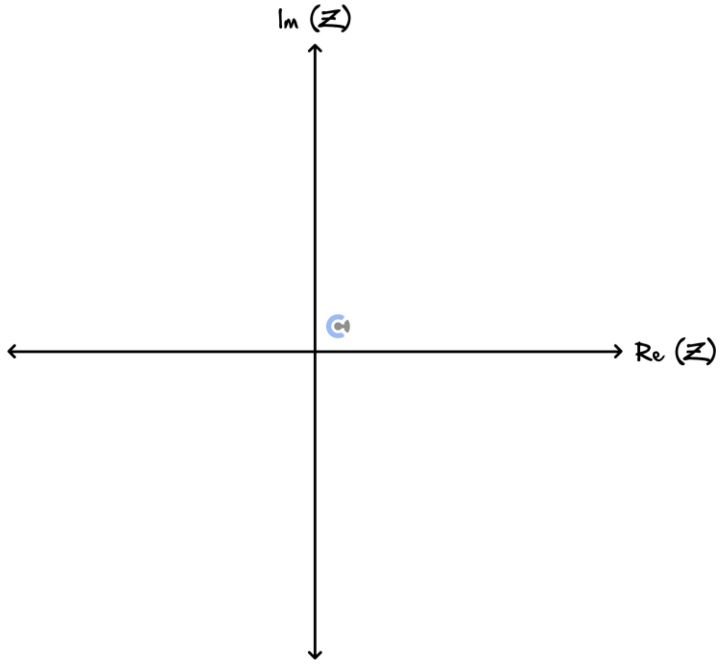
Section B: Representation of Complex Numbers

Sub-Section: Argand Diagram

How do we visualise complex numbers?

Argand Diagram



- We can position each complex number as a point on an Argand diagram.
- The vertical axis is the imaginary part of a complex number.
- The horizontal axis is the real part of a complex number.

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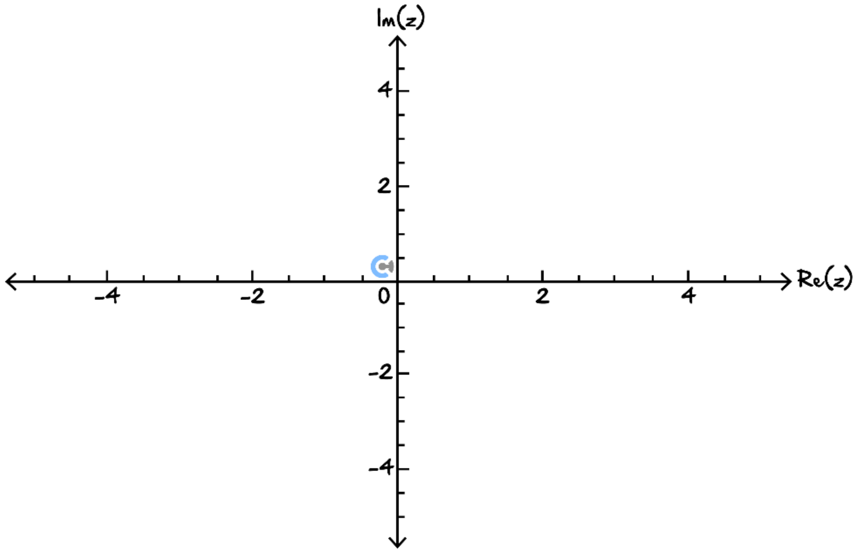
Question 5

Consider the Argand diagram below.



Plot the following points on the above Argand diagram.

a. $-1 - i$

b. $2 + i$

c. $-3 - 4i$

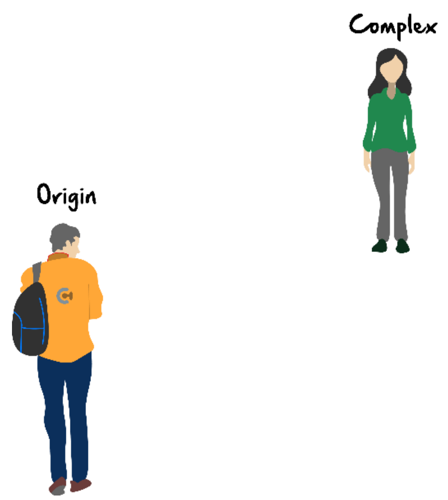
<u>Discussion:</u> How many points could a single complex number occupy on an Argand diagram?

<u>Discussion:</u> When we represent complex numbers, what are we actually representing?

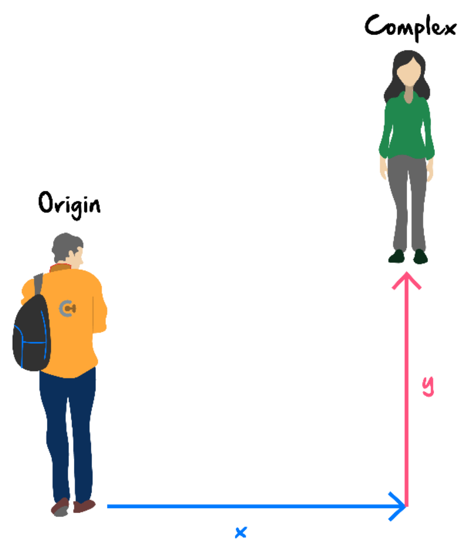
Sub-Section: Rectangular and Polar Forms

Analogy: Rectangular Form

- "Origin" wants to walk towards his love at first sight "Complex."



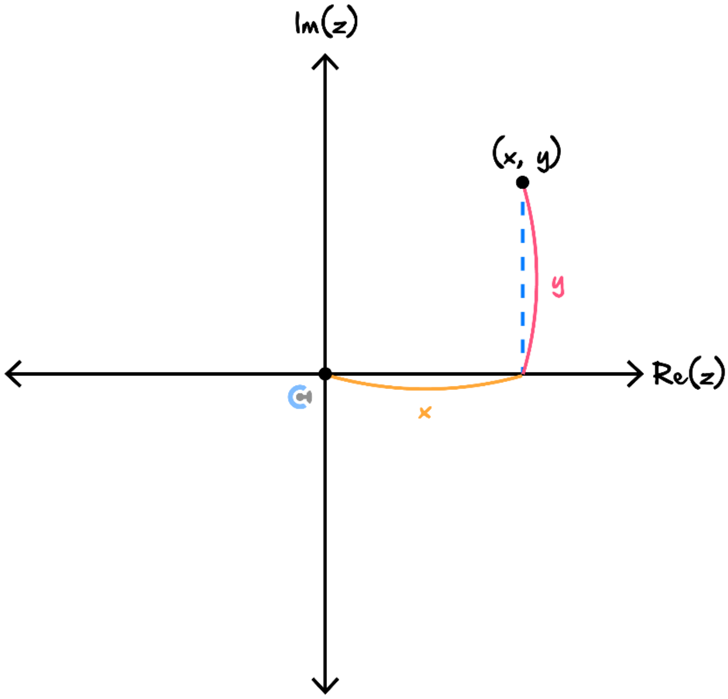
- To do so, "Origin" can only walk horizontally and vertically.



- How can we represent "Complex's" position from the "Origin"?

- What shape does origin's path form?

Rectangular Form



- It is simply a way to represent a complex number's position on an Argand diagram.

$$z = x + yi = Re(z) + Im(z)i$$

NOTE: We have been using the rectangular form!

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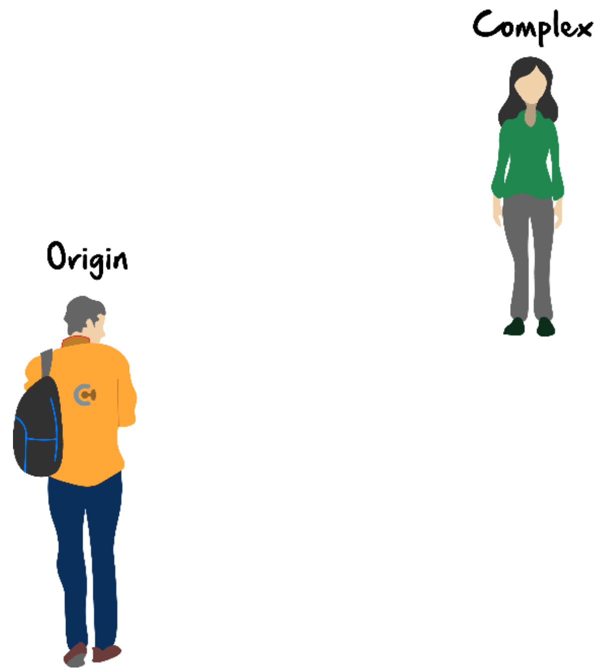
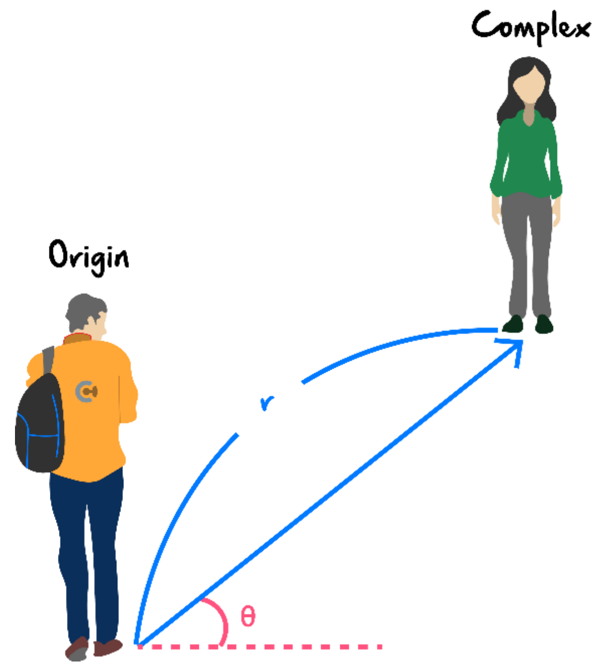
<u>Analogy</u>: **Polar Form**

- "Origin" wants to walk towards his love at first sight "Complex."



- Now unlike last time, "Origin" is able to walk diagonally at a certain bearing.

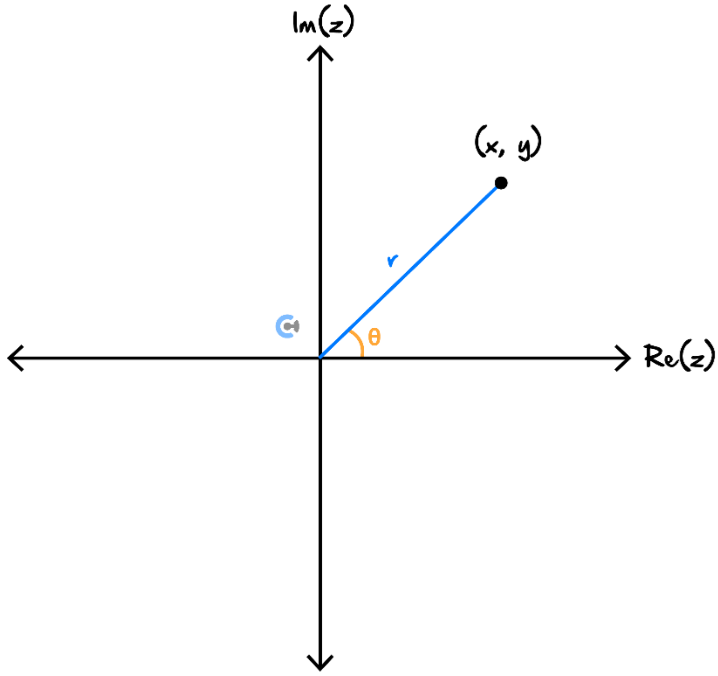


- The distance "Origin" needs to walk is called the ____________________.

- The angle "Origin" needs to walk is called the ______________________________.

Polar Form

- It is simply a way to represent a complex number's position on an Argand diagram.



$$z = r\text{cis}(\theta)$$

- The distance from the origin to the complex number is called the ____________________.
- The angle from the origin to the complex number is called the ____________________.

- Argument must be within the principal argument for final answers $(-\pi, \pi]$.

$$\text{Principal Argument} = (-\pi, \pi]$$

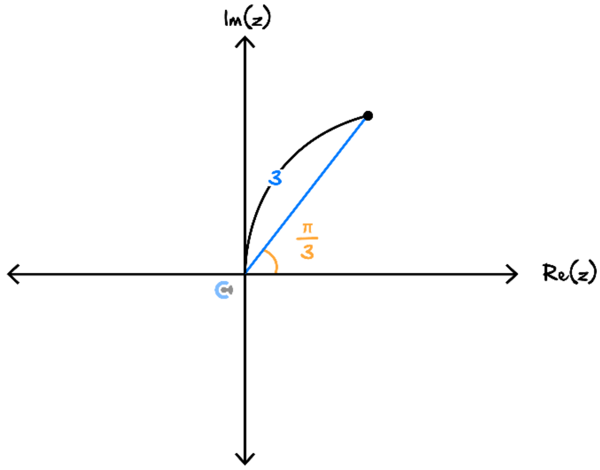
Discussion: Why do we only need the angles $(-\pi, \pi]$?

Question 6

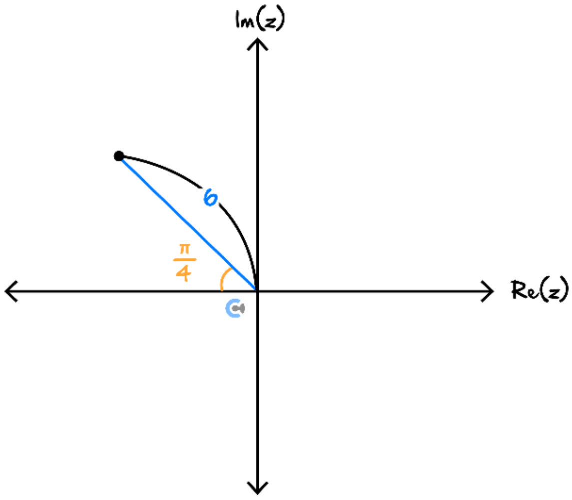
Consider the following complex numbers on the Argand diagram.

Evaluate the polar form for z.

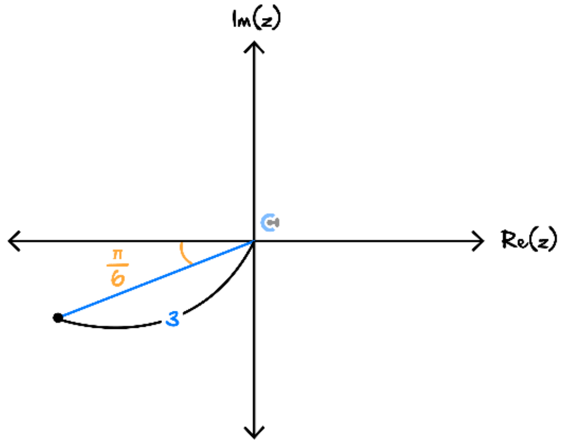
a.



b.



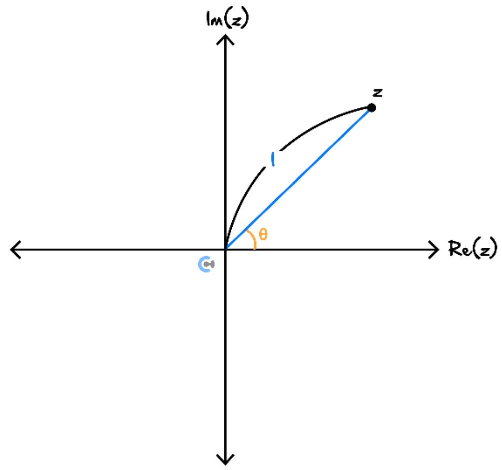
c.



What does cis stand for?

Exploration: Derivation of Polar form of Complex Numbers

- Consider a complex number below.

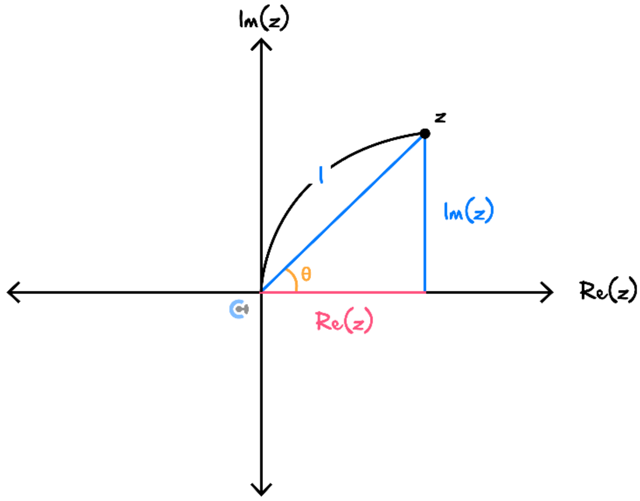


- The polar form of the complex number is:

$$z = \_\_\_\_\_\_\_\_\_\_\_\_$$

- What would the $\text{Re}(z)$ and $\text{Im}(z)$ equal to?

  HINT: Use trigonometry.



$$\textbf{Re}(z) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\textbf{Im}(z) = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Hence,

$$z = \cos(\theta) + i\sin(\theta)$$

- What do you think c i s stand for?

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Sub-Section: Converting Forms

Converting Polar Form to Rectangular Form

- Simply change cis into $\cos + i\ \sin$.

$$\mathbf{cis}(\boldsymbol{\theta}) = \mathbf{cos}(\boldsymbol{\theta}) + \boldsymbol{i}\mathbf{sin}(\boldsymbol{\theta})$$

Question 7 Walkthrough.

Let $z = 4\text{cis}\left(\frac{\pi}{3}\right)$. Convert z into Cartesian form.

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Question 8

Let $u = 2\text{cis}\left(-\frac{5\pi}{6}\right)$. Convert u into Cartesian form.

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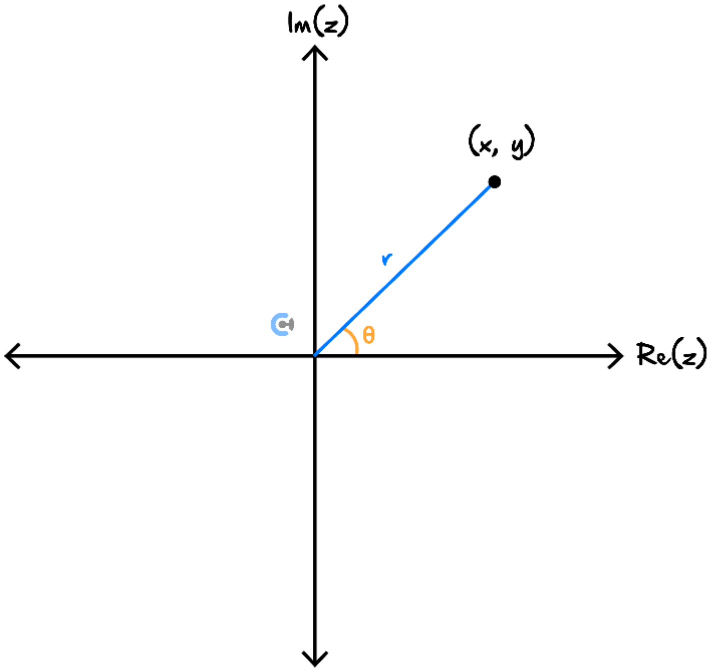
Now, the other way around!

Converting Rectangular Form to Polar Form



- For,

$$z = a + bi$$

- Radius is simply the size of the complex number.

$$r = |z| = \sqrt{a^2 + b^2}$$

- And its argument:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \text{ for } 1^{\text{st}} \text{ and } 4^{\text{th}} \text{ quadrant}$$

$$\theta = \pi - \tan^{-1}\left(\frac{b}{a}\right) \text{ for } 2^{\text{nd}} \text{ quadrant}$$

$$\theta = -\pi + \tan^{-1}\left(\frac{b}{a}\right) \text{ for } 3^{\text{rd}} \text{ quadrant}$$

Question 9 Walkthrough.

Convert $\sqrt{3} - i$ into polar form with principal arguments.

Space for Personal Notes

Your turn!

Question 10

Convert the following into polar form with principal arguments:

a. $\sqrt{3}+i$

b. $-2+2\sqrt{3}\,i$

c. $-4-4i$

d. $1-\sqrt{3}i$

Calculator Commands: Form Conversion

Mathematica

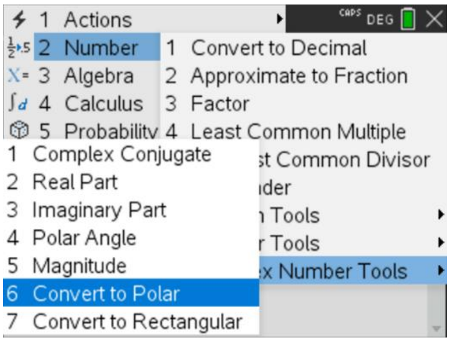
$$\text{ToPolarCoordinates}[1+\sqrt{3}\text{I}]$$

$$\left\{2,\frac{\pi}{3}\right\}$$

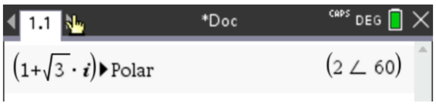
$$\text{FromPolarCoordinates}\left[\left\{2,\frac{\pi}{3}\right\}\right]$$

$$\{1+\sqrt{3}\}$$

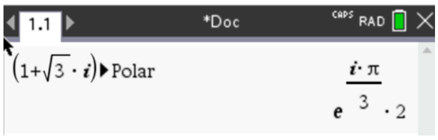
TI-Nspire



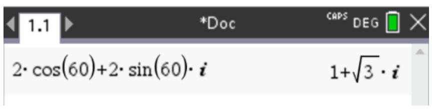
- Rectangular → Polar.Menu 2 9 6
  "i" is under the pi button.



- If CAS is in degrees

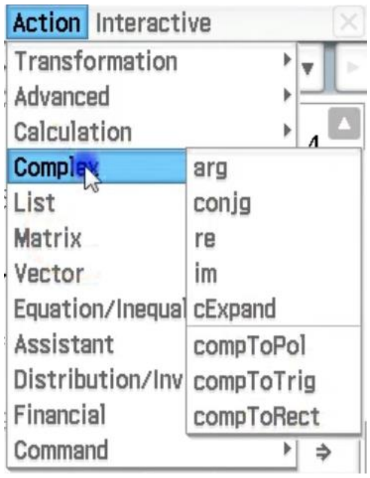


- In Radian form note: $*e^{i}*$ is the same as CIS.



- Polar → Rectangular. Don't bother with menu 297, just split the complex into two pieces.

Casio Classpad



- Complex → compToPol (Changes to Polar Form)
- Complex → compToRect (Changes to Rectangular Form)

Space for Personal Notes

Question 11 Tech-Active.

Convert $10 - 10\sqrt{3}i$ into polar form with principal arguments.

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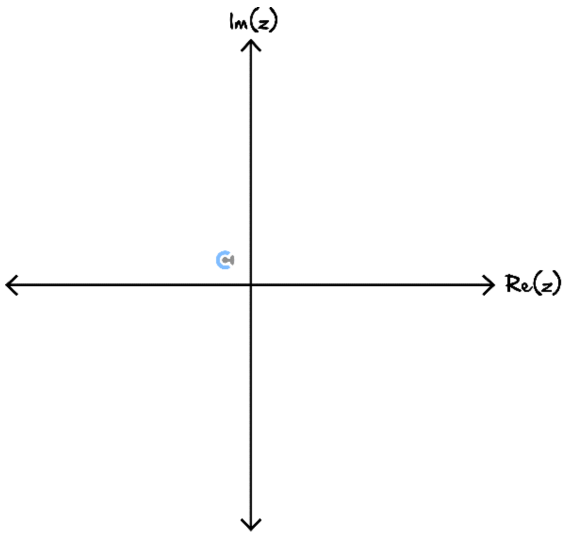
Sub-Section: Purely Real and Imaginary Numbers

Discussion: What arguments do real and purely imaginary numbers have?

- Real: ____________________
- Purely Imaginary: ____________________

Arguments for Purely Imaginary and Real



Real: πk

Positive Real: $2\pi k$

Negative Real: $\pi + 2\pi k$

Imaginary: $\pi/2 + \pi k$

Positive Imaginary: $\pi/2 + 2\pi k$

Negative Imaginary: $-\pi/2 + 2\pi k$

where, $k \in \mathbb{Z}$

Question 12 Walkthrough.

Solve for the value(s) of n such that, z is a real number.

$$z = 3\text{cis}\left(\frac{\pi}{4}n\right)$$

Space for Personal Notes

Your turn!

Question 13

Solve for the value(s) of n such that, z is a purely imaginary number.

$$z = 2\text{cis}\left(\frac{\pi}{6}n\right)$$

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Sub-Section: Conjugate

Let's look at conjugate complex numbers!

Complex Conjugate

- In Cartesian form:

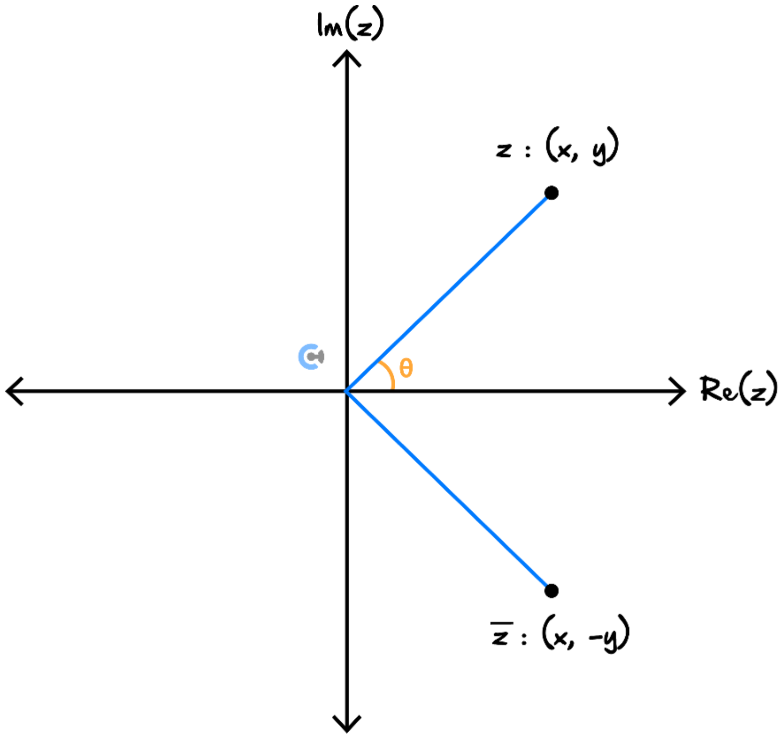
$$z = x + yi$$

$$\bar{z} = \underline{\hspace{6cm}}$$

- In polar form:

$$z = r\,\text{cis}(\theta)$$

$$\bar{z} = \underline{\hspace{6cm}}$$



Conjugates are merely ______________________________.

Question 14

State the conjugate of $2 - 3i$.

<u>Exploration</u>: Why are conjugates important?

$$z + \bar{z} \in R$$

$$z\bar{z} \in R$$

- Sums and multiplications of conjugate pairs are always ______________________________.

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Question 15

Consider $z = 3 + 2i$.

a. Find $z + \bar{z}$.

b. Find $z\bar{z}$.

c. Find $|z|$. What do you notice?

Conjugate Properties

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{kz} = k\bar{z}$$

$$z + \bar{z} = 2\text{Re}(z)$$

$$z - \bar{z} = 2\text{Im}(z)$$

$$z\bar{z} = |z|^2 = |\bar{z}|^2$$

Section C: Operations of Complex Numbers

Sub-Section: Multiplication

Multiplication

- Polar Form:

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

- Rectangular Form:

$$(x + yi)(u + vi) = xu + yui + xvi + yvi^2 = xu - yv + (yu + xv)i$$

Discussion: Which form do we prefer for multiplication?

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How does the multiplication work using polar form?

Question 16

Let $z_1 = \text{cis}(a)$ and $z_2 = \text{cis}(b)$.

a. Convert z_1 and z_2 into a rectangular form.

b. Hence, find $z_1 z_2$ in rectangular form.

c. Simplify the terms using the compound angle formulae. What do you notice?

NOTE: The polar form multiplication is derived from the compound angle formulae!

Question 17

Evaluate the following:

a. $(2 + 3i)(1 - i)$

b. $2\text{cis}\left(\frac{\pi}{6}\right) \cdot 3\text{cis}\left(\frac{\pi}{3}\right)$

Space for Personal Notes

Sub-Section: Division

Now, division!

Division

- Polar Form:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$$

- In rectangular form, we multiply the ____________________________________ on the top and bottom.

$$\frac{x+yi}{u+vi} = \frac{x+yi}{u+vi} \times \underline{\qquad\qquad}$$

Discussion: Why would we want to multiply by the conjugate of the denominator? What does that achieve?

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Question 18 Walkthrough.

Evaluate the following:

a. $\frac{2+5i}{3-2i}$

b. $\frac{6\text{cis}\left(\frac{\pi}{6}\right)}{2\text{cis}\left(\frac{\pi}{4}\right)}$

Space for Personal Notes

Question 19

Evaluate the following:

a. $\frac{3+7i}{1+i}$

b. $\frac{1+4i}{2-i}$

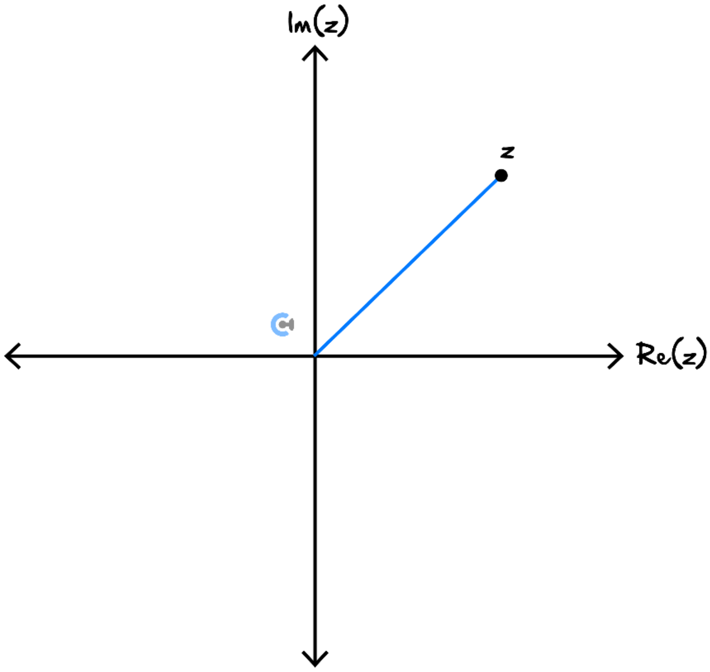
c. $\frac{4\text{cis}\left(\frac{\pi}{6}\right)}{3\text{cis}\left(-\frac{\pi}{3}\right)}$

Sub-Section: Multiplication and Division of i

Discussion: What is the polar form of i?

Exploration: Multiplication of i

- Consider:



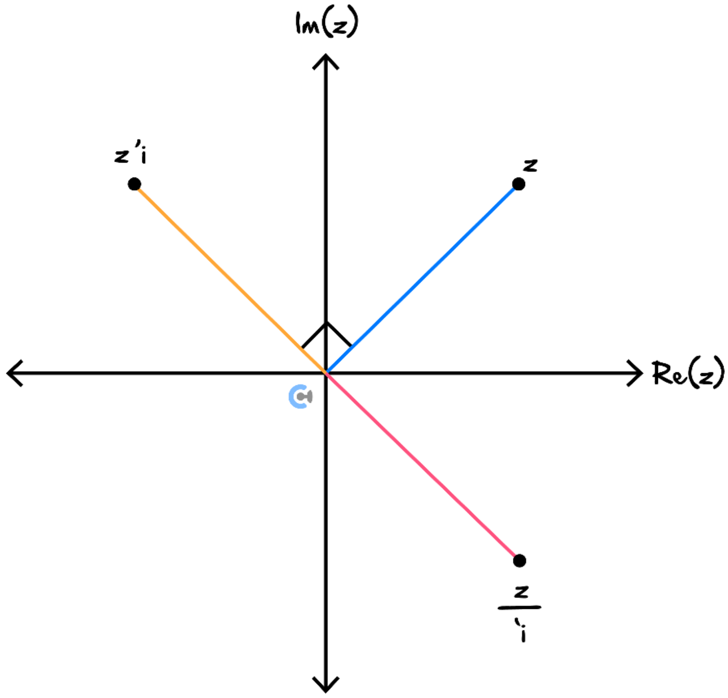
$$z = r \operatorname{cis}(\theta)$$

- What does zi equal to, given that $i = \operatorname{cis}\left(\frac{\pi}{2}\right)$?

$$zi = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Sketch the zi on the axes above.

Multiplication and Division by i



- Multiplication by i rotates the complex number by $\frac{\pi}{2}$ in the ____________________ direction.
- Division by i rotates the complex number by $\frac{\pi}{2}$ in the ____________________ direction.

Discussion: What would happen if we multiply z by i 4 times?

Space for Personal Notes

<u>Complex Number Properties</u>

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$$

Multiplication by i rotates anticlockwise 90 degrees.

Space for Personal Notes

Sub-Section: De Moivre's Theorem

What happens if we multiply the same complex number n times? z^n?

De Moivre's Theorem

- For,

$$z = r\text{cis}(\theta)$$

- z^n equals to:

$$z^n = r^n\text{cis}(n\theta)$$

Exploration: De Moivre's Theorem

- It is derived from the polar form multiplication.

$$z^n = z \cdot z \cdots\cdots z$$

$$= r\ \text{cis}(\theta) \cdot r\ \text{cis}(\theta) \cdots\cdots r\ \text{cis}(\theta)$$

$$= r \cdot r \cdots\cdots r\text{cis}(\theta + \theta + \cdots + \theta)$$

$$= r^n\text{cis}(n\theta)$$

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Question 20 Walkthrough.

It is known that, $z = 1 - \sqrt{3}i$.

Find z^3 in rectangular form.

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Question 21

Evaluate the following:

a. z^4 for $z = 3\text{cis}\left(\frac{\pi}{12}\right)$.

b. z^3 for $z = 1 - \sqrt{3}i$.

c. z^2 for $z = -2\sqrt{3} - 2i$.

d. z^4 for $z = 2 - 2i$.

NOTE: Even if your answer is in rectangular form, it is easier to use De Moivre's theorem via polar form.

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Cheat Sheet

[8.1.1] – Find polar and rectangular forms of complex numbers

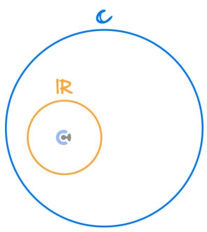
- **Imaginary Number:**

$$\sqrt{-1} = i$$

$$-1 = i^2$$

  - Imaginary number is simply the square root of a negative number.

- **Complex Number:**
  - The set of complex numbers is given by $\mathbb{C}$.
  - It is a combination of real and imaginary numbers.



$$z = x + yi$$

- **Real and Imaginary Part of Complex Numbers:**

$$z = x + yi$$

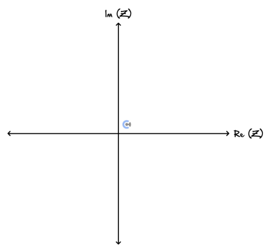
  - For complex number $x + yi$,

$$\text{Re}(z) = x$$

$$\text{Im}(z) = y$$

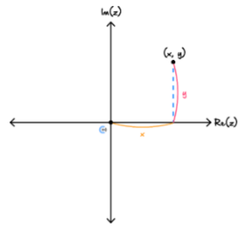
  - Real part $= x$ and Imaginary part $= y$.

- **Argand Diagram:**



  - We can position each complex number as a point on an argand diagram.
  - The vertical axis is the imaginary part of a complex number.
  - The horizontal axis is the real part of a complex number.

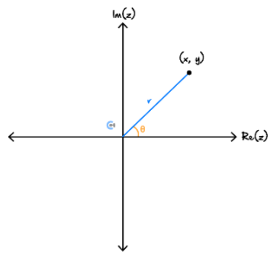
- **Rectangular Form:**



  - It is simply a way to represent a complex number's position on an argand diagram.

$$z = x + yi = \text{Re}(z) + \text{Im}(z)i$$

- **Polar Form:**
  - It is simply a way to represent a complex number's position on an argand diagram.



$$z = r\text{cis}(\theta)$$

  - The distance from the origin to the complex number is called the radius.
  - The angle from the origin to the complex number is called the argument.
  - Argument must be within the principal argument for final answers $(-\pi, \pi]$.

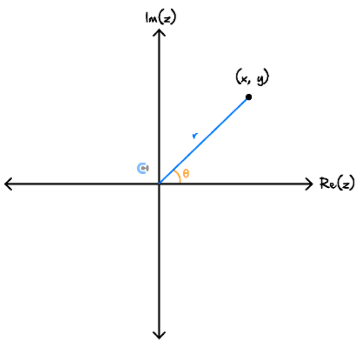
$$\text{Principal Argument} = (-\pi, \pi]$$

- **Converting Polar Form to Rectangular Form:**
  - Simply change cis into $\cos + i\sin$.

$$\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$$

Cheat Sheet

➤ **Converting Rectangular Form to Polar Form:**



- For,

$$z = a + bi$$

- Radius is simply the size of the complex number.

$$r = |z| = \sqrt{a^2 + b^2}$$

- And its argument:

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$ for 1^{st} and 4^{th} quadrant.

$\theta = \pi - \tan^{-1}\left(\frac{b}{a}\right)$ for 2^{nd} quadrant.

$\theta = -\pi + \tan^{-1}\left(\frac{b}{a}\right)$ for 3^{rd} quadrant.

➤ **Arguments for Purely Imaginary and Real:**

$$\textbf{Real}: \pi k$$

$$\textbf{Positive Real}: 2\pi k$$

$$\textbf{Negative Real}: \pi + 2\pi k$$

$$\textbf{Imaginary}: \pi/2 + \pi k$$

$$\textbf{Positive Imaginary}: \pi/2 + 2\pi k$$

$$\textbf{Negative Imaginary}: -\pi/2 + 2\pi k$$

where, $k \in \mathbb{Z}$

[8.1.2] – Evaluate operations of complex numbers

➤ **Complex Conjugate:**

- In cartesian form:

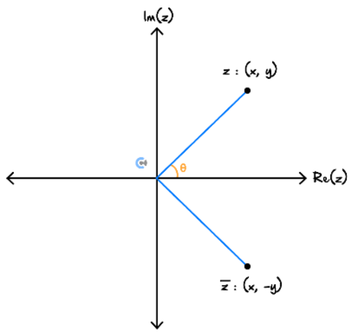
$$z = x + yi$$

$$\overline{z} = x - yi$$

- In polar form:

$$z = r\,\text{cis}(\theta)$$

$$\overline{z} = r\,\text{cis}(-\theta)$$



- Conjugates are merely reflected around Re-axis.

➤ **Conjugate Properties:**

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{kz} = k\overline{z}$$

$$z + \overline{z} = 2\text{Re}(z)$$

$$z - \overline{z} = 2\text{Im}(z)$$

$$z\overline{z} = |z|^2 = |\overline{z}|^2$$

➤ **Multiplication:**

- Polar Form:

$$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$$

- Rectangular Form:

$$(x + yi)(u + vi) = xu + yui + xvi + yvi^2$$
$$= xu - yv + (yu + xv)i$$

Cheat Sheet

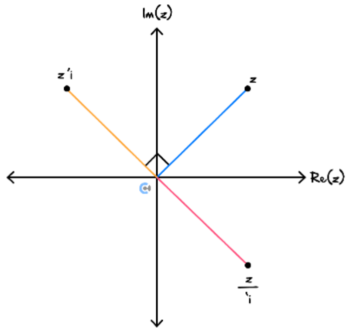
- **Division:**
  - Polar Form:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$$

  - In rectangular form, we multiply the conjugate of the denominator on the top and bottom.

$$\frac{x+yi}{u+vi} = \frac{x+yi}{u+vi} \times \frac{u-vi}{u-vi}$$

- **Multiplication and Division by i:**



  - Multiplication by i rotates the complex number by $\frac{\pi}{2}$ in the anticlockwise direction.
  - Division by i rotates the complex number by $\frac{\pi}{2}$ in the clockwise direction.

- **Complex Number Properties:**

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$$

Multiplication by i rotates anticlockwise 90 degrees.

[8.1.3] – Apply De Moivre's Theorem

- **De Moivre's Theorem:**
  - For,

$$z = r\ \text{cis}(\theta)$$

  - z^n equals to:

$$z^n = r^n\text{cis}(n\theta)$$