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VCE Specialist Mathematics ½

Vectors I [6.1]

Homework Solutions

Admin Info & Homework Outline:

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Section A: Compulsory Questions

Sub-Section [6.1.1]: Basics of Vectors

Question 1

The vector $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is defined by the directed line segment from $(-2, 6)$ to $(4, -2)$. Find a and b.

$$a = 6, \quad b = -8$$

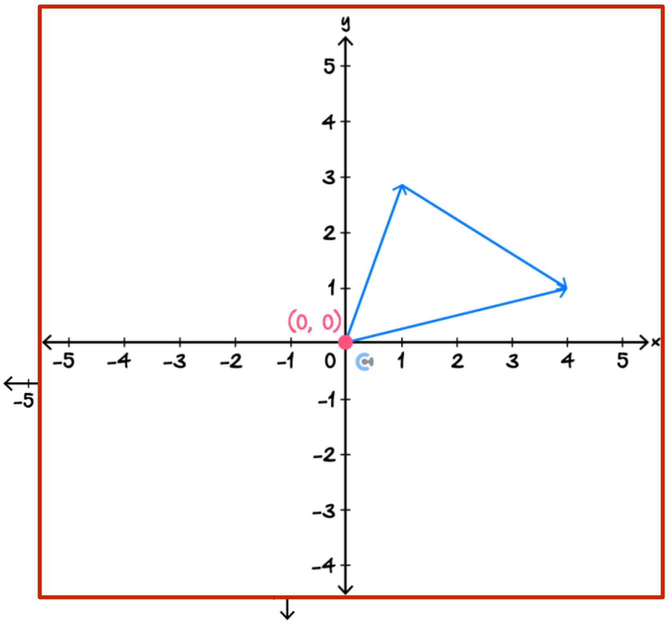
Question 2

$A = (1, 3)$, $B = (4, 1)$ and O is the origin. Sketch the following vectors:

a. $\overrightarrow{OA}$

b. $\overrightarrow{OB}$

c. $\overrightarrow{AB}$



Question 3

If $\boldsymbol{a} = \boldsymbol{i} - 5\boldsymbol{j}$ and $\boldsymbol{b} = -3\boldsymbol{i} + 6\boldsymbol{j}$ find in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

a. $a + b$

$$\mathbf{a} + \mathbf{b} = -2\mathbf{i} + \mathbf{j}$$

b. $2a - 3b$

$$2\mathbf{a} - 3\mathbf{b} = 11\mathbf{i} - 28\mathbf{j}$$

Question 4

In the triangle OAB, $\overrightarrow{OA} = 4\boldsymbol{i} + 2\boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} + 3\boldsymbol{j}$. If M is the midpoint of AB, find $\overrightarrow{OM}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1-4)\mathbf{i} + (3-2)\mathbf{j} = (-3)\mathbf{i} + (1)\mathbf{j}$$

$$\overrightarrow{AM} = \frac{1}{2}((-3)\mathbf{i} + (1)\mathbf{j}) = (-\frac{3}{2})\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = (4\mathbf{i} + 2\mathbf{j}) + \left(-\frac{3}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)$$

$$= \left(4 - \frac{3}{2}\right)\mathbf{i} + \left(2 + \frac{1}{2}\right)\mathbf{j}$$

$$= \left(\frac{8}{2} - \frac{3}{2}\right)\mathbf{i} + \left(\frac{4}{2} + \frac{1}{2}\right)\mathbf{j}$$

$$= \left(\frac{5}{2}\right)\mathbf{i} + \left(\frac{5}{2}\right)\mathbf{j}$$

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Sub-Section [6.1.2]: Magnitude and Unit Vectors

Question 5

a. Find the length of vector $\boldsymbol{v} = 3\boldsymbol{i} - 2\boldsymbol{j}$.

$$|\mathbf{v}| = \sqrt{(3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$

b. Find the unit vector parallel to $\boldsymbol{v}$.

$$\mathbf{u} = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j}) = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}$$

Question 6

Let $\boldsymbol{a} = -\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j}$ .Vector $\boldsymbol{c}$ is parallel to $2\boldsymbol{a} + \boldsymbol{b}$, and has a magnitude of 10.

Find $\boldsymbol{c}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

$$2\mathbf{a} + \mathbf{b} = (-2\mathbf{i} + 4\mathbf{j}) + (3\mathbf{i} + 5\mathbf{j}) = (1\mathbf{i}) + (9\mathbf{j})$$

$$|\mathbf{i} + 9\mathbf{j}| = \sqrt{(1)^2 + (9)^2} = \sqrt{1+81} = \sqrt{82}$$

$$\mathbf{c} = k(\mathbf{i} + 9\mathbf{j})$$

$$|k(\mathbf{i} + 9\mathbf{j})| = |k||\mathbf{i} + 9\mathbf{j}|$$

$$|k|\sqrt{82} = 10$$

$$|k| = \frac{10}{\sqrt{82}}$$

$$k = \pm\frac{10}{\sqrt{82}}$$

$$\mathbf{c} = \pm\left(\frac{10}{\sqrt{82}}\mathbf{i} + \frac{90}{\sqrt{82}}\mathbf{j}\right)$$

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Question 7

$A(2,3)$, $B(4,5)$ and $C(7,2)$ are the vertices of a triangle ABC.

a. Find

i. $|\overrightarrow{AB}|$

$\overrightarrow{AB} = (4-2)\mathbf{i} + (5-3)\mathbf{j} = 2\mathbf{i} + 2\mathbf{j}$

$|\overrightarrow{AB}| = \sqrt{(2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

ii. $|\overrightarrow{BC}|$

$\overrightarrow{BC} = (7-4)\mathbf{i} + (2-5)\mathbf{j} = 3\mathbf{i} - 3\mathbf{j}$

$|\overrightarrow{BC}| = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

iii. $\overrightarrow{AC}$

$\overrightarrow{AC} = (7-2)\mathbf{i} + (2-3)\mathbf{j} = 5\mathbf{i} - 1\mathbf{j}$

$|\overrightarrow{AC}| = \sqrt{(5)^2 + (-1)^2} = \sqrt{25+1} = \sqrt{26}$

b. Identify the type of triangle.

The triangle is a right-angled triangle

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Sub-Section [6.1.3]: Dot Product

Question 8

If $\boldsymbol{u} = -3\boldsymbol{i} + \boldsymbol{j}$ and $= 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, find $\boldsymbol{u} \bullet \boldsymbol{v}$.

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Question 9

If $|\boldsymbol{u}| = 5$ and $|\boldsymbol{v}| = 6$, and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is $60°$, find $\boldsymbol{u} \bullet \boldsymbol{v}$.

$$\mathbf{u} \cdot \mathbf{v} = (5)(6)\left(\frac{1}{2}\right) = 15$$

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Question 10 Tech-Active.

Find the angle of the vector $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ makes with the positive direction of the z-axis in degrees, correct to two decimal places.

36.70°

Question 11

A position vector in two dimensions has a magnitude of 4 and a direction of 120° measured anticlockwise from the x-axis. Find the vector.

$$\mathbf{v} = 4\cos(120^\circ)\mathbf{i} + 4\sin(120^\circ)\mathbf{j}$$

$$\boxed{\mathbf{v} = -2\mathbf{i} + 2\sqrt{3}\mathbf{j}}$$

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Sub-Section: Problem Solving

Question 12

If $\overrightarrow{OA} = 5\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $\overrightarrow{OB} = 9\boldsymbol{i} + 6\boldsymbol{j} + 7\boldsymbol{k}$. Find $\overrightarrow{AB}$ and hence, show that $\overrightarrow{AB}$ is parallel to the x-y plane.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (9\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}) - (5\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$$

$$= (9-5)\mathbf{i} + (6-2)\mathbf{j} + (7-7)\mathbf{k}$$

$$= 4\mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$$

$$\overrightarrow{AB} = 4\mathbf{i} + 4\mathbf{j}$$

Since the k-component (the z-component) of $\overrightarrow{AB}$ is 0, the vector lies completely in the xy-plane, meaning it is **parallel to the xy-plane**.

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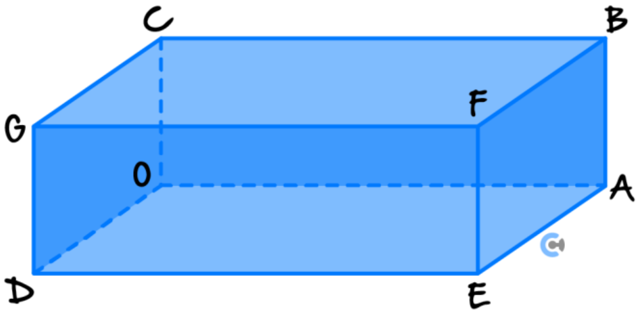
Question 13

Suppose $OABCDEFG$ is a cuboid and that $\overrightarrow{OA} = 5\boldsymbol{j}$, $\overrightarrow{OC} = 2\boldsymbol{k}$, and $\overrightarrow{OD} = 3\boldsymbol{i}$.



a. Express, in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

i. $\overrightarrow{OE}$

$$\overrightarrow{OE} = \overrightarrow{OD} + \overrightarrow{DE} = \overrightarrow{OD} + \overrightarrow{OA} = 3\boldsymbol{i} + 5\boldsymbol{j}$$

ii. $\overrightarrow{OF}$

$$\overrightarrow{OF} = \overrightarrow{OE} + \overrightarrow{EF} = \overrightarrow{OE} + \overrightarrow{OC} = 3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$$

iii. $\overrightarrow{GA}$

$$\overrightarrow{GA} = \overrightarrow{GD} + \overrightarrow{DA} = \overrightarrow{CO} + \overrightarrow{DO} + \overrightarrow{OA} = -3\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$$

b. Let M be the midpoint of face $FEAB$. Find $\overrightarrow{OM}$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

M is the **midpoint of face** $FEAB$.

The corners of face $FEAB$ are:

- F at $(3, 5, 2)$
- E at $(3, 5, 0)$
- A at $(0, 5, 0)$
- B at $(0, 5, 2)$

Take the midpoint:

$$\overrightarrow{OM} = \text{average of the four position vectors}$$

Add the coordinates:

- x coordinate: $\frac{3+3+0+0}{4} = \frac{6}{4} = \frac{3}{2}$
- y coordinate: always 5
- z coordinate: $\frac{2+0+0+2}{4} = \frac{4}{4} = 1$

Thus:

$$\overrightarrow{OM} = \frac{3}{2}\mathbf{i} + 5\mathbf{j} + 1\mathbf{k}$$

Question 14

Let $\boldsymbol{u} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} - \boldsymbol{j}$.

a. Find the length of vector $\boldsymbol{u}$.

$$|\mathbf{u}| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

b. Find unit vector parallel to $\boldsymbol{u}$.

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j}) = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$$

c. Find a vector of length 10 parallel to $\boldsymbol{u}$.

$$10 \times \left(\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}\right) = 8\mathbf{i} - 6\mathbf{j}$$

d. Find the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$(4)(2) + (-3)(-1) = 8 + 3 = 11$$

$$|\mathbf{v}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$\cos\theta = \frac{11}{5\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{11}{5\sqrt{5}}\right)$$

e. Write $\boldsymbol{u}$ as the sum of two vectors, one parallel to $\boldsymbol{v}$ and the other perpendicular to $\boldsymbol{v}$.

$$\mathbf{u} = \mathbf{p} + \mathbf{q}$$

where:

- $\mathbf{p}$ is parallel to $\mathbf{v}$, so $\mathbf{p} = k(2\mathbf{i} - \mathbf{j})$
- $\mathbf{q}$ is perpendicular to $\mathbf{v}$, meaning $\mathbf{q}\cdot\mathbf{v} = 0$

Thus:

$$\mathbf{p} = (2k)\mathbf{i} - (k)\mathbf{j}$$

$$\mathbf{q} = \mathbf{u} - \mathbf{p} = (4-2k)\mathbf{i} + (-3+k)\mathbf{j}$$

Now, since $\mathbf{q}$ is perpendicular to $\mathbf{v}$:

$$(4-2k)(2) + (-3+k)(-1) = 0$$

Expand:

$$8 - 4k + 3 - k = 0$$

$$11 - 5k = 0$$

$$k = \frac{11}{5}$$

Thus:

$$\mathbf{p} = \frac{11}{5}(2\mathbf{i} - \mathbf{j}) = \frac{22}{5}\mathbf{i} - \frac{11}{5}\mathbf{j}$$

and

$$\mathbf{q} = (4\mathbf{i} - 3\mathbf{j}) - \left(\frac{22}{5}\mathbf{i} - \frac{11}{5}\mathbf{j}\right) = \left(4 - \frac{22}{5}\right)\mathbf{i} + \left(-3 + \frac{11}{5}\right)\mathbf{j} = -\frac{2}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

Question 15

Let $t \in \mathbb{R}$ and suppose $\overrightarrow{OA} = t\boldsymbol{i} + t\boldsymbol{j} + 8\boldsymbol{k}$ and $\overrightarrow{OB} = t\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$. Find the values of t for which $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$.

$\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$

$t^2 - 2t - 24 = 0$

$\boxed{t = 6 \quad \text{or} \quad t = -4}$

Question 16

$A, B, C,$ and D are the vertices of a parallelogram.

Given that $A = (2, 3)$, $B = (7, 6)$ and $C = (10, 1)$, find the coordinates of D in vector form.

(5, -2)

(15, 4)

(-1, 8)

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Question 17

The points A and B have position vectors $\begin{bmatrix}-1\\1\end{bmatrix}$ and $\begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}$ respectively.

a.

i. Find the vector $\overrightarrow{AB}$.

$$\begin{bmatrix}\frac{5}{2}\\\frac{1}{2}\end{bmatrix}$$

ii. Find $|\overrightarrow{AB}|$.

$$|\overrightarrow{AB}| = \frac{\sqrt{26}}{2}$$

The point D has a position vector $\begin{bmatrix}d\\0\end{bmatrix}$.

b. Find the vectors $\overrightarrow{AD}$ and $\overrightarrow{DB}$ in terms of d.

$$\overrightarrow{AD} = \begin{pmatrix}d+1\\-1\end{pmatrix}, \quad \overrightarrow{DB} = \begin{pmatrix}\frac{3}{2}-d\\\frac{3}{2}\end{pmatrix}$$

c. If angle $\angle ADB$ is $90°$, find the two possible values of $\boldsymbol{d}$.

$$\overrightarrow{AD} \cdot \overrightarrow{DB} = 0$$

$$(d+1)\left(\frac{3}{2}-d\right) + (-1)\left(\frac{3}{2}\right) = 0$$

$$d = 0 \quad \text{or} \quad d = \frac{1}{2}$$

d. For the smaller value of $\boldsymbol{d}$ find the area of the triangle ADB.

$$\text{Area} = \frac{3}{2}$$

e. For the larger value of $\boldsymbol{d}$ show that the triangle is isosceles.

$$|\vec{AD}| = \sqrt{\left(\frac{3}{2}\right)^2 + (-1)^2} = \sqrt{\frac{9}{4} + 1} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$$

$$|\vec{DB}| = \sqrt{(1)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{1 + \frac{9}{4}} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$$

$$|\vec{AD}| = |\vec{DB}|$$

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Sub-Section: The Tech-Free "Final Boss" [VCAA Level]

Question 18

Points A, B, and C have position vectors:

$$\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}, \boldsymbol{b} = 5\boldsymbol{i} + 2\boldsymbol{j}, \boldsymbol{c} = 3\boldsymbol{i} + 6\boldsymbol{j}$$

Let point D lie on line segment AC, and suppose its position vector is $\boldsymbol{d} = (1-k)\boldsymbol{a} + k\boldsymbol{c}$, where $0 \leq k \leq 1$.

a.

i. Find the vectors $\overrightarrow{AD}$ and $\overrightarrow{DB}$ in terms of k.

$$\vec{AD} = k(2\vec{i} + 4\vec{j}), \quad \vec{DB} = (4-2k)\vec{i} - 4k\vec{j}$$

ii. Hence, write an expression of $\overrightarrow{AD} \cdot \overrightarrow{DB}$.

$$\vec{AD} \cdot \vec{DB} = k[(2)(4-2k) + (4)(-4k)] = k[8 - 4k - 16k] = k(8-20k)$$

b. If angle $\angle ADB = 90^\circ$, use your expression from **part a.** to find the exact value(s) of k that satisfies this condition.

$$\boxed{k = \frac{2}{5}}$$

c. For the value(s) of k found in **part b.**, compute the lengths of $\overrightarrow{AD}$ and $\overrightarrow{DB}$, and hence find the area of triangle ADB.

$$\text{Area} = \frac{1}{2}|\vec{AD}||\vec{DB}| = \frac{1}{2} \cdot \frac{4\sqrt{5}}{5} \cdot \frac{8\sqrt{5}}{5} = \frac{1}{2} \cdot \frac{32 \cdot 5}{25} = \frac{160}{50} = \boxed{3.2}$$

d. Let $\boldsymbol{m} = \boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{n} = \boldsymbol{c} - \boldsymbol{a}$.

Use the dot product to find the angle between vectors $\boldsymbol{m}$ and $\boldsymbol{n}$, in degrees.

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

e. Suppose point E lies on the line AB, with position vector $\boldsymbol{e} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$.

If the angle between $\boldsymbol{e}$ and $\boldsymbol{c}$ is $60°$, find the value of t.

$$t = \frac{15 \pm 10\sqrt{3}}{4}$$

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Sub-Section: The Tech-Active "Final Boss" [VCAA Level]

Question 19

A runner sets off on a bearing of 120° (assume east is in the direction of $\boldsymbol{i}$ and north is in the direction of $\boldsymbol{j}$).

a. Find a unit vector for the direction the runner goes.

$$\cos(120^\circ)\mathbf{i} + \sin(120^\circ)\mathbf{j}$$

$$\cos(120^\circ) = -\frac{1}{2}, \quad \sin(120^\circ) = \frac{\sqrt{3}}{2}$$

$$\text{Unit vector} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

b. If the runner runs in this direction for 3 km, find the position of the runner with respect to their starting point.

$$\vec{r} = 3 \times \left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = \left(-\frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}\right)$$

c. If the runner now turns and runs 6 km south, find the position of the runner with respect to the original starting point.

$$\vec{r}_{\text{new}} = \left(-\frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}\right) + (0\mathbf{i} - 6\mathbf{j}) = \left(-\frac{3}{2}\mathbf{i} + \left(\frac{3\sqrt{3}}{2} - 6\right)\mathbf{j}\right)$$

$$\vec{r}_{\text{new}} = -\frac{3}{2}\mathbf{i} + \left(\frac{3\sqrt{3} - 12}{2}\right)\mathbf{j}$$

d. Find the distance of the runner from the starting point.

$$\text{Distance} = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3} - 12}{2}\right)^2}$$

$$\text{Distance} = \sqrt{\frac{9}{4} + \frac{171 - 72\sqrt{3}}{4}} = \sqrt{\frac{180 - 72\sqrt{3}}{4}} = \frac{1}{2}\sqrt{180 - 72\sqrt{3}}$$

e. Find the bearing of the runner from their starting point, correct to 2 decimal places.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$\tan\theta = \frac{\frac{3\sqrt{3}-12}{2}}{-\frac{3}{2}} = \frac{3\sqrt{3}-12}{-3} = -\sqrt{3}+4$$

$$\tan\theta = 4-\sqrt{3}$$

$\theta = 66.21^\circ$

f. Another runner sets off from the same starting point and runs directly **east** for 4 km.

Find the **angle between the two runners' final displacement vectors**. Round your answer to **2 decimal places**.

113.79°

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Section B: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 20

P is the point $(-1, 3)$, Q is the point $(12, 4)$ and R is the midpoint of PQ.

a. Calculate the lengths of OP and OQ.

$$|OP| = \sqrt{(-1)^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$$

$$\boxed{|OP| = \sqrt{10}}$$

$$|OQ| = \sqrt{12^2 + 4^2} = \sqrt{144+16} = \sqrt{160} = 4\sqrt{10}$$

$$\boxed{|OQ| = 4\sqrt{10}}$$

b. Find $\overrightarrow{PQ}$ and hence determine the length PQ.

$$\overrightarrow{PQ} = Q - P = (12 - (-1), 4 - 3) = (13, 1)$$

$$\boxed{\overrightarrow{PQ} = (13, 1)}$$

$$|PQ| = \sqrt{13^2 + 1^2} = \sqrt{169+1} = \sqrt{170}$$

$$\boxed{|PQ| = \sqrt{170}}$$

c. Show that ΔPOQ is a right-angled triangle.

$$|OP|^2 + |OQ|^2 = 10 + 160 = 170$$

$$|PQ|^2 = 170$$

$$|PQ|^2 = |OP|^2 + |OQ|^2$$

d. Find $\overrightarrow{OR}$ and hence show R is equidistant from the three vertices of ΔPOQ.

$$R = \left(\frac{-1+12}{2}, \frac{3+4}{2}\right) = \left(\frac{11}{2}, \frac{7}{2}\right)$$

Show: $|PR| = |QR| = |OR| = \frac{\sqrt{170}}{2}$

Question 21

Points A, B, C, and D are defined by position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, and $\boldsymbol{d}$ respectively. If $\overrightarrow{AB} + \overrightarrow{CD} = 0$:

a. Express $\boldsymbol{d}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

$$\overrightarrow{AB} + \overrightarrow{CD} = \mathbf{0}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad \overrightarrow{CD} = \mathbf{d} - \mathbf{c}$$

$$(\mathbf{b} - \mathbf{a}) + (\mathbf{d} - \mathbf{c}) = \mathbf{0}$$

$$\mathbf{b} + \mathbf{d} = \mathbf{a} + \mathbf{c}$$

b. Show that AC and BD bisect each other.

$$\text{Midpoint of } AC = \frac{\mathbf{a}+\mathbf{c}}{2}$$

$$\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$$

$$\mathbf{b} + \mathbf{d} = \mathbf{b} + (\mathbf{a} + \mathbf{c} - \mathbf{b}) = \mathbf{a} + \mathbf{c}$$

$$\text{Midpoint of } BD = \frac{\mathbf{b}+\mathbf{d}}{2} = \frac{\mathbf{a}+\mathbf{c}}{2}$$

c. Prove that $ABCD$ is a rhombus if $|\boldsymbol{a}| = |\boldsymbol{c}|$ and angles AOB and BOC are equal.

Since:

$$\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = (\mathbf{a} + \mathbf{c} - \mathbf{b}) - \mathbf{c} = \mathbf{a} - \mathbf{b}$$

Thus:

$$\overrightarrow{CD} = \mathbf{a} - \mathbf{b} = -(\mathbf{b} - \mathbf{a}) = -\overrightarrow{AB}$$

Similarly, $\overrightarrow{DA}$:

$$\overrightarrow{DA} = \mathbf{a} - \mathbf{d} = \mathbf{a} - (\mathbf{a} + \mathbf{c} - \mathbf{b}) = \mathbf{b} - \mathbf{c} = -(\mathbf{c} - \mathbf{b}) = -\overrightarrow{BC}$$

Thus, $\overrightarrow{CD}$ and $\overrightarrow{AB}$ have the same length, and $\overrightarrow{DA}$ and $\overrightarrow{BC}$ have the same length.

Because $|\mathbf{a}| = |\mathbf{c}|$ and angles AOB and BOC are equal, the vectors are symmetric, so:

- All four sides have equal length.
- Therefore, $ABCD$ is a **rhombus**.

Question 22

A pyramid $ABCDV$ has a square base $ABCD$, with vertices $A,\ B$, and D having position vectors: $\mathbf{i} - \boldsymbol{j} + \boldsymbol{k}$, $11\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{i} + 5\boldsymbol{j} + 9\boldsymbol{k}$.

a. Verify that sides AB and AD are equal in length and are perpendicular to each other.

$$|\overrightarrow{AB}| = \sqrt{10^2 + 0^2 + 0^2} = \sqrt{100} = 10$$

$$|\overrightarrow{AD}| = \sqrt{0^2 + 6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

$$\overrightarrow{AB} \cdot \overrightarrow{AD} = (10)(0) + (0)(6) + (0)(8) = 0$$

b. Determine the coordinates of C, the fourth vertex of the square base.

$$\boxed{C = (11, 5, 9)}$$

c. Find the coordinates of P, the point where the diagonals of the square $ABCD$ intersect.

$$P = \left(\frac{1+11}{2}, \frac{-1+5}{2}, \frac{1+9}{2}\right)$$

$$= (6, 2, 5)$$

d. If V is defined by $x\boldsymbol{i} + y\boldsymbol{j} + 2\boldsymbol{k}$, and if $\overrightarrow{VP}$ is perpendicular to the two diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ of the base, find x and y.

$$\vec{VP} = (6 - x)\mathbf{i} + (2 - y)\mathbf{j} + (5 - 2)\mathbf{k} = (6 - x, 2 - y, 3)$$

$$\vec{VP} \cdot \overrightarrow{AC} = 0 \qquad \vec{VP} \cdot \overrightarrow{BD} = 0$$

$$\boxed{x = 6, \quad y = 6}$$

e. Let M be the midpoint of side AB. Calculate the angle VMP (in degrees).

$$M = \left(\frac{1+11}{2}, \frac{-1+(-1)}{2}, \frac{1+1}{2}\right) = (6, -1, 1)$$

$$\overrightarrow{VM} = (6-6, -1-6, 1-2) = (0, -7, -1)$$

$$\overrightarrow{VP} = (6-6, 2-6, 5-2) = (0, -4, 3)$$

$$\cos\theta = \frac{\overrightarrow{VM} \cdot \overrightarrow{VP}}{|\overrightarrow{VM}||\overrightarrow{VP}|}$$

$$(0)(0) + (-7)(-4) + (-1)(3) = 0 + 28 - 3 = 25$$

$$|\overrightarrow{VM}| = \sqrt{0^2 + (-7)^2 + (-1)^2} = \sqrt{49+1} = \sqrt{50}$$

$$|\overrightarrow{VP}| = \sqrt{0^2 + (-4)^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\cos\theta = \frac{25}{\sqrt{50} \times 5} = \frac{25}{5\sqrt{50}} = \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

f. Find the exact volume of the pyramid.

$$V = \frac{1}{3} \times \text{Area of base} \times \text{Height}$$

$$V = \frac{1}{3} \times 100 \times 3 = 100$$

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Sub-Section: Exam 2 (Tech-Active)

Question 23

If vector $\overrightarrow{AB} = \boldsymbol{u}$ and vector $\overrightarrow{BC} = \boldsymbol{v}$ then vector $\overrightarrow{AC}$ is equal to:

A. $\boldsymbol{u} + \boldsymbol{v}$

B. $\boldsymbol{v} - \boldsymbol{u}$

C. $\boldsymbol{u} - \boldsymbol{v}$

D. $\boldsymbol{u} \times \boldsymbol{v}$

Question 24

$ABCD$ is a parallelogram. If $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$, then in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, $\overrightarrow{CA}$ equals:

A. $\boldsymbol{a} + \boldsymbol{b}$

B. $\boldsymbol{a} - \boldsymbol{b}$

C. $-\boldsymbol{b} - \boldsymbol{a}$

D. $\boldsymbol{b} \bullet \boldsymbol{a}$

Question 25

If $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$, then the unit vector parallel to $\boldsymbol{a}$ is:

A. $3\boldsymbol{i} + 4\boldsymbol{j}$

B. $\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

C. $\frac{1}{\sqrt{5}}(3\boldsymbol{i} + 4\boldsymbol{j})$

D. $\frac{1}{\sqrt{3}}(3\boldsymbol{i} + 4\boldsymbol{j})$

Question 26

If $\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ then $\hat{\boldsymbol{a}}$ is:

A. $\frac{1}{6}(3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

B. $\frac{1}{\sqrt{17}}(3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

C. $\frac{1}{7}(3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

D. $\frac{1}{\sqrt{13}}(3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

Question 27

If vector $a = 3\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$ is parallel to vector $\boldsymbol{b} = a\boldsymbol{i} + b\boldsymbol{j} - 4\boldsymbol{k}$, then:

A. $a = 3$ and $b = 5$

B. $a = 5$ and $b = 10$

C. $a = 6$ and $b = 10$

D. $a = 1$ and $b = 2$

E. $a = 9$ and $b = 15$

Question 28

A and B are points on a plane such that $\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OB} = 2\boldsymbol{i} - 5\boldsymbol{j}$. If M is the midpoint of the line segment B, then $\overrightarrow{MO}$ equals:

A. $\frac{3}{2}\boldsymbol{i} - 4\boldsymbol{j}$

B. $-3\boldsymbol{i} + \boldsymbol{j}$

C. $-\frac{3}{2}\boldsymbol{i} + 4\boldsymbol{j}$

D. $3\boldsymbol{i} - \boldsymbol{j}$

Question 29

Given $|\boldsymbol{a}| = 3, |\boldsymbol{b}| = 4$ and $\boldsymbol{a} \bullet \boldsymbol{b} = 5$, the value of $|\boldsymbol{a} - \boldsymbol{b}|$ is:

A. $\sqrt{7}$

B. $\sqrt{15}$

C. 1

D. 15

Question 30

If vector $\boldsymbol{b} = 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$, the angle $\boldsymbol{b}$ makes with the z-axis is closest to:

A. 38°

B. 52°

C. 101°

D. 54°

Question 31

A two-dimensional unit vector that is perpendicular to $3\boldsymbol{i} + 4\boldsymbol{j}$, is:

A. $3\boldsymbol{i} - 4\boldsymbol{j}$

B. $\frac{4}{5}\boldsymbol{i} + \frac{3}{5}\boldsymbol{j}$

C. $\frac{4}{5}\boldsymbol{i} - \frac{3}{5}\boldsymbol{j}$

D. $\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$

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Question 32

If $(2x\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}) \bullet (-3\boldsymbol{i} + 2x\boldsymbol{j} - 4\boldsymbol{k}) = 8$, then x is equal to:

A. $\frac{4}{3}$

B. 2

C. 3

D. -2

Question 33

Two drones depart from the same base.

Drone M moves in the direction $4\boldsymbol{i} + 3\boldsymbol{j}$ and Drone N moves in the direction $-6\boldsymbol{i} + 8\boldsymbol{j}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the East and North directions, respectively (with 1 unit representing 1 kilometre).

a. Find the unit vector representing the direction of:

i. Drone M.

$$\mathbf{u}_M = \frac{1}{5}(4\mathbf{i} + 3\mathbf{j}) = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$$

ii. Drone N.

$$\mathbf{u}_N = \frac{1}{10}(-6\mathbf{i} + 8\mathbf{j}) = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

b. Using the vector method, find the distance between the two drones if Drone M travels for 12 km and Drone N travels for 16 km.

$$\mathbf{r}_M = 12\left(\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}\right) = \left(\frac{48}{5}\mathbf{i} + \frac{36}{5}\mathbf{j}\right)$$

$$\mathbf{r}_N = 16\left(-\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = \left(-\frac{48}{5}\mathbf{i} + \frac{64}{5}\mathbf{j}\right)$$

$$\mathbf{r} = \mathbf{r}_M - \mathbf{r}_N$$

$$= \left(\frac{48}{5} - \left(-\frac{48}{5}\right)\right)\mathbf{i} + \left(\frac{36}{5} - \frac{64}{5}\right)\mathbf{j}$$

$$= \left(\frac{96}{5}\right)\mathbf{i} + \left(-\frac{28}{5}\right)\mathbf{j}$$

$$\text{Distance} = \sqrt{\left(\frac{96}{5}\right)^2 + \left(-\frac{28}{5}\right)^2}$$

$$= \sqrt{\frac{9216}{25} + \frac{784}{25}}$$

$$= \sqrt{\frac{10000}{25}}$$

$$= \sqrt{400}$$

$$= 20$$

c. Find the angle between the directions of the two drones.

$$\cos\theta = \frac{\mathbf{d}_M \cdot \mathbf{d}_N}{|\mathbf{d}_M||\mathbf{d}_N|}$$

$$(4)(-6) + (3)(8) = -24 + 24 = 0$$

$$\cos\theta = \frac{0}{5 \times 10} = 0$$

$$\theta = 90^\circ$$

d. On a different mission, Drone M flies in the direction $5\boldsymbol{i} - \sqrt{7}\boldsymbol{j}$.

Drone N must fly at 90° to Drone M's direction.

Find two possible unit vectors for Drone N's direction in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

Suppose Drone N's unit vector is $\mathbf{n} = a\boldsymbol{i} + b\boldsymbol{j}$.

$$5a - \sqrt{7}b = 0 \quad \Rightarrow \quad b = \frac{5}{\sqrt{7}}a$$

$$a^2 + b^2 = 1$$

$$a = \pm\frac{\sqrt{14}}{8}$$

$$b = \pm\frac{5}{\sqrt{7}} \times \frac{\sqrt{14}}{8} = \pm\frac{5\sqrt{2}}{8}$$

$$\boxed{\frac{\sqrt{14}}{8}\mathbf{i} + \frac{5\sqrt{2}}{8}\mathbf{j}} \quad \text{or} \quad \boxed{-\frac{\sqrt{14}}{8}\mathbf{i} - \frac{5\sqrt{2}}{8}\mathbf{j}}$$

Question 34

Two ships leave from the same port.
Ship X sails in the direction of $5\boldsymbol{i} + 12\boldsymbol{j}$, and Ship Y sails in the direction of $7\boldsymbol{i} - 24\boldsymbol{j}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors pointing East and North, respectively (with 1 unit representing 1 kilometre).

a. Find the unit vector representing the direction of:

i. Ship X.

$$\mathbf{u}_X = \frac{1}{13}(5\mathbf{i} + 12\mathbf{j}) = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$$

ii. Ship Y.

$$\mathbf{u}_Y = \frac{1}{25}(7\mathbf{i} - 24\mathbf{j}) = \frac{7}{25}\mathbf{i} - \frac{24}{25}\mathbf{j}$$

b. Using a vector method, find the distance between the two ships if Ship X travels for 15 km and Ship Y travels for 10 km. Round your answer to one decimal place.

$$\mathbf{r}_X = 15\left(\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}\right) = \left(\frac{75}{13}\mathbf{i} + \frac{180}{13}\mathbf{j}\right)$$

$$\mathbf{r}_Y = 10\left(\frac{7}{25}\mathbf{i} - \frac{24}{25}\mathbf{j}\right) = \left(\frac{70}{25}\mathbf{i} - \frac{240}{25}\mathbf{j}\right) = \left(\frac{14}{5}\mathbf{i} - \frac{48}{5}\mathbf{j}\right)$$

$$\mathbf{r} = \mathbf{r}_X - \mathbf{r}_Y$$

$$= \left(\frac{75}{13} - \frac{14}{5}\right)\mathbf{i} + \left(\frac{180}{13} - \left(-\frac{48}{5}\right)\right)\mathbf{j}$$

Distance = 23.6

c. Use a vector method to find the angle between the directions of the two ships. Round your answer to one decimal place.

$$\cos\theta = \frac{\mathbf{d}_X \cdot \mathbf{d}_Y}{|\mathbf{d}_X||\mathbf{d}_Y|}$$

$$\theta = \cos^{-1}\left(-\frac{253}{13 \times 25}\right) = 141.1°$$

d. On another journey, Ship X again sails in the direction $2\boldsymbol{i} + \sqrt{5}\boldsymbol{j}$. Ship Y departs from and sails at $60°$ in the direction of Ship X. Find two possible unit vectors for Ship Y's direction in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

Suppose Ship Y's unit vector is:

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j}$$

Because the angle between $\mathbf{d}_X$ and $\mathbf{n}$ is $60°$, we use the **dot product formula**:

$$\cos 60° = \frac{\mathbf{d}_X \cdot \mathbf{n}}{|\mathbf{d}_X||\mathbf{n}|}$$

Since $|\mathbf{n}| = 1$ (unit vector), this simplifies to:

$$\cos 60° = \frac{2a + \sqrt{5}b}{3}$$

$$\text{solve}\left(\begin{cases}\frac{1}{2} = \frac{2\cdot a + \sqrt{5}\cdot b}{3} \\ a^2 + b^2 = 1\end{cases}, \{a, b\}\right) \quad a = \frac{-(\sqrt{15}-2)}{6} \text{ and } b = \frac{\sqrt{5}+2\cdot\sqrt{3}}{6} \text{ or } a = \frac{\sqrt{15}+2}{6} \text{ and } b = \frac{\sqrt{5}-2\cdot\sqrt{3}}{6}$$

Question 35

Two cyclists leave from the same place, O. Cyclist A heads in a direction of $3\boldsymbol{i} + 4\boldsymbol{j}$ and Cyclist B in a direction of $2.5\boldsymbol{i} - 6\boldsymbol{j}$ where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the East and North directions, respectively with 1 unit representing 1 kilometre.

a. Find the unit vector representing the direction of:

i. Cyclist A.

$$\mathbf{u}_A = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

ii. Cyclist B.

$$\mathbf{u}_B = \frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$$

b. Find the **midpoint** between the two cyclists after they finish their journey where each cyclist ends up after travelling for 2.5 km (Cyclist A) and 26 km (Cyclist B) respectively. Express the coordinates.

$$\mathbf{r}_A = 2.5 \times \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$$

$$= \left(\frac{7.5}{5}\mathbf{i} + \frac{10}{5}\mathbf{j}\right)$$

$$= (1.5\mathbf{i} + 2\mathbf{j})$$

$$(1.5, 2)$$

$$\mathbf{r}_B = 26 \times \left(\frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}\right)$$

$$= 2 \times (5\mathbf{i} - 12\mathbf{j})$$

$$= (10\mathbf{i} - 24\mathbf{j})$$

$$(10, -24)$$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\frac{1.5 + 10}{2} = \frac{11.5}{2} = 5.75$$

$$\frac{2 + (-24)}{2} = \frac{-22}{2} = -11$$

$$\boxed{(5.75, -11)}$$

c. Use a vector method to find the angle between the direction of travel of the two cyclists. Round to one decimal place.

$$\theta = \cos^{-1}\left(-\frac{16.5}{5 \times 6.5}\right) = 120.5°$$

d. Another time, the cyclists again leave from the same place, O. Cyclist A heads in the direction $3\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$.

Cyclist B sets off at 120° to Cyclist A. Give two possible unit vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$ for the direction of Cyclist B.

Suppose Cyclist B's unit vector is:

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j}$$

We know:

$$\cos 120° = \frac{\mathbf{d}_A \cdot \mathbf{n}}{|\mathbf{d}_A||\mathbf{n}|}$$

$$\text{solve}\left(\begin{cases} \frac{-1}{2} = \frac{3 \cdot a + \sqrt{3} \cdot b}{2 \cdot \sqrt{3}} \\ a^2 + b^2 = 1 \end{cases}, \{a, b\}\right)$$

$$a = \frac{-\sqrt{3}}{2} \text{ and } b = \frac{1}{2} \text{ or } a = 0 \text{ and } b = -1$$

$$\boxed{0\mathbf{i} - 1\mathbf{j}} \text{ and } \boxed{-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}}$$

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