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VCE Specialist Mathematics ½
Graph Theory II [5.4]
Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Compulsory Questions	Pg 2-Pg 19
Supplementary Questions	Pg 20-Pg 34

Section A: Compulsory Questions

Sub-Section: Recap

Cheat Sheet

[5.4.1] - Walk of a graph

Walk of a Graph

$$v_1, e_1, \cdots, v_{n-1}, e_1, v_n$$

- Order of vertices (points) and edges (lines) we are visiting.
- Alternates between a vertex and an edge.
- If there is only one edge between two vertices, stating an edge can be abbreviated.

Length of a Walk

$$Length\ of\ a\ Walk = Number\ of\ Edges\ in\ a\ Walk$$

- Number of edges in a walk.
- If we use the same edge multiple times, we count it multiple times for the length.

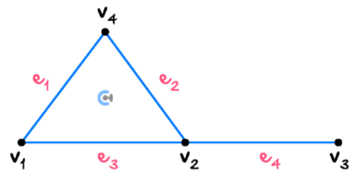
Number of Walks with the Length of n

$$A^n = \begin{bmatrix} v_1, v_1 & v_1, v_2 & v_1, v_3 \\ v_1, v_2 & v_2, v_2 & v_2, v_3 \\ v_1, v_3 & v_3, v_2 & v_3, v_3 \end{bmatrix}$$

- For walk of length n, we power the adjacency matrix by n.
- The element gives the number of walks with length of n between two vertices.

[5.4.2] - Euler trail and circuits

Euler Trail



- Euler Trail is a walk of a graph where all the edges are used exactly once.
- Whether a vertex is used exactly once is irrelevant.

Existence of an Euler Trail

- A graph needs to satisfy one of the following two rules:

  1. Every vertex has an even degree.

  OR

  2. Exactly two vertices have an odd degree.

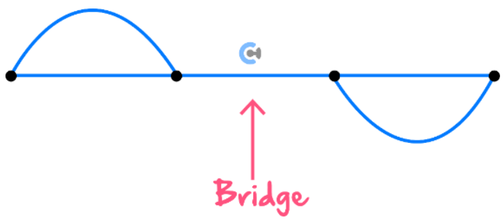
Fleury's Algorithm

- Allows us to find the Euler's trail efficiently to the graphs with Euler trails.
- The steps are as follows:

  1. If there are two vertices of an odd degree, then start from one of them. Otherwise, start from any vertex.
  2. Move from the current vertex across an edge to an adjacent vertex. Always choose a non-bridge edge unless there is no alternative.
  3. Delete the edge that you have just traversed.
  4. Repeat from Step 2 until there are no edges left.

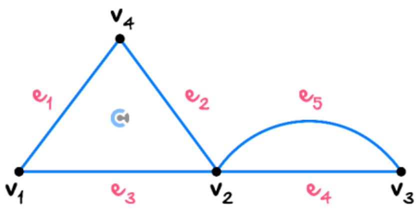
Cheat Sheet

- **Bridge**



- Bridge is an edge which if you delete will cause a graph to be cut into two graphs.

- **Euler Circuits**



- Euler Circuit is a walk of a graph where all the edges are used exactly once.

  Starting vertex and ending vertex must be the same.

- Whether a vertex is used exactly once is irrelevant.

- Euler circuits are a subset of Euler trails.

- **Existence of an Euler Circuit**

- A graph needs to satisfy the following rule:
  - Every vertex has an even degree.

[5.4.3] – Hamiltonian paths and cycles

- **Hamiltonian Path**

- Path which covers all the vertices exactly once.

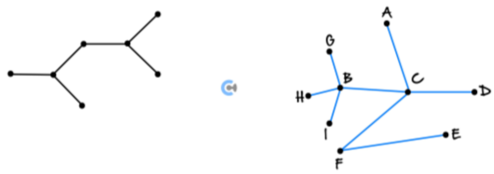
- **Hamiltonian cycle**

- Path which covers all the vertices exactly once. (Except the starting one.)

  Starting position and ending vertex must be identical.

[5.4.4] – Trees

- **Trees**



- Tree is a connected graph (can go from one vertex to any other vertex) without any cycle.

- We cannot come back to the same vertex by not repeating the edges.

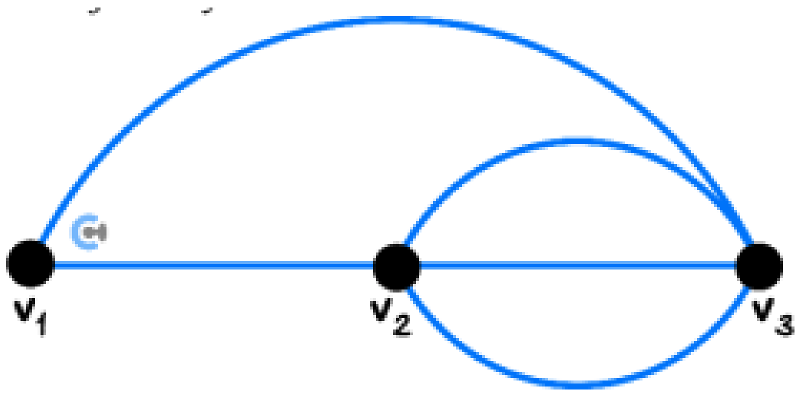
- **Equivalent conditions for a tree**

- A simple graph with n vertices is a tree if ANY of the following conditions are met:
  - It is connected and contains no cycles.
  - It is connected and has $(n-1)$ edges.
  - It has no cycles and has $(n-1)$ edges.
  - It is connected but would become disconnected if any edge is removed.
  - It is connected and would form a cycle if any edge is added.
  - Any two vertices are connected by only a single path.

Sub-Section [5.4.1]: Walk of a Graph

Question 1

Consider the following graph and its adjacency matrix.



Adjacency matrix, $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

a. Find the number of walks with length of 1 between v_2 and v_3.

3 walks

b. Find the number of walks with length of 2 from v_1 to v_3.

3 walks

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}^2 \qquad \begin{bmatrix} 2 & 3 & 3 \\ 3 & 10 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

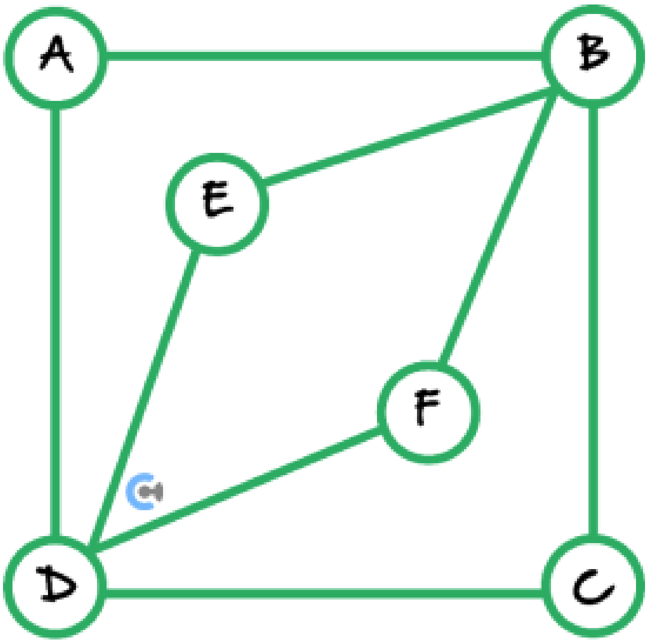
c. Can a walk contain repeated vertices or edges? Provide an example of such a walk in this graph.

Yes, a **walk** can contain **repeated vertices or edges**.
$v_2 \rightarrow v_3 \rightarrow v_2$

Sub-Section [5.4.2]: Euler Trail and Circuits

Question 2

Consider the following graph.



a. List the degree of each vertex in the graph.

$$\begin{aligned} \deg(A) &= 2 \\ \deg(B) &= 4 \\ \deg(C) &= 2 \\ \deg(D) &= 4 \\ \deg(E) &= 2 \\ \deg(F) &= 2 \end{aligned}$$

b. Does this graph contain an Euler Circuit? Justify your answer.

Euler Circuit exists **if all vertices have even degree** and the graph is connected.
So yes, the graph contains an **Euler Circuit** because it is connected and all vertices have even degrees.

c. Does this graph contain an Euler Trail? If yes, write the sequence of vertices in one possible trail.

Yes, the graph also contains an **Euler Trail** because it is connected and all vertices have even degrees.

$$A \to D \to F \to B \to C \to D \to E \to B \to A$$

(There are many such trails; this is just one possible example.)

d. If you were to remove the edge between D and F, would the graph still have an Euler trail or circuit? Explain.

After removing ***D*–*F***, the degrees change:

$\deg(D) = 3$

$\deg(F) = 1$

Now we have two **odd-degree** vertices: D and F → this **allows an Euler Trail**, but **not a circuit**.

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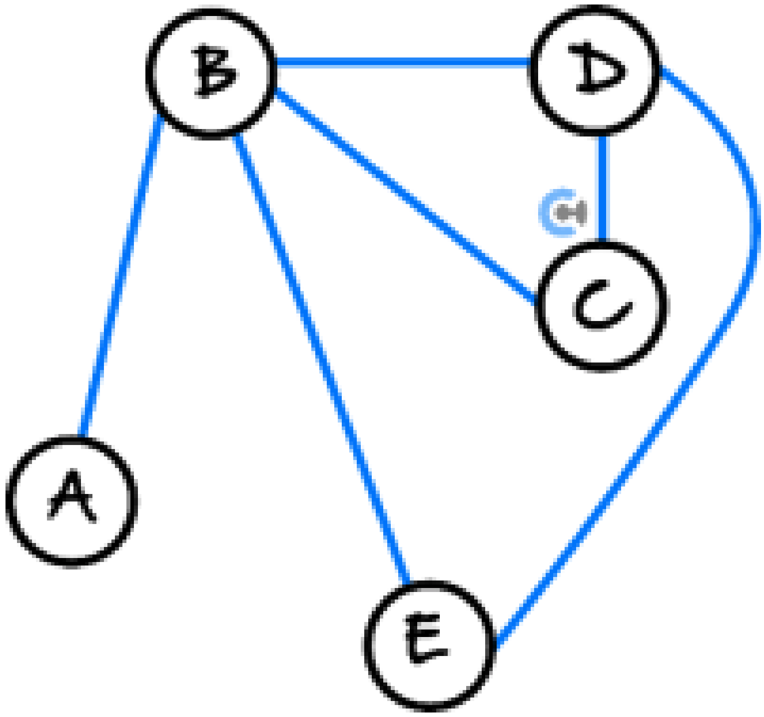
Sub-Section [5.4.3]: Hamiltonian Paths and Cycles

Question 3

Consider the following graph.



a. Does this graph have a Hamiltonian Path? If yes, list one possible path.

A **Hamiltonian Path** visits **every vertex exactly once**, without necessarily returning to the starting point.
So **yes**, the graph does have a Hamiltonian Path.
Example path: $A \rightarrow B \rightarrow D \rightarrow C \rightarrow E$

b. Does this graph have a Hamiltonian Cycle? If yes, provide one possible cycle.

No, this graph **does NOT** have a **Hamiltonian Cycle** because there is no way to **return to any vertices** after visiting all vertices exactly once.

c. Is it possible to start at vertex A and visit every vertex exactly once without returning to A? Why or why not?

Yes — this is exactly what a **Hamiltonian Path** does.

d. If the edge between A and E is added, would a Hamiltonian cycle exist? Justify your answer.

Yes, if the edge between A and E is added, the graph **will have a Hamiltonian Cycle**. One such cycle is: $\boldsymbol{A \rightarrow B \rightarrow D \rightarrow C \rightarrow E \rightarrow A}$

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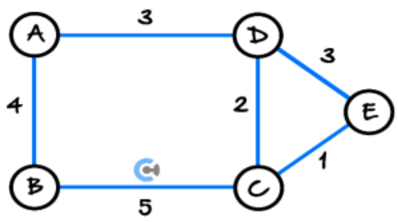
Sub-Section [5.4.4]: Trees

Question 4

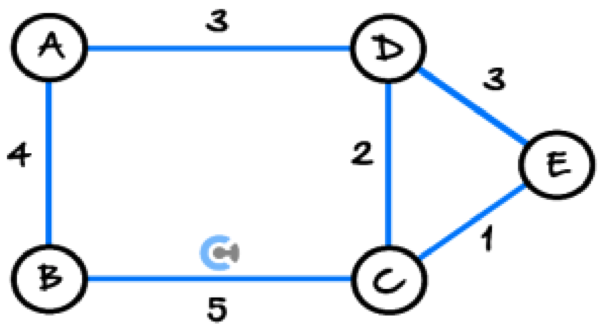
Consider the following graph.

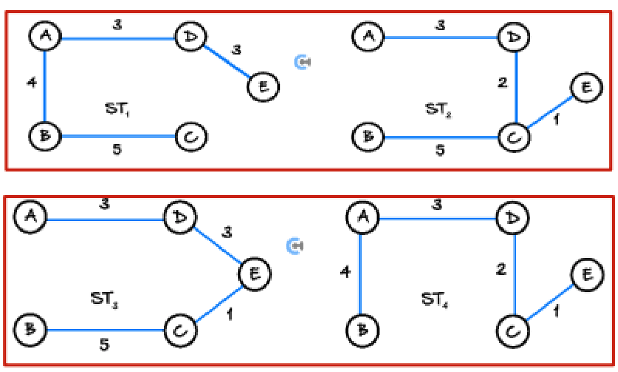


a. How many edges does a spanning tree for this graph have?

A **spanning tree** for a graph with **5 vertices** must have **4 edges** and **connect all vertices** without forming cycles.

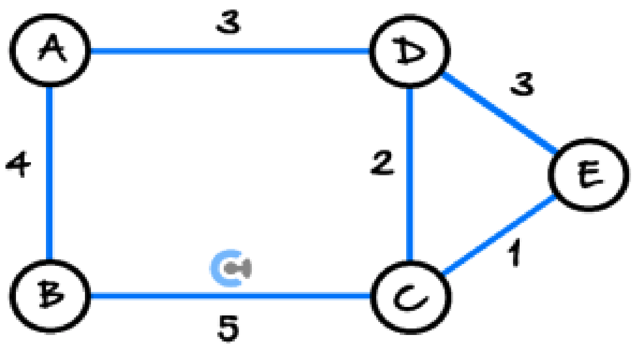
b. Identify one of the spanning trees.





Not exhaustive — more possible.

c. Identify the minimum spanning tree with its total weight.



C-E, D-C, A-B, A-D

$1 + 2 + 3 + 4 = 10$

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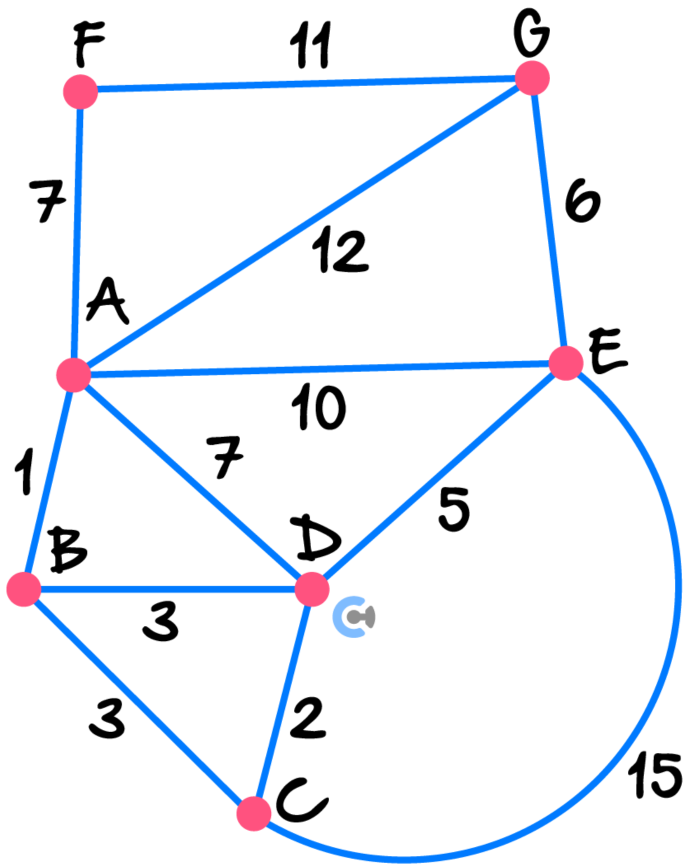
Sub-Section: Problem Solving

Question 5

Consider the following graph.



a. Write down the adjacency matrix A for this graph.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

b. How many walks of length 4 are there from vertex B to C? (Treat all edges as unweighted for now.)

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}^4 \quad \begin{bmatrix} 48 & 24 & 33 & 35 & 32 & 16 & 27 \\ 24 & 26 & 17 & 27 & 31 & 15 & 17 \\ 33 & 17 & 25 & 26 & 21 & 9 & 18 \\ 35 & 27 & 26 & 37 & 32 & 16 & 25 \\ 32 & 31 & 21 & 32 & 39 & 20 & 21 \\ 16 & 15 & 9 & 16 & 20 & 12 & 12 \\ 27 & 17 & 18 & 25 & 21 & 12 & 21 \end{bmatrix}$$

17 walks

c. Explain your answer with reference to the degrees of the vertices. Determine whether this graph has:

i. An Euler trail.

No, the graph does not have an Euler trail. Many vertices have odd degree.

ii. An Euler circuit.

No, since for an Euler circuit, all vertices must have **even degree**.

d. If you want to find a path that uses every road exactly once, how many roads must you duplicate (if any) to convert this graph into one that admits an Euler circuit?

To have an Euler circuit, all degrees must be **even**. Currently **4 vertices** have odd degree ($ABCG$).
To make all degrees even, we must **pair up** the 4 odd-degree vertices and add an edge for each pair.
4 odd-degree vertices → needs **2 more edges**
Answer: Duplicate 2 roads to convert it into an Euler circuit.

e. A salesperson wants to visit each city (vertex) exactly once and return to the starting city.

i. List one Hamiltonian cycle from the graph.

Example: $A - B - C - D - E - G - F - A$

ii. Compute its total cost using the edge weights.

$1 + 3 + 2 + 5 + 6 + 11 + 7 = 35$

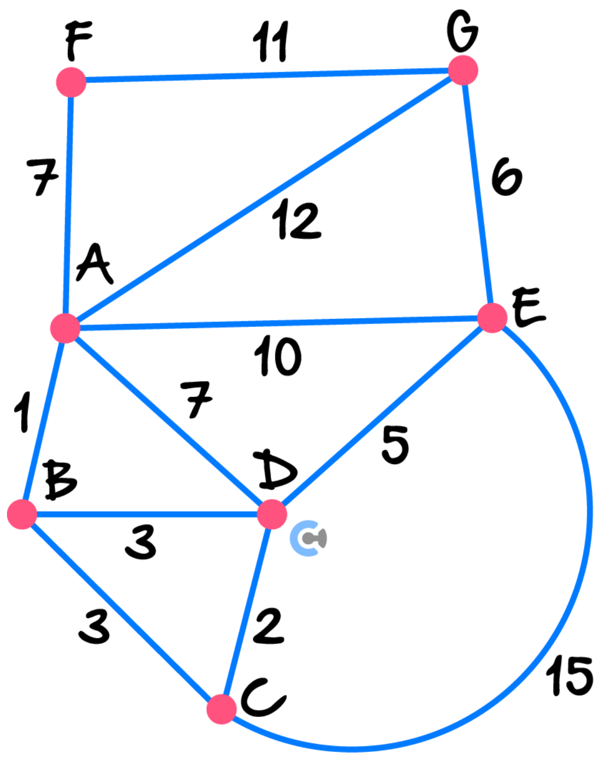
iii. Suggest a different Hamiltonian cycle with a smaller cost if possible.

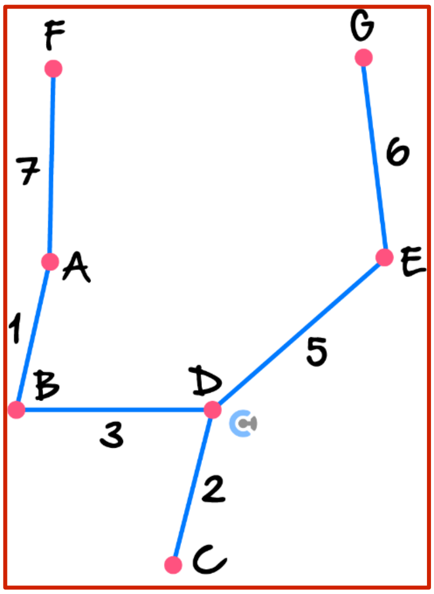
The one listed in (a) is the smallest one.

f. Is it possible to find a Hamiltonian path that does not return to the starting point? Justify your answer.

Yes, Hamiltonian paths exist. One example: $A - B - C - D - E - G - F$

g. Draw the minimum spanning tree for the graph. What is the distance of the minimum spanning tree?





$1 + 2 + 3 + 5 + 6 + 7 = 24$

Question 6

Prove that if a connected undirected graph has an Euler trail, then it has exactly 0 or 2 odd-degree vertices.

- In an Euler trail, we traverse every edge exactly once.
- At each vertex (except possibly the start and end), every time we enter via one edge, we must leave via another. So these vertices must have even degrees.
- The start and end vertices can have odd degrees, because they have one unmatched edge (one extra entrance or exit).
- Therefore, at most two vertices can have odd degrees.
- So:
  - If the trail is open (different start and end), there are exactly 2 odd-degree vertices.
  - If the trail is closed (Euler circuit), there are 0 odd-degree vertices.

Thus, Euler trail $\Rightarrow$ exactly 0 or 2 odd-degree vertices.

Question 7

Prove that if a connected graph has an Euler circuit, then all its vertices have even degrees.

An Euler circuit is a closed trail that starts and ends at the same vertex, and uses every edge exactly once.
In such a trail, every time we enter a vertex, we must also leave it via another edge.
This means edges at each vertex are used in pairs: one to enter, one to exit.
Therefore, every vertex must have an even number of edges (degree), so that all edges can be paired up.

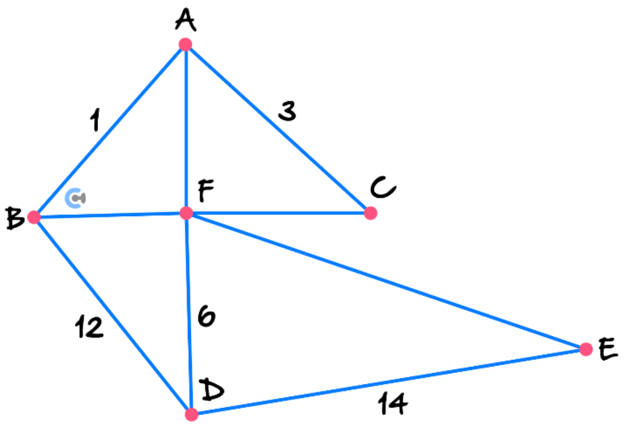
Question 8

Suppose that for some vertex X, the diagonal entry $(A^n)_{xx} = 0$ for all odd n. What can you say about the structure of the graph around vertex X?

This tells us that every **closed walk from X back to X** must have **even length**. (This happens **only if** the graph is **bipartite**.)

Question 9

Consider the following graph.



a. Write a walk from vertex A to E that uses at least 4 edges.

Example: $A \to B \to F \to C \to F \to E$

b. Find a walk starting and ending at vertex F of length 5. (You may repeat edges or vertices.)

Example: $F \to D \to B \to A \to C \to F$

c. List the degree of each vertex.

$A = 3, B = 3, C = 2, D = 3, E = 2, F = 5$

d. Based on your answer in **part c.**, does the graph have an Euler trail? Justify your answer.

No, the graph does not have an Euler trail — it has more than two odd-degree vertices.

e. Does this graph have an Euler circuit? Explain why or why not.

No, the graph does not have an Euler circuit — not all vertices have even degree.

f. Is there a Hamiltonian path in this graph (a path that visits every vertex exactly once)? If so, give one.

Yes. Example: $A \to B \to D \to E \to F \to C$

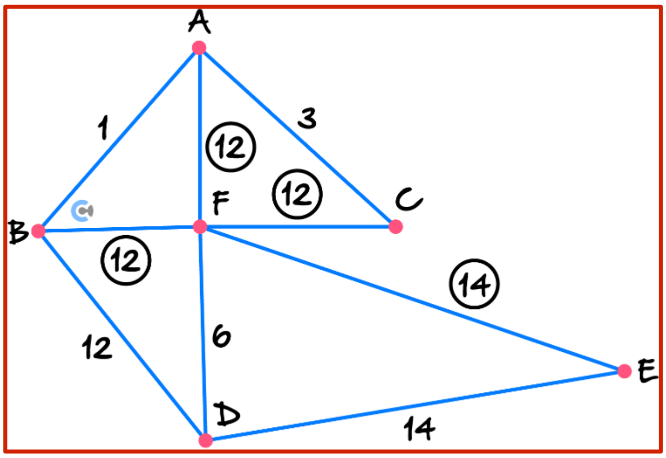
g. Is there a Hamiltonian cycle in this graph (a cycle that visits every vertex exactly once and returns to the start)? Why or why not?

Yes. Example: $A \to B \to D \to E \to F \to C \to A$

h. What is the minimum number of edges needed to connect all vertices without forming any cycles?

That's the definition of a **spanning tree:**
For $\boldsymbol{n} = \mathbf{6}$ vertices, a tree has exactly $\boldsymbol{n} - \mathbf{1} = \mathbf{5}$ edges.

i. The edges which have weights in the following graph define a minimum spanning tree. What is the minimum possible sum of the remaining edges (without weights), which will maintain the present minimum spanning tree? Explain your answer.



$$12 + 12 + 12 + 14 = 50$$

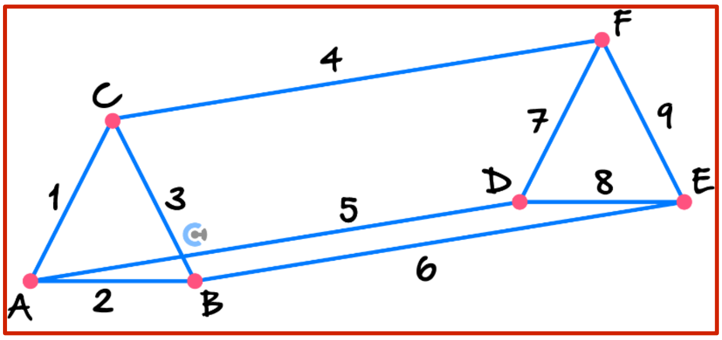
Remember that you have the edges that make up the minimum spanning tree, and the other edges will need to be ignored.

Sub-Section: The Tech-Free "Final Boss"

Question 10

a. Draw the polyhedral graph of a triangular prism and label the vertices as:

- A, B, C (top triangle) and D, E, F (bottom triangle).
- Connect each top vertex to its corresponding bottom vertex (i.e. $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$).



b. How many walks of **length 2** exist from vertex A to vertex E?

2 walks of length 2 from A to E. ($A - B - E$ and $A - D - E$)

c. Can you trace the edges of the triangular prism graph you drew in Q1 without lifting your pencil or retracing any edge? Explain your answer.

No, you cannot trace the graph without lifting your pencil or retracing edges. All vertices have odd degree, so no Euler trail or circuit exists.

d. Find one **Hamiltonian path** in the graph.

Example Hamiltonian Path: $\boldsymbol{A \rightarrow B \rightarrow C \rightarrow F \rightarrow E \rightarrow D}$

e. What is the **minimum number of edges** you need to **remove** from the Hamiltonian cycle to form a **spanning tree**?

- A Hamiltonian cycle has **6 edges** (for 6 vertices).
- A **spanning tree** has $\boldsymbol{n - 1 = 5}$ **edges** (for 6 vertices)

Answer: You need to remove **1 edge** from the Hamiltonian cycle to form a spanning tree.

f. Write down a **spanning tree** for the **triangular prism graph.**

Spanning Tree edges:

$\boldsymbol{A-B, B-C, A-D, B-E, C-F}$

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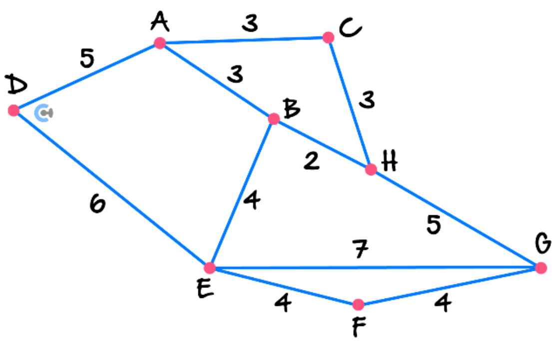
Section B: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 11

The following graph represents the distances (in km) of roads between towns.



a. Construct the adjacency matrix A for the graph. (Treat all edges as unweighted for now.)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b. Explain the meaning of the diagonal entry $(A^n)_{ii}$.

It's the number of walks of length n that begin and end at vertex i.

c. If $(A^3)_{HH} = 0$, what does that tell you about walks starting and ending at H?

There is **no walk of length 3** that starts and ends at H.

d. Does it have an Euler circuit? Justify your answers using the degrees of vertices.

Euler circuit: All vertices have even degree
Let's compute degrees:

$$\begin{aligned} \deg(A) &= 3 \\ \deg(B) &= 3 \\ \deg(C) &= 2 \\ \deg(D) &= 2 \\ \deg(E) &= 4 \\ \deg(F) &= 2 \\ \deg(G) &= 3 \\ \deg(H) &= 3 \end{aligned}$$

Odd-degree vertices: $\boldsymbol{A, B, G, H}$
Euler trail: No; **Euler circuit:** No

e. Find a **Hamiltonian path** in this graph.

One example:
$\boldsymbol{D \to E \to F \to G \to H \to C \to A \to B}$

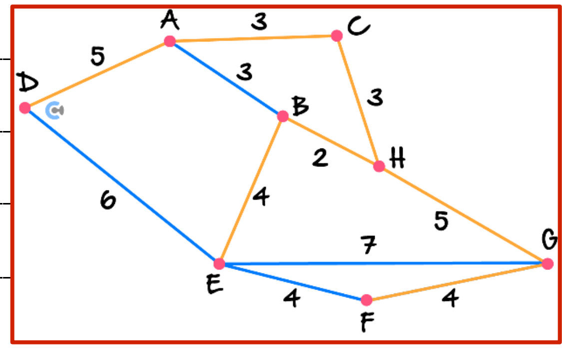
f. Does the graph contain a **Hamiltonian cycle**? Justify your answer.

No — no Hamiltonian cycle exists (can't return to start without repeating).

g. What is the distance of the shortest route from town D to town G?

$\text{DEG} = 6 + 7 = 13$

h. Find the **minimum spanning tree and its total weight.**



$5 + 3 + 3 + 2 + 5 + 4 + 4 = 26$

A postal worker starts at vertex A and wants to visit every street **exactly once** without repeating edges. He does not need to go back to A.

i. Is this possible?

No, an Euler trail is not possible.

j. If not, how many **edges must be duplicated** to make it possible, and how would you decide **which** to duplicate?

1 edge
Duplicate edge: $A–B$ or $G–H$ or B-H

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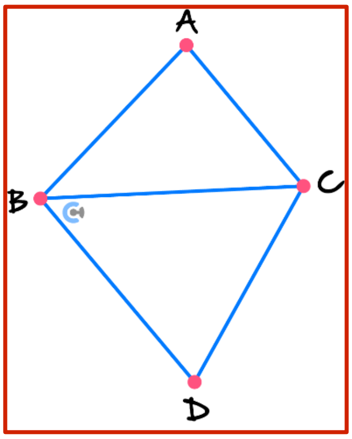
Question 12

Is it possible to draw a Hamiltonian path that also has an Euler trail? If yes, draw a graph, and explain why this is possible. If no, explain why this is not possible.



Yes. A Hamiltonian path must reach all the vertices of a graph exactly once. An Euler trail has to have exactly two vertices of odd degree, and the rest of the vertices need to be of even degree. A Hamiltonian path can exist in circumstances with two odd-degree vertices.

Hamiltonian path = $ABCD$

Euler trail = $CBACDB$

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Sub-Section: Exam 2 (Tech-Active)

Question 13

If a network has 9 vertices and 16 edges, how many edges will it take to form a minimum spanning tree?

A. 9 edges.

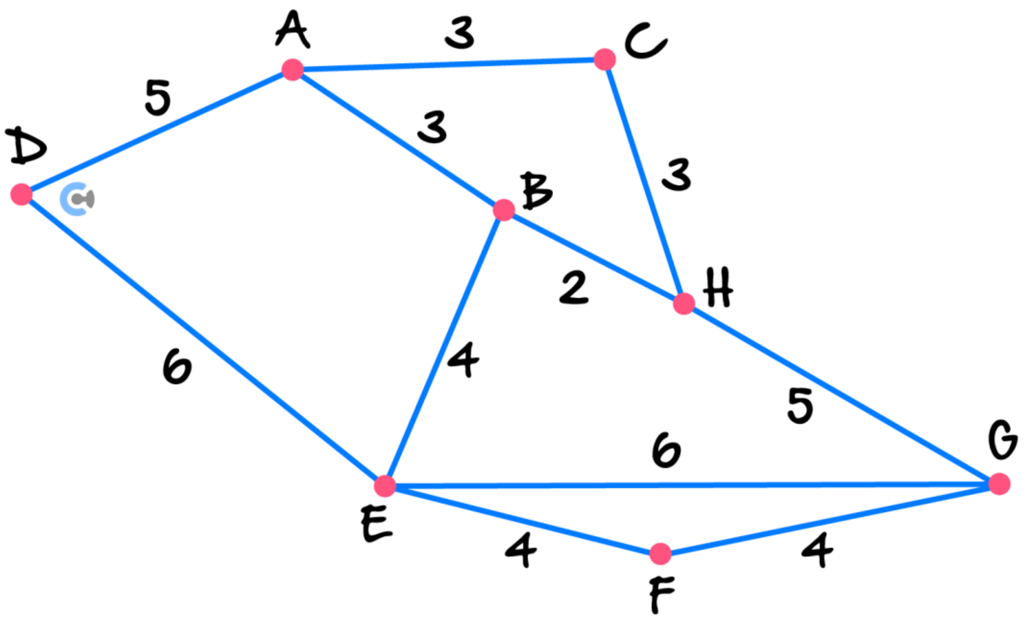
B. 8 edges.

C. 7 edges.

D. 6 edges.

Question 14

The following graph represents the distances (in km) of roads between towns. What is the distance of the longest route from town D to town G if each road can only be used at most once?



A. 12 km

B. 18 km

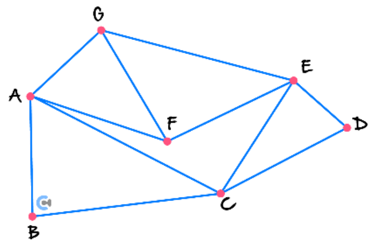
C. 24 km

D. 25 km

$DACHBEFG$

Question 15

Which combination of vertices represents a Hamiltonian cycle in this graph?



A. $CAFGEDCB$

B. $FEDCBAGF$

C. $ABCDEFAG$

D. All of the above.

Question 16

If the sum of the degrees in a graph is 14, how many edges does the graph have?

A. 7

B. 9

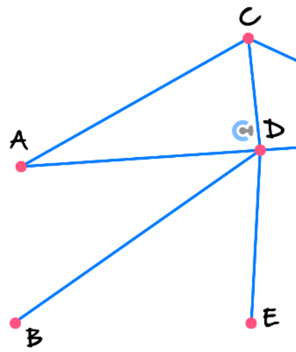
C. 8

D. 28

Space for Personal Notes

Question 17

Which vertices must you add an edge or edges to, in order to create a Hamiltonian cycle that commences from vertex C?



A. Only BE.

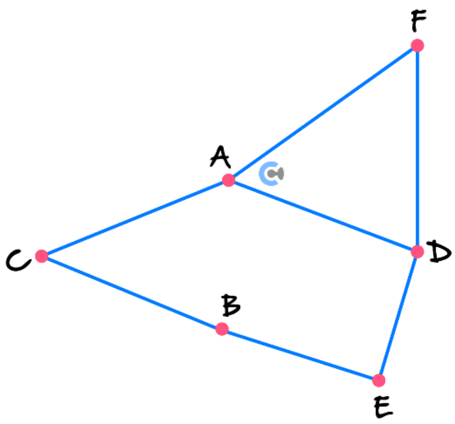
B. Only FE.

C. FE and BE.

D. Only AE.

Question 18

Which combination of vertices in the following graph represents an Euler trail?



A. $FADEBC$

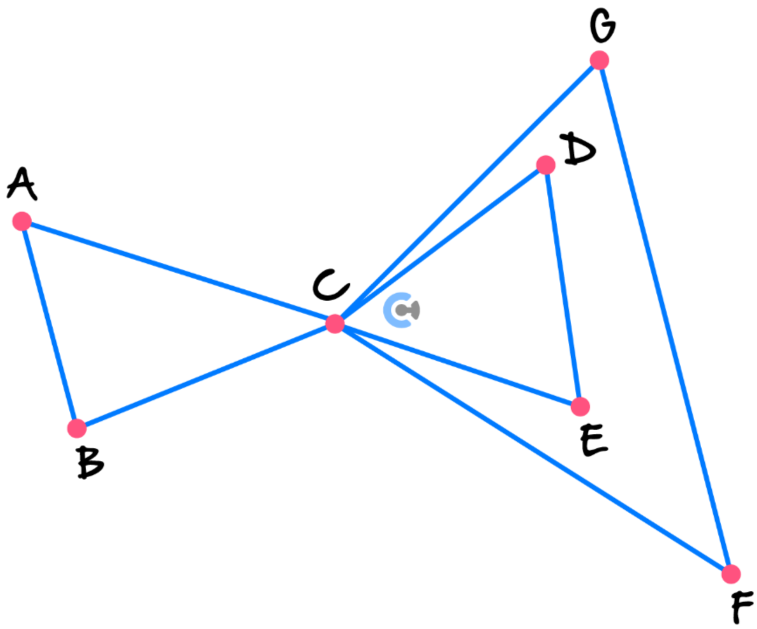
B. $FDACBE$

C. $DAFDEBCA$

D. $CAFDEBCA$

Question 19

Which combination of vertices represents an Euler circuit in the following graph?



A. $CDECBACFG$

B. $ABCFGCDECA$

C. $DECBACGFCD$

D. $EDCFGCABC$

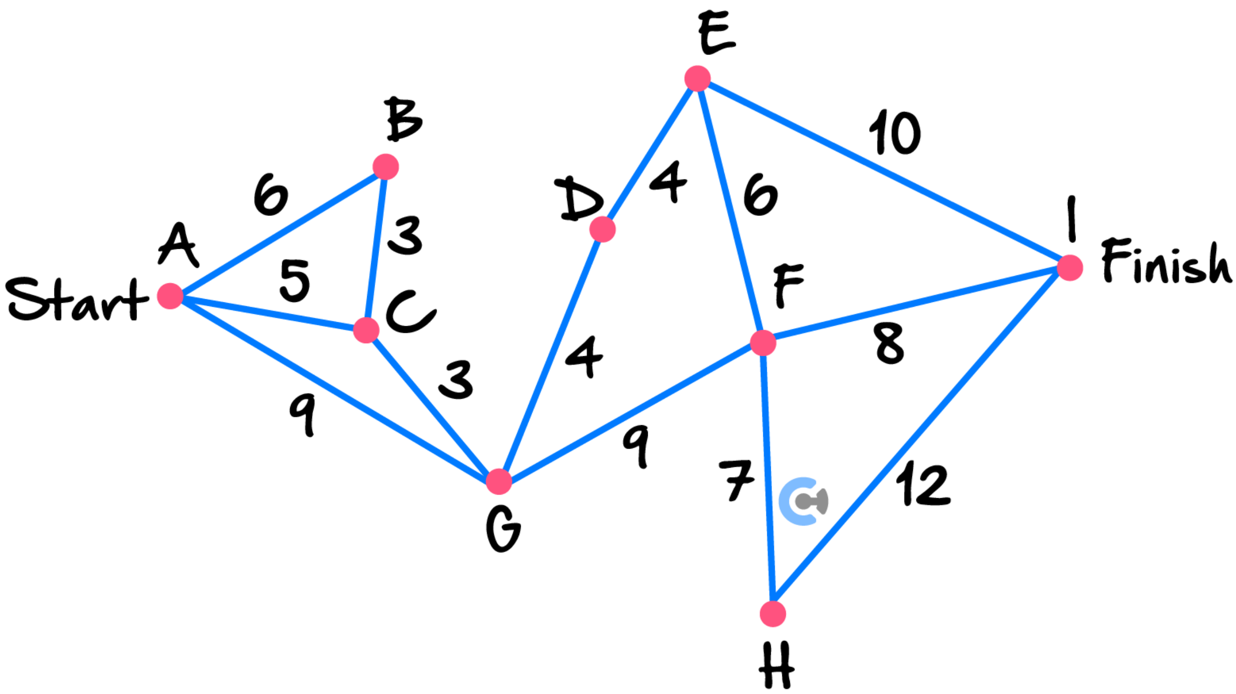
Question 20

Prove that every connected graph has at least one spanning tree.

- A spanning tree is a subgraph that:
  - Includes all vertices of the original graph
  - Is connected
  - Contains no cycles
- Start with a connected graph G.
- If G has no cycles, then it's already a tree, and hence a spanning tree.
- If G has cycles, we can remove edges from those cycles one by one without disconnecting the graph.
- Continue this process until no cycles remain — what's left is a connected acyclic subgraph that includes all vertices.

Conclusion: Every connected graph contains at least one spanning tree.

Question 21



a. Fill in the missing part of the adjacency matrix A of the graph.

(Ignore weights for now; just mark 1 for each undirected connection.)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

b. Use the adjacency matrix method to determine how many walks of length 5 exist between vertex A (Start) and vertex C.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}^5 \quad \begin{bmatrix} 36 & 33 & 37 & 14 & 24 & 20 & 53 & 15 & 16 \\ 33 & 18 & 33 & 18 & 6 & 22 & 22 & 4 & 8 \\ 37 & 33 & 36 & 14 & 24 & 20 & 53 & 15 & 16 \\ 14 & 18 & 14 & 6 & 36 & 18 & 45 & 25 & 22 \\ 24 & 6 & 24 & 36 & 26 & 62 & 20 & 23 & 44 \\ 20 & 22 & 20 & 18 & 62 & 46 & 66 & 46 & 49 \\ 53 & 22 & 53 & 45 & 20 & 66 & 32 & 17 & 35 \\ 15 & 4 & 15 & 25 & 23 & 46 & 17 & 20 & 35 \\ 16 & 8 & 16 & 22 & 44 & 49 & 35 & 35 & 40 \end{bmatrix}$$

37 walks

c. Suppose you observe that there are 6 walks between D and G.
You suspect they are all walks of length m.
Use the adjacency matrix approach to determine a possible value of m.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}^3 \quad \begin{bmatrix} 4 & 5 & 5 & 1 & 2 & 1 & 7 & 1 & 1 \\ 5 & 2 & 5 & 2 & 0 & 2 & 2 & 0 & 0 \\ 5 & 5 & 4 & 1 & 2 & 1 & 7 & 1 & 1 \\ 1 & 2 & 1 & 0 & 5 & 1 & 6 & 3 & 2 \\ 2 & 0 & 2 & 5 & 2 & 8 & 1 & 2 & 6 \\ 1 & 2 & 1 & 1 & 8 & 4 & 8 & 6 & 6 \\ 7 & 2 & 7 & 6 & 1 & 8 & 2 & 1 & 3 \\ 1 & 0 & 1 & 3 & 2 & 6 & 1 & 2 & 5 \\ 1 & 0 & 1 & 2 & 6 & 6 & 3 & 5 & 4 \end{bmatrix}$$

$m = 3$

d. Suppose a robot is navigating from Start (A) to Finish (I).

i. What is the route the robot will need to traverse if the robot wants to begin at the start, arrive at every checkpoint only once, and end at the finish? What is the distance of the path?

There is only one Hamiltonian path between these two vertices: $ABCGDEFHI$.
$6 + 3 + 3 + 4 + 4 + 6 + 7 + 12 = 45\ km$

ii. Could the robot return to A after 4 steps? Justify your answer.

Example: $A \to C \to A \to G \to A$
Valid walk (repeats A), length 4.
Yes, the robot can return to A after 4 steps.

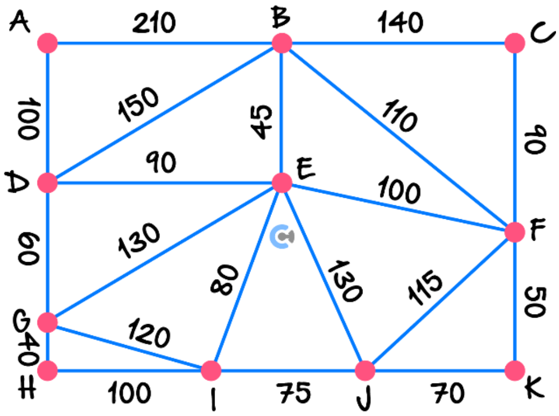
e. What is the shortest path from Start to Finish? What is the length of the shortest path?

In order to find the shortest path from start to finish, it is necessary to examine every route between the two vertices. The shortest route is $ACGFI$. $5 + 3 + 9 + 8 = 25\ km$

Space for Personal Notes

Question 22

A group of tourists are exploring the Carlton Garden in Melbourne. There are a number of feature sites in the gardens that the tourists would like to see. They are represented as vertices in the network below, with edges representing the available pathways and their lengths in metres between each of the feature sites. The spaces between the paths are garden beds.



a. Explain why there are only zeros on the leading diagonal of the adjacency matrix.

There are no paths that go directly from one feature site and directly back to the same site without passing through another site.

b. Explain why the adjacency matrix is symmetric around the leading diagonal.

The network is undirected, so every path from one feature to another is also a path in reverse.

Some of the tourists would like to make sure that they see every feature site of the gardens exactly once, starting and finishing at E.

c. State the name given to the route that would need to be taken by these tourists.

Hamiltonian cycle.

d. Write down a route that could be taken by the tourists to make sure that they see all the feature sites exactly once starting and finishing at E.

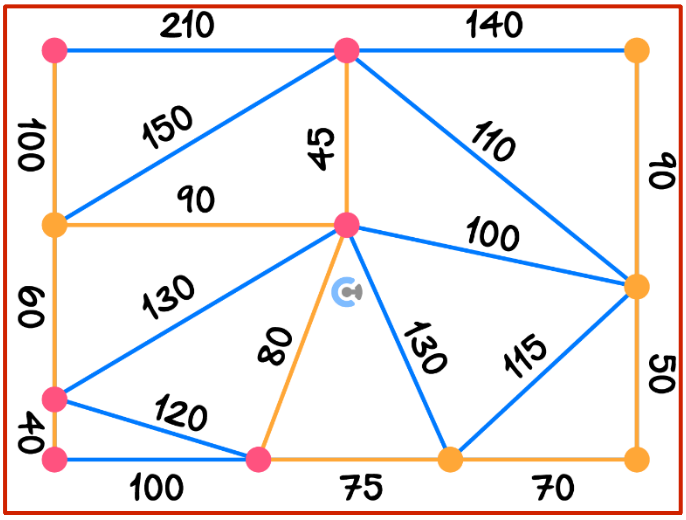
$E - I - H - G - D - A - B - C - F - K - J - E$

One of the tour guides is concerned that just visiting the feature sites of the gardens would mean that the tourists would miss the gardens themselves that are in the spaces between the edges in the network. He suggests that it would be better for the group to start at A and travel along every edge exactly once.

e. Given that the group would want to meet at B again, explain why it is possible for the group to do this in this graph without repeating an edge. Include in your answer where the group would finish if they started at B and used every edge exactly once.

It is not possible because the required route is an Eulerian circuit. (1 mark) and the degree of every vertex in the network must be even for there to be an Eulerian circuit. (1 mark)
Both vertices B and F have odd degrees with all other vertices having even degrees. (1 mark)
and as such only an Euler Path exists starting at B and finishing at F. (1 mark)

f. Add the minimum spanning tree to the copy of the graph below.



g. What is the minimum length of cable required to connect these points?

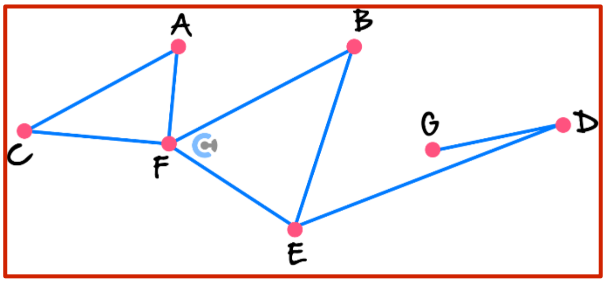
$40 + 60 + 100 + 90 + 80 + 75 + 70 + 50 + 90 + 45 = 700\ m$

Question 23

A city planner is designing a small connected road network between 7 key locations, labeled ***A* through *G***. The planned road network will be represented by a **connected undirected graph**. The graph must satisfy the following conditions:

- The degree of each vertex is:
  - A: 2
  - B: 2
  - C: 2
  - D: 2
  - E: 3
  - F: 4
  - G: 1
- The graph has **8 edges**.
- Vertex ***E* is directly connected to *F*, *B*, *D***.
- Vertex ***C* is directly connected to *A* and *F***.
- Vertex ***G* is directly connected to *D***.

a. Draw a graph that satisfies all the conditions above. Label each vertex clearly.

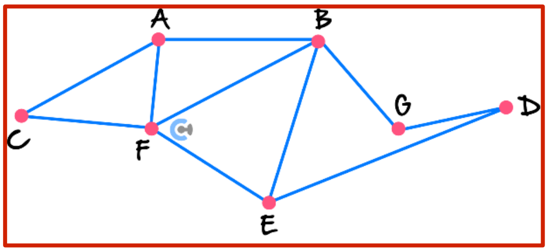


Does the graph contain a:

b. Hamiltonian path? If yes, then describe the path. If no, add an edge (or edges) to create a Hamiltonian path, and describe the new path.

Yes. A Hamiltonian path must travel to each vertex only once. Four different Hamiltonian paths are possible: $ACFBEDG, GDEBFCA, CAFBEDG$ or $GDEBFAC$.

c. Hamiltonian cycle? If yes, then describe the cycle. If no, add an edge (or edges) to create a Hamiltonian cycle, and describe the new cycle.



No. A Hamiltonian cycle must travel to each vertex only once and return to the beginning vertex. It is impossible to return to either of the starting vertices without traversing the same vertices more than once. If edges AB and BG are added, then many Hamiltonian circuits are possible: $ACFEDGBA, CABGDEFC$, etc.

d. Euler circuit? If yes, then describe the circuit. If no, add an edge (or edges) to create a Euler circuit, and describe the new circuit.

No Euler circuit (need 0 odd-degree vertices). Add an edge E-G

$$C - A - F - B - E - G - D - E - F - C$$

e. **Euler trail?** If yes, then describe the trail. If no, add an edge (or edges) to create an **Euler trail**, and describe the new trail.

Yes, only E and G have odd degree → **2 odd-degree vertices**

$$\boldsymbol{G - D - E - B - F - A - C - F - E}$$