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VCE Specialist Mathematics ½
Graph Theory II [5.4]
Homework

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Compulsory Questions	Pg 2-Pg 19
Supplementary Questions	Pg 20-Pg 34

Section A: Compulsory Questions

Sub-Section: Recap

Cheat Sheet

[5.4.1] - Walk of a graph

- **Walk of a Graph**

$$v_1, e_1, \cdots, v_{n-1}, e_1, v_n$$

  - Order of vertices (points) and edges (lines) we are visiting.
  - Alternates between a vertex and an edge.
  - If there is only one edge between two vertices, stating an edge can be abbreviated.

- **Length of a Walk**

$$Length\ of\ a\ Walk = Number\ of\ Edges\ in\ a\ Walk$$

  - Number of edges in a walk.
  - If we use the same edge multiple times, we count it multiple times for the length.

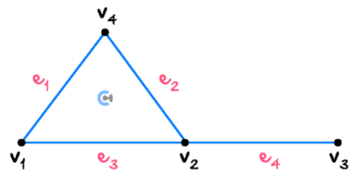
- **Number of Walks with the Length of n**

$$A^n = \begin{bmatrix} v_1, v_1 & v_1, v_2 & v_1, v_3 \\ v_1, v_2 & v_2, v_2 & v_2, v_3 \\ v_1, v_3 & v_3, v_2 & v_3, v_3 \end{bmatrix}$$

  - For walk of length n, we power the adjacency matrix by n.
  - The element gives the number of walks with length of n between two vertices.

[5.4.2] - Euler trail and circuits

- **Euler Trail**



  - Euler Trail is a walk of a graph where all the edges are used exactly once.
  - Whether a vertex is used exactly once is irrelevant.

- **Existence of an Euler Trail**

  - A graph needs to satisfy one of the following two rules:

    1. Every vertex has an even degree.

    OR

    2. Exactly two vertices have an odd degree.

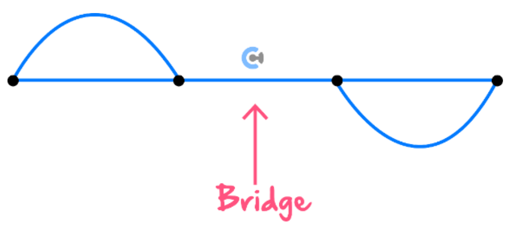
- **Fleury's Algorithm**

  - Allows us to find the Euler's trail efficiently to the graphs with Euler trails.
  - The steps are as follows:

    1. If there are two vertices of an odd degree, then start from one of them. Otherwise, start from any vertex.
    2. Move from the current vertex across an edge to an adjacent vertex. Always choose a non-bridge edge unless there is no alternative.
    3. Delete the edge that you have just traversed.
    4. Repeat from Step 2 until there are no edges left.

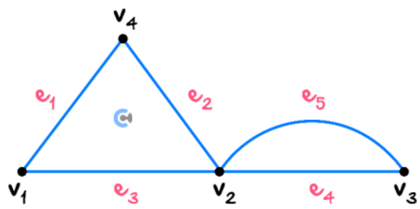
Cheat Sheet

- **Bridge**



  - Bridge is an edge which if you delete will cause a graph to be cut into two graphs.

- **Euler Circuits**



  - Euler Circuit is a walk of a graph where all the edges are used exactly once.

    Starting vertex and ending vertex must be the same.

  - Whether a vertex is used exactly once is irrelevant.

  - Euler circuits are a subset of Euler trails.

- **Existence of an Euler Circuit**

  - A graph needs to satisfy the following rule:

    - Every vertex has an even degree.

[5.4.3] - Hamiltonian paths and cycles

- **Hamiltonian Path**

  - Path which covers all the vertices exactly once.

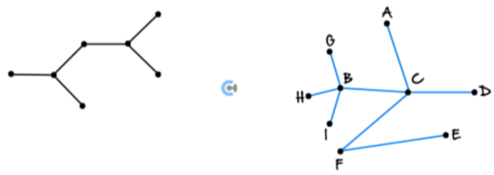
- **Hamiltonian cycle**

  - Path which covers all the vertices exactly once. (Except the starting one.)

    Starting position and ending vertex must be identical.

[5.4.4] - Trees

- **Trees**



  - Tree is a connected graph (can go from one vertex to any other vertex) without any cycle.

  - We cannot come back to the same vertex by not repeating the edges.

- **Equivalent conditions for a tree**

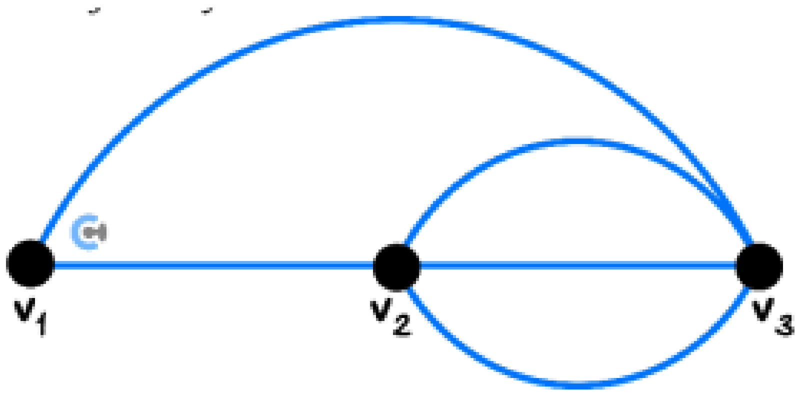
  - A simple graph with n vertices is a tree if ANY of the following conditions are met:

    - It is connected and contains no cycles.
    - It is connected and has $(n-1)$ edges.
    - It has no cycles and has $(n-1)$ edges.
    - It is connected but would become disconnected if any edge is removed.
    - It is connected and would form a cycle if any edge is added.
    - Any two vertices are connected by only a single path.

Sub-Section [5.4.1]: Walk of a Graph

Question 1

Consider the following graph and its adjacency matrix.



Adjacency matrix, $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

a. Find the number of walks with length of 1 between v_2 and v_3.

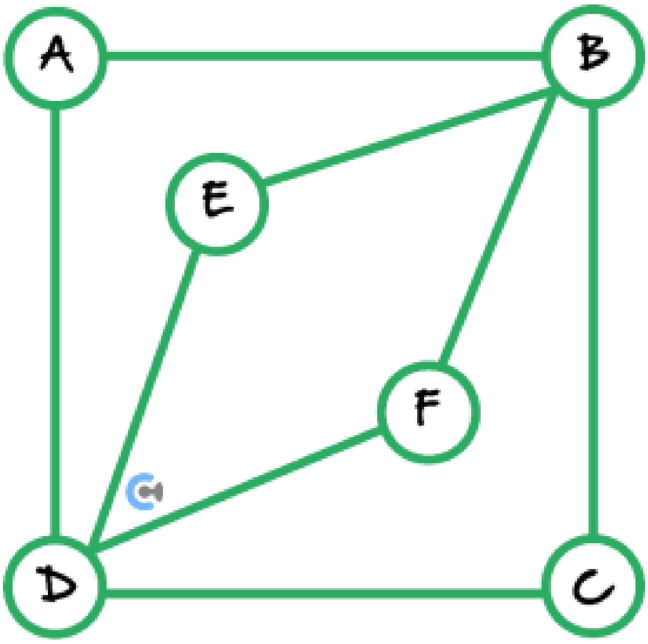
b. Find the number of walks with length of 2 from v_1 to v_3.

c. Can a walk contain repeated vertices or edges? Provide an example of such a walk in this graph.

Sub-Section [5.4.2]: Euler Trail and Circuits

Question 2

Consider the following graph.



a. List the degree of each vertex in the graph.

b. Does this graph contain an Euler Circuit? Justify your answer.

c. Does this graph contain an Euler Trail? If yes, write the sequence of vertices in one possible trail.

d. If you were to remove the edge between D and F, would the graph still have an Euler trail or circuit? Explain.

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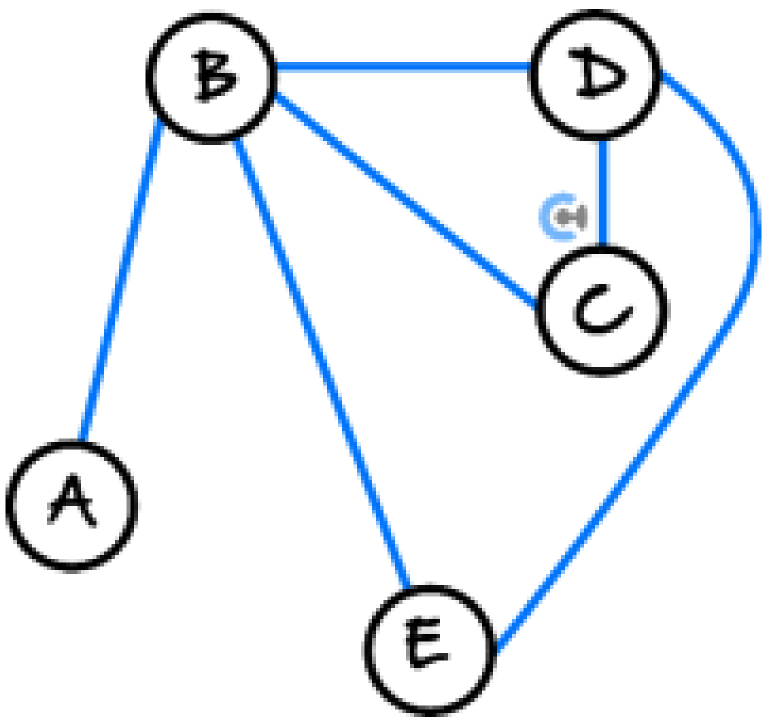
Sub-Section [5.4.3]: Hamiltonian Paths and Cycles

Question 3

Consider the following graph.



a. Does this graph have a Hamiltonian Path? If yes, list one possible path.

b. Does this graph have a Hamiltonian Cycle? If yes, provide one possible cycle.

c. Is it possible to start at vertex A and visit every vertex exactly once without returning to A? Why or why not?

d. If the edge between A and E is added, would a Hamiltonian cycle exist? Justify your answer.

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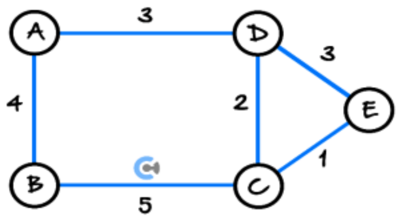
Sub-Section [5.4.4]: Trees

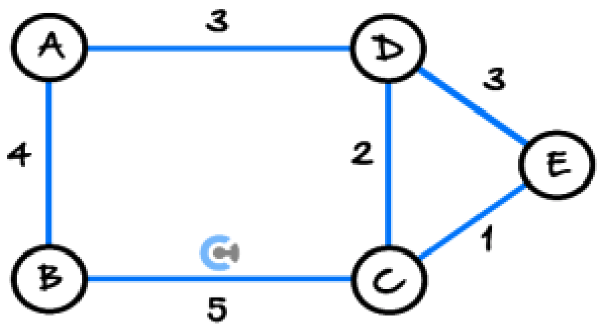
Question 4

Consider the following graph.

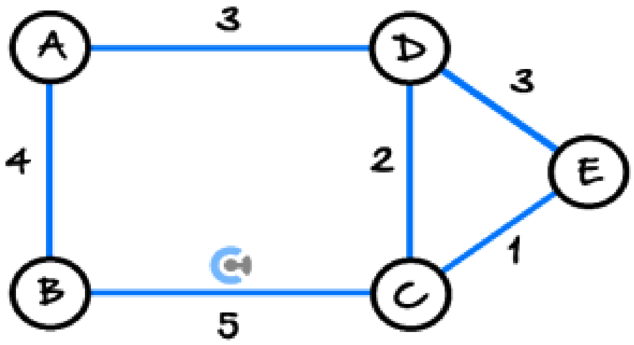


a. How many edges does a spanning tree for this graph have?

b. Identify one of the spanning trees.



c. Identify the minimum spanning tree with its total weight.



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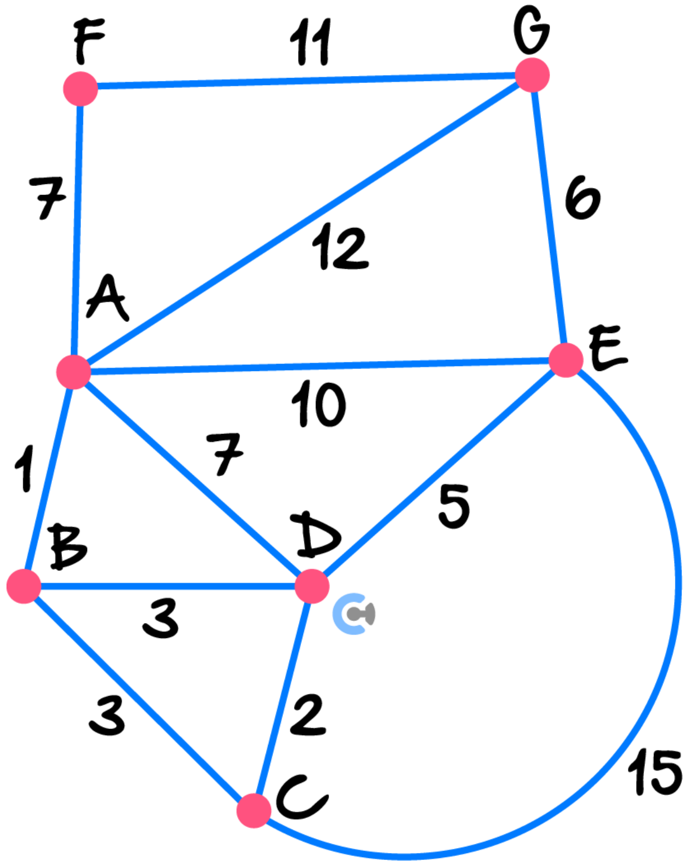
Sub-Section: Problem Solving

Question 5

Consider the following graph.



a. Write down the adjacency matrix A for this graph.

b. How many walks of length 4 are there from vertex B to C? (Treat all edges as unweighted for now.)

---

c. Explain your answer with reference to the degrees of the vertices. Determine whether this graph has:

i. An Euler trail.

ii. An Euler circuit.

d. If you want to find a path that uses every road exactly once, how many roads must you duplicate (if any) to convert this graph into one that admits an Euler circuit?

e. A salesperson wants to visit each city (vertex) exactly once and return to the starting city.

i. List one Hamiltonian cycle from the graph.

---

ii. Compute its total cost using the edge weights.

---

iii. Suggest a different Hamiltonian cycle with a smaller cost if possible.

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f. Is it possible to find a Hamiltonian path that does not return to the starting point? Justify your answer.

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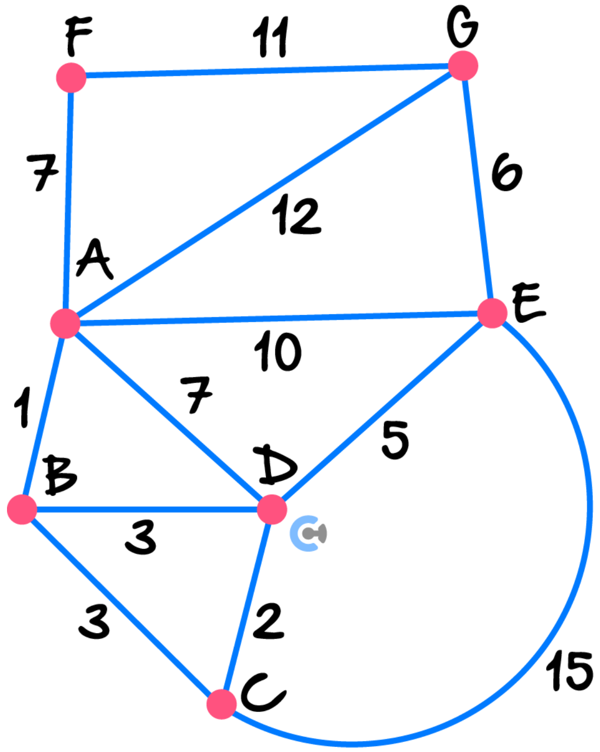
g. Draw the minimum spanning tree for the graph. What is the distance of the minimum spanning tree?



Question 6

Prove that if a connected undirected graph has an Euler trail, then it has exactly 0 or 2 odd-degree vertices.

Question 7

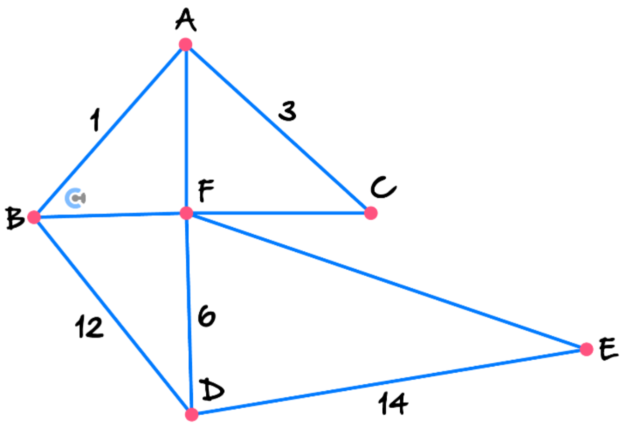
Prove that if a connected graph has an Euler circuit, then all its vertices have even degrees.

Question 8

Suppose that for some vertex X, the diagonal entry $(A^n)_{xx} = 0$ for all odd n. What can you say about the structure of the graph around vertex X?

Question 9

Consider the following graph.



a. Write a walk from vertex A to E that uses at least 4 edges.

---

b. Find a walk starting and ending at vertex F of length 5. (You may repeat edges or vertices.)

---

c. List the degree of each vertex.

---

d. Based on your answer in **part c.**, does the graph have an Euler trail? Justify your answer.

---

e. Does this graph have an Euler circuit? Explain why or why not.

---

f. Is there a Hamiltonian path in this graph (a path that visits every vertex exactly once)? If so, give one.

---

g. Is there a Hamiltonian cycle in this graph (a cycle that visits every vertex exactly once and returns to the start)? Why or why not?

---

h. What is the minimum number of edges needed to connect all vertices without forming any cycles?

---

i. The edges which have weights in the following graph define a minimum spanning tree. What is the minimum possible sum of the remaining edges (without weights), which will maintain the present minimum spanning tree? Explain your answer.

Sub-Section: The Tech-Free "Final Boss"

Question 10

a. Draw the polyhedral graph of a triangular prism and label the vertices as:

- A, B, C (top triangle) and D, E, F (bottom triangle).
- Connect each top vertex to its corresponding bottom vertex (i.e. $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$).

b. How many walks of **length 2** exist from vertex A to vertex E?

---

c. Can you trace the edges of the triangular prism graph you drew in Q1 without lifting your pencil or retracing any edge? Explain your answer.

---

d. Find one **Hamiltonian path** in the graph.

---

e. What is the **minimum number of edges** you need to **remove** from the Hamiltonian cycle to form a **spanning tree**?

---

f. Write down a **spanning tree** for the **triangular prism graph.**

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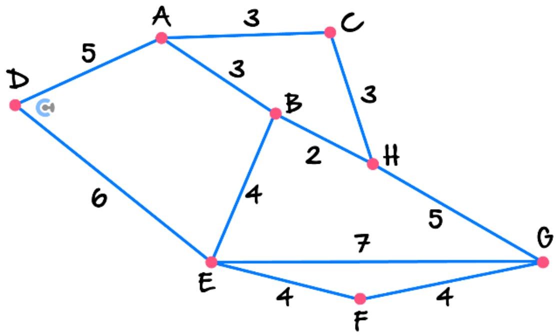
Section B: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 11

The following graph represents the distances (in km) of roads between towns.



a. Construct the adjacency matrix A for the graph. (Treat all edges as unweighted for now.)

b. Explain the meaning of the diagonal entry $(A^n)_{ii}$.

c. If $(A^3)_{HH} = 0$, what does that tell you about walks starting and ending at H?

d. Does it have an Euler circuit? Justify your answers using the degrees of vertices.

e. Find a **Hamiltonian path** in this graph.

f. Does the graph contain a **Hamiltonian cycle**? Justify your answer.

g. What is the distance of the shortest route from town D to town G?

h. Find the **minimum spanning tree and its total weight.**

A postal worker starts at vertex A and wants to visit every street **exactly once** without repeating edges. He does not need to go back to A.

i. Is this possible?

j. If not, how many **edges must be duplicated** to make it possible, and how would you decide **which** to duplicate?

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Question 12

Is it possible to draw a Hamiltonian path that also has an Euler trail? If yes, draw a graph, and explain why this is possible. If no, explain why this is not possible.

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Sub-Section: Exam 2 (Tech-Active)

Question 13

If a network has 9 vertices and 16 edges, how many edges will it take to form a minimum spanning tree?

A. 9 edges.

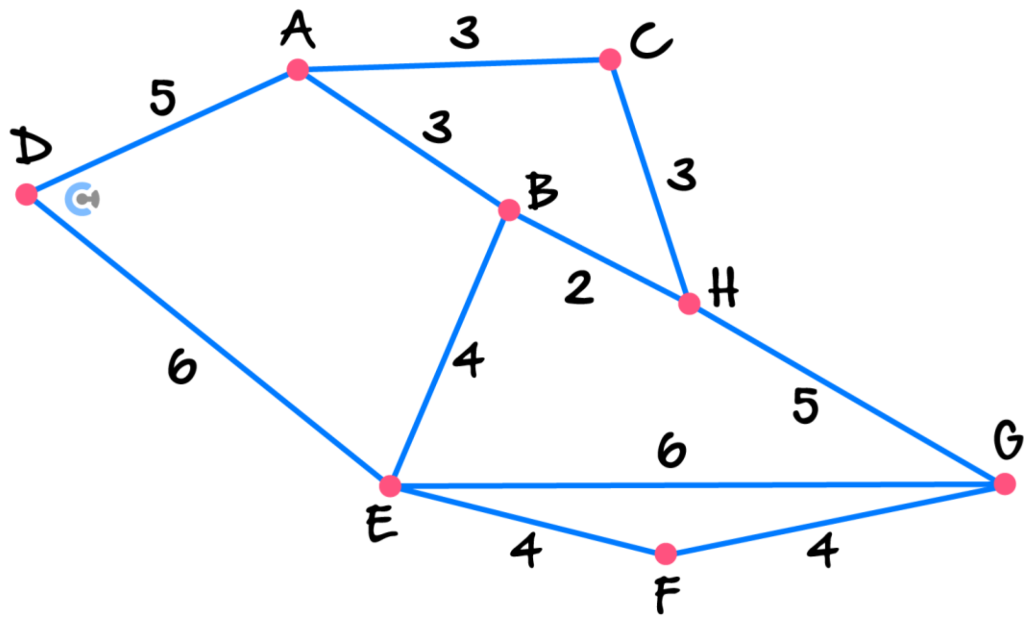
B. 8 edges.

C. 7 edges.

D. 6 edges.

Question 14

The following graph represents the distances (in km) of roads between towns. What is the distance of the longest route from town D to town G if each road can only be used at most once?



A. 12 km

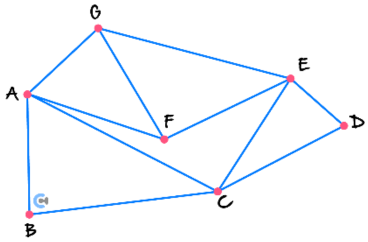
B. 18 km

C. 24 km

D. 25 km

Question 15

Which combination of vertices represents a Hamiltonian cycle in this graph?



A. $CAFGEDCB$

B. $FEDCBAGF$

C. $ABCDEFAG$

D. All of the above.

Question 16

If the sum of the degrees in a graph is 14, how many edges does the graph have?

A. 7

B. 9

C. 8

D. 28

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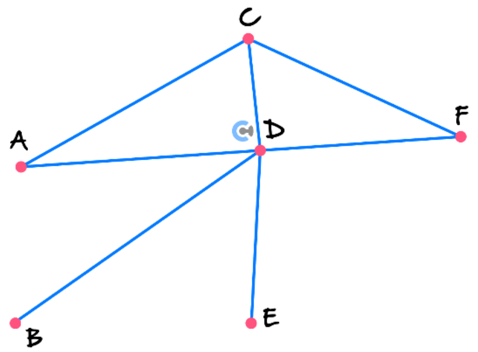
Question 17

Which vertices must you add an edge or edges to, in order to create a Hamiltonian cycle that commences from vertex C?



A. Only BE.

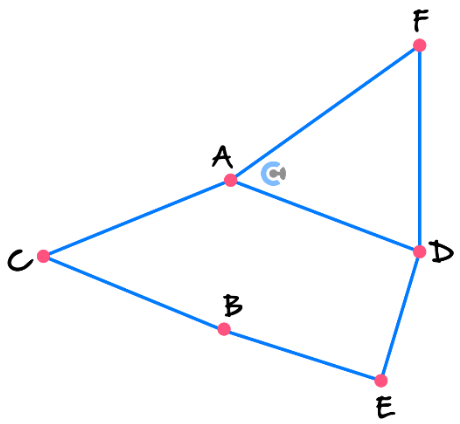
B. Only FE.

C. FE and BE.

D. Only AE.

Question 18

Which combination of vertices in the following graph represents an Euler trail?



A. $FADEBC$

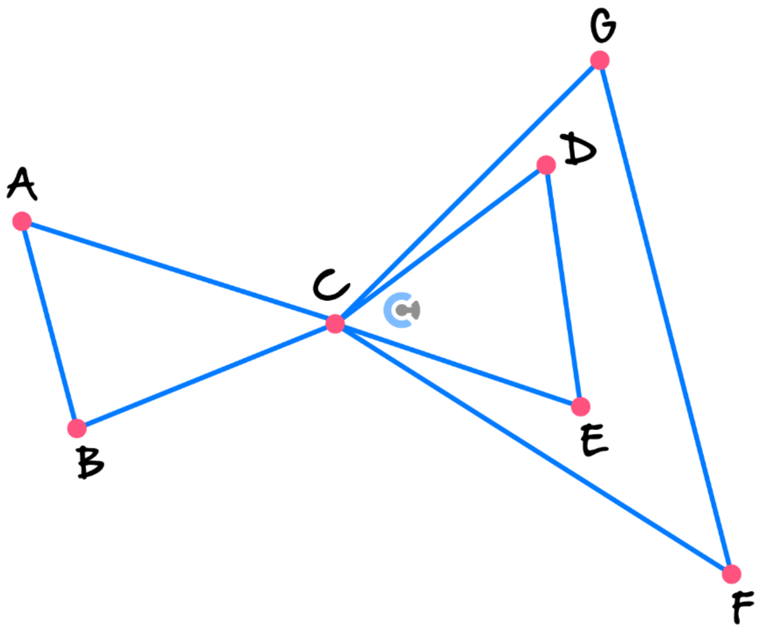
B. $FDACBE$

C. $DAFDEBCA$

D. $CAFDEBCA$

Question 19

Which combination of vertices represents an Euler circuit in the following graph?



A. $CDECBACFG$

B. $ABCFGCDECA$

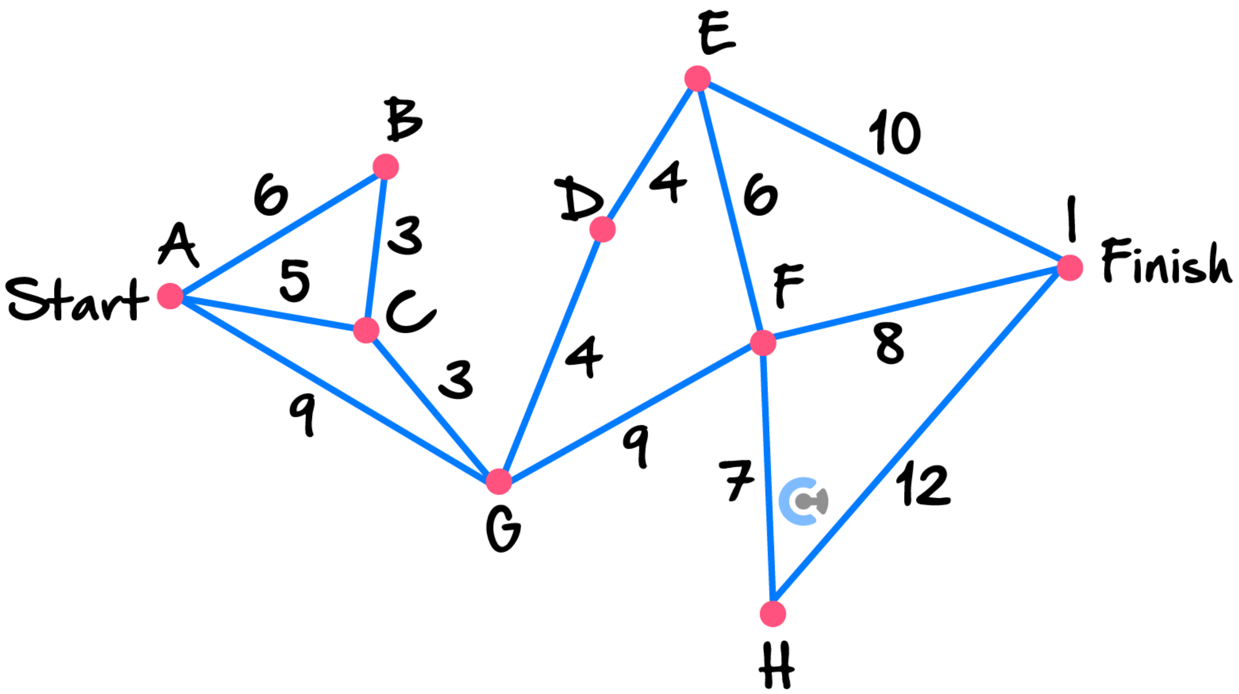
C. $DECBACGFCD$

D. $EDCFGCABC$

Question 20

Prove that every connected graph has at least one spanning tree.

Question 21



a. Fill in the missing part of the adjacency matrix A of the graph.

(Ignore weights for now; just mark 1 for each undirected connection.)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & \square & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \square & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \square & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \square & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & \square & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \square & 1 & 1 \\ \square & \square & \square & \square & \square & \square & \square & \square & \square \\ 0 & 0 & 0 & 0 & 0 & 1 & \square & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & \square & 1 & 0 \end{bmatrix}$$

b. Use the adjacency matrix method to determine how many walks of length 5 exist between vertex A (Start) and vertex C.

c. Suppose you observe that there are 6 walks between D and G.
You suspect they are all walks of length m.
Use the adjacency matrix approach to determine a possible value of m.

d. Suppose a robot is navigating from Start (A) to Finish (I).

i. What is the route the robot will need to traverse if the robot wants to begin at the start, arrive at every checkpoint only once, and end at the finish? What is the distance of the path?

ii. Could the robot return to A after 4 steps? Justify your answer.

e. What is the shortest path from Start to Finish? What is the length of the shortest path?

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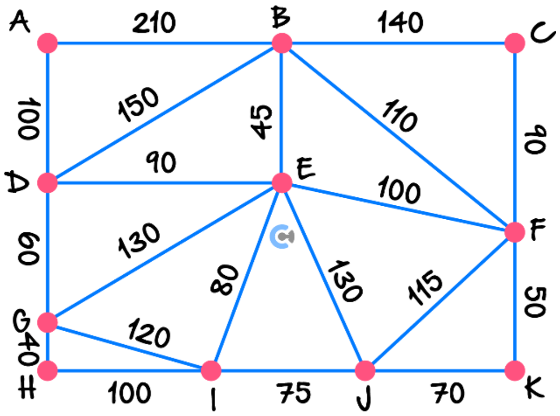
Question 22

A group of tourists are exploring the Carlton Garden in Melbourne. There are a number of feature sites in the gardens that the tourists would like to see. They are represented as vertices in the network below, with edges representing the available pathways and their lengths in metres between each of the feature sites. The spaces between the paths are garden beds.



a. Explain why there are only zeros on the leading diagonal of the adjacency matrix.

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b. Explain why the adjacency matrix is symmetric around the leading diagonal.

Some of the tourists would like to make sure that they see every feature site of the gardens exactly once, starting and finishing at E.

c. State the name given to the route that would need to be taken by these tourists.

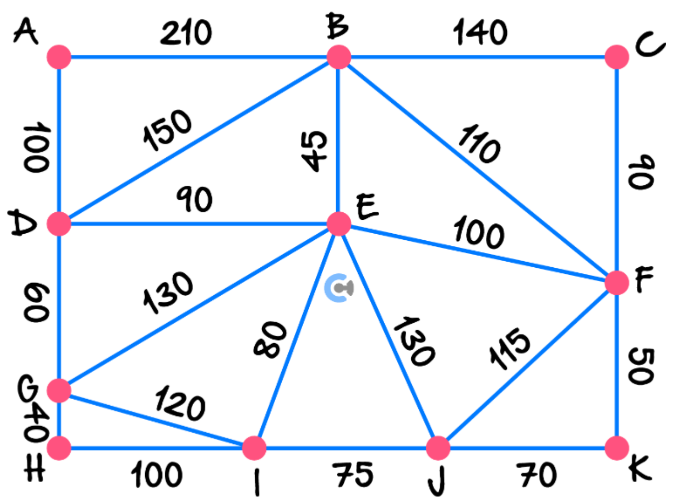
---

d. Write down a route that could be taken by the tourists to make sure that they see all the feature sites exactly once starting and finishing at E.

One of the tour guides is concerned that just visiting the feature sites of the gardens would mean that the tourists would miss the gardens themselves that are in the spaces between the edges in the network. He suggests that it would be better for the group to start at A and travel along every edge exactly once.

e. Given that the group would want to meet at B again, explain why it is possible for the group to do this in this graph without repeating an edge. Include in your answer where the group would finish if they started at B and used every edge exactly once.

f. Add the minimum spanning tree to the copy of the graph below.



g. What is the minimum length of cable required to connect these points?

Question 23

A city planner is designing a small connected road network between 7 key locations, labeled ***A* through *G***. The planned road network will be represented by a **connected undirected graph**. The graph must satisfy the following conditions:

- The degree of each vertex is:
  - A: 2
  - B: 2
  - C: 2
  - D: 2
  - E: 3
  - F: 4
  - G: 1
- The graph has **8 edges**.
- Vertex ***E* is directly connected to *F*, *B*, *D***.
- Vertex ***C* is directly connected to *A* and *F***.
- Vertex ***G* is directly connected to *D***.

a. Draw a graph that satisfies all the conditions above. Label each vertex clearly.

Does the graph contain a:

b. Hamiltonian path? If yes, then describe the path. If no, add an edge (or edges) to create a Hamiltonian path, and describe the new path.

c. Hamiltonian cycle? If yes, then describe the cycle. If no, add an edge (or edges) to create a Hamiltonian cycle, and describe the new cycle.

d. Euler circuit? If yes, then describe the circuit. If no, add an edge (or edges) to create a Euler circuit, and describe the new circuit.

e. **Euler trail?** If yes, then describe the trail. If no, add an edge (or edges) to create an **Euler trail**, and describe the new trail.