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VCE Specialist Mathematics ½

Graph Theory I [5.3]

Workbook

Outline:

Learning Objectives:

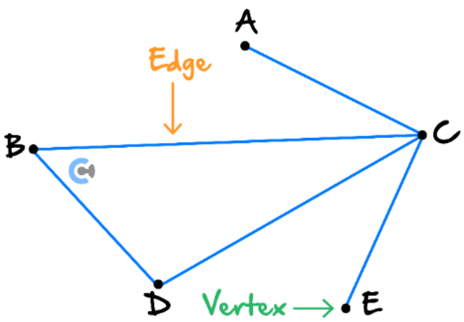
- [ ] **SM12 [5.3.1]** – Graph Theory Fundamentals – Vertices, Edges, Degree, Adjacency Lists, and Matrices
- [ ] **SM12 [5.3.2]** – Types of Graphs
- [ ] **SM12 [5.3.3]** – Isomorphisms and Subgraphs

Section A: Graphs

Sub-Section: Vertices and Edges

What does the graph have?

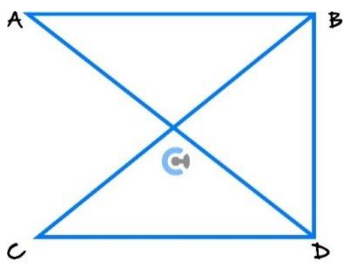
Vertices and Edges



- A graph consists of a set of points called vertices and a set of unordered pairs of vertices, called edges.

Question 1 Walkthrough.

Consider a graph below.



Write down the vertex set and edge set of the given graph.

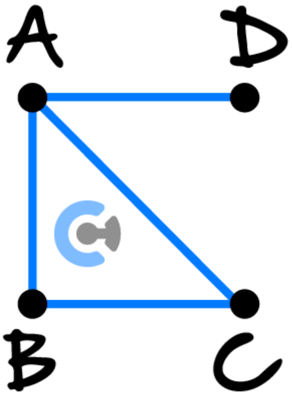
vertex = $\{A, B, C, D\}$

edges = $\{AB, AD, BD, BC, CD\}$

Question 2

Write the vertex sets and edge sets for the graphs corresponding to the following pictures.

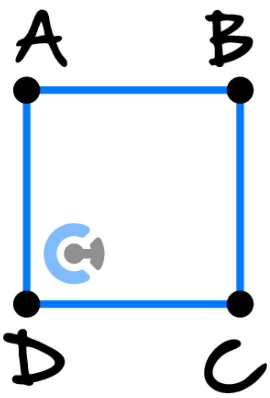
a.



vertices = {A, B, C, D}

edges = {AD, AB, AC, BC}

b.



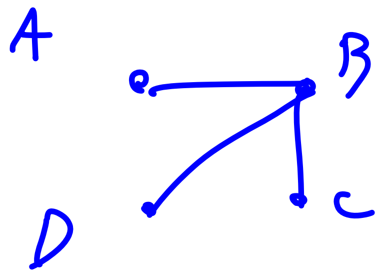
vertices = {A, B, C, D}

edges = {AB, BC, CD, AD}

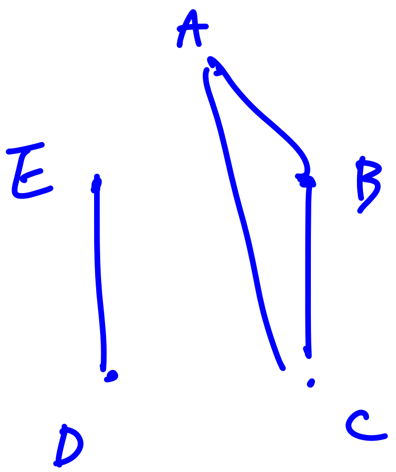
Question 3

Draw pictures of 2 graphs with the following vertex and edge sets.

a. Vertex set: $\{A, B, C, D\}$
Edge set: $\{AB, BC, BD\}$



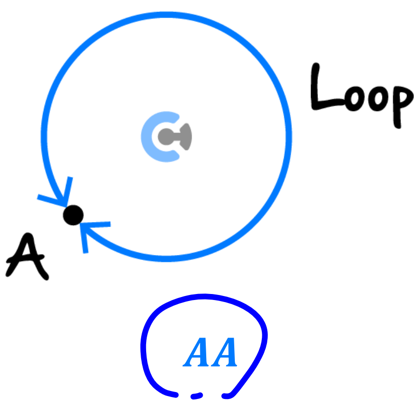
b. Vertex set: $\{A, B, C, D, E\}$
Edge set: $\{AB, BC, CA, DE\}$



What if an edge connects A to A?

Definition

<u>Loops</u>

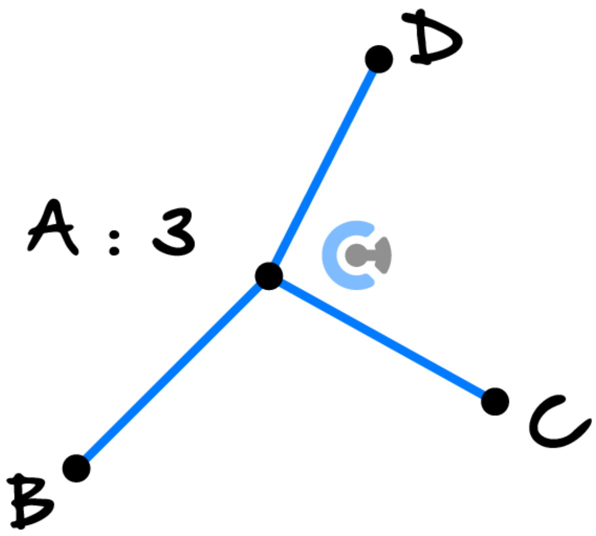


- Loop is an edge which connects to the same vertex.

Sub-Section: Degree of a Vertex

Let's consider the degree of a vertex!

Degree of a Vertex



- Degree of a vertex is the number of edges connected to the vertex.

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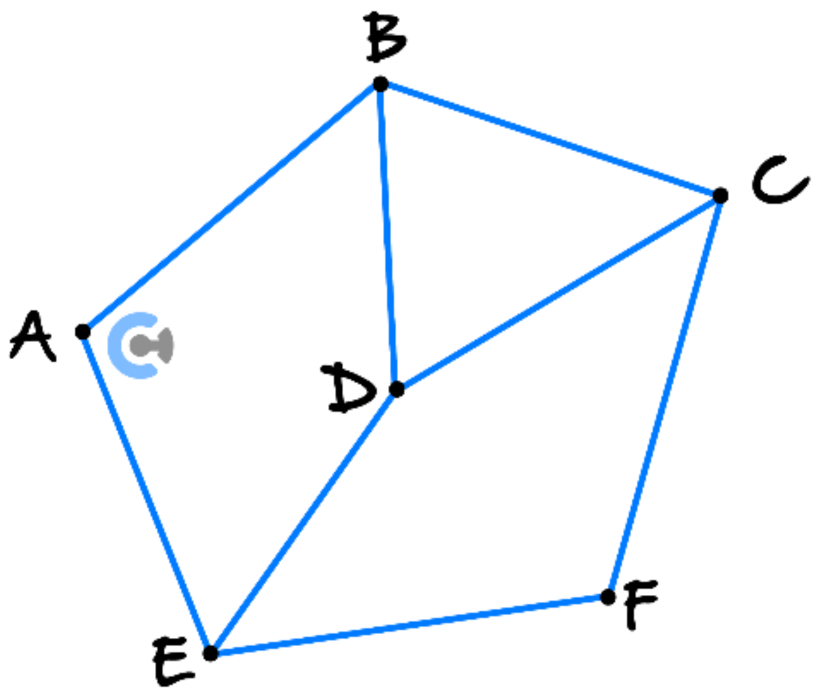
Question 4 Walkthrough.

Fill in the following information for the graph below.

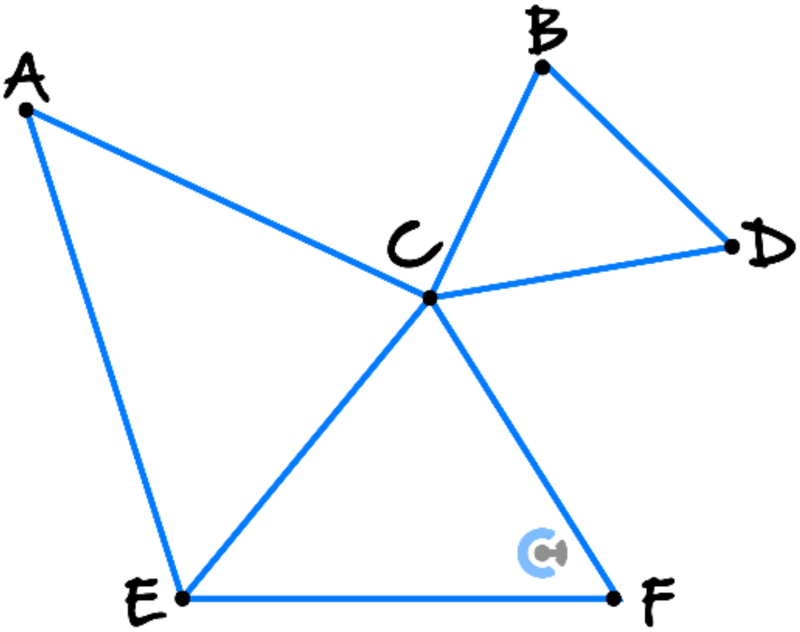


Vertex	Degree of Vertex
A	2
B	3
C	3
D	3
E	3
F	2
Number of Edges: 8	Sum of Degrees: 16

Question 5

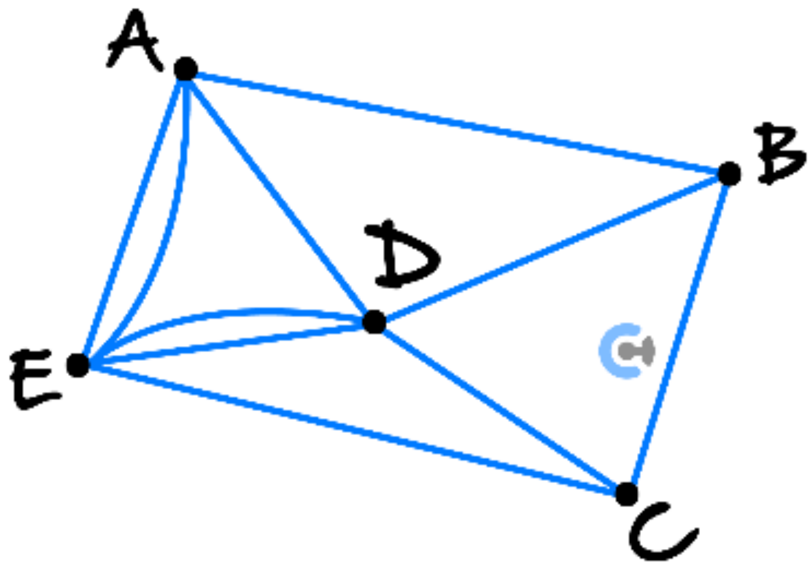
Fill in the following information for the graphs below.

a.



Vertex	Degree of Vertex
A	2
B	2
C	5
D	2
E	3
F	2
Number of Edges: 8	**Sum of Degrees:** 16

b.



Vertex	Degree of Vertex
A	4
B	3
C	3
D	5
E	5
Number of Edges: 10	**Sum of Degrees:** 20

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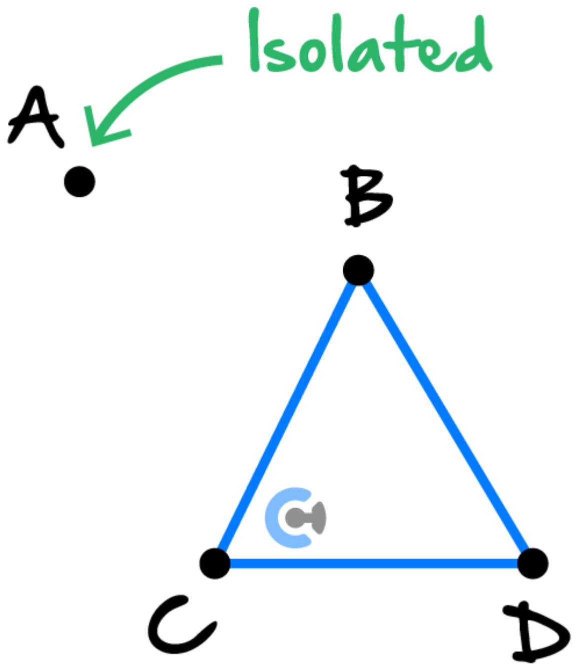
What about if a vertex is not connected to any other point (including itself)?

Definition

Isolated Vertex



- Isolated vertex has no edges connected to it.
- Its degree is equal to zero.

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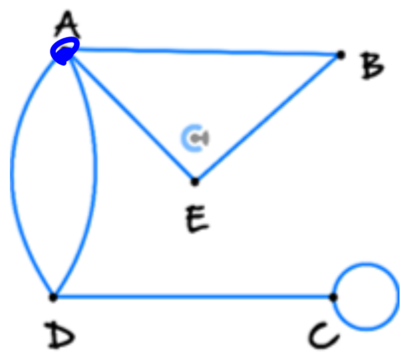
Section B: Adjacency List and Matrix

Sub-Section: Adjacency List

Discussion: What do we call two points that are connected by an edge?

adjacent

Adjacency Lists

Graph	Adjacency List
	$A \to (B, D, D, E)$ $B \to (A, E)$ $C \to (C, D)$ $D \to (A, A, C)$ $E \to (A, B)$

- Adjacency list contains all the vertices a given vertex is connected to.
- If the point is connected multiple times, we write the vertex multiple times.
- If a point is loop with itself, we write the vertex to be adjacent to itself.

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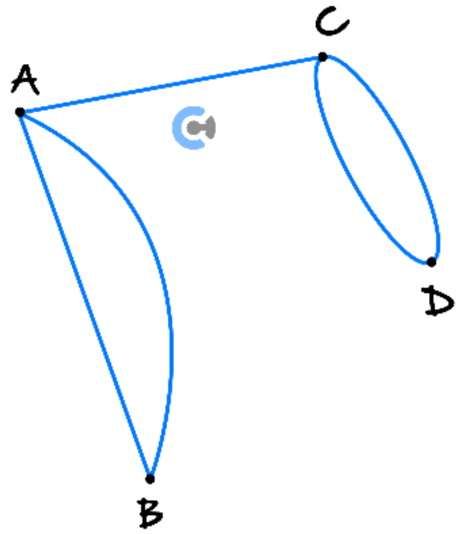
Question 6

Create an adjacency list that describes the following graph.



A → (B, B, C) Graph
B → (A, A)
C → (A, D, D)
D → (C, C)

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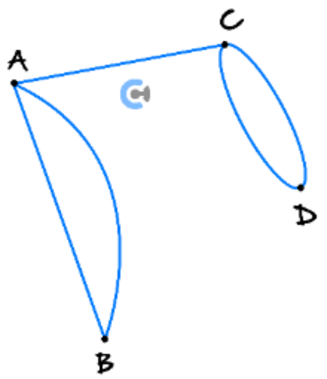
Sub-Section: Adjacency Matrix

Discussion: Is there a way to convert the adjacency list into a matrix form?

$$\begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Question 7 Walkthrough.

Consider the previous question with its graph and adjacency list.



Graph

$$A \to (B, B, C)$$

$$B \to (A, A)$$

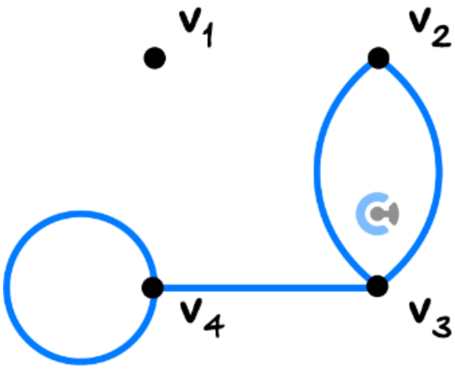
$$C \to (A, D, D)$$

$$D \to (C, C)$$

Convert the given adjacency list into an adjacency matrix.

$$\begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix



$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

- A matrix that represents the vertices and edges that connect the vertices of a graph.

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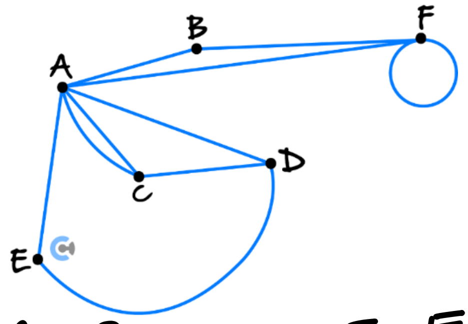
Question 8

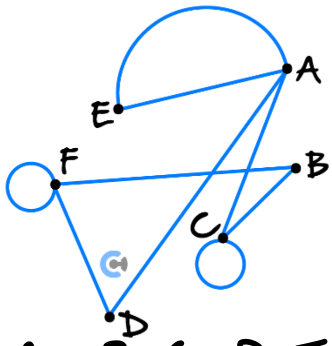
Construct the adjacency matrix for the given graphs.

a.



$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

b.



$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Discussion: Could the adjacency matrix be a non-square matrix? (Same number of rows and columns)

No. Rows = columns

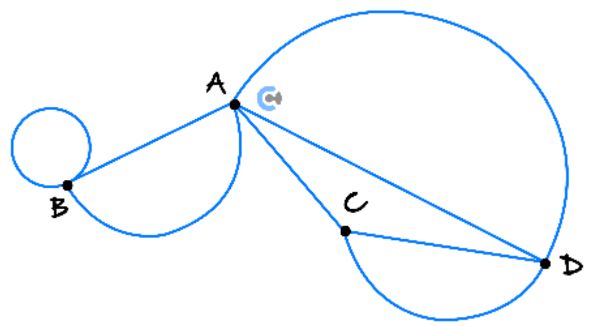
Discussion: What does it mean when there is a non-zero value in the diagonal element of the adjacent matrix?

there is a loop

Properties of Adjacency Matrices

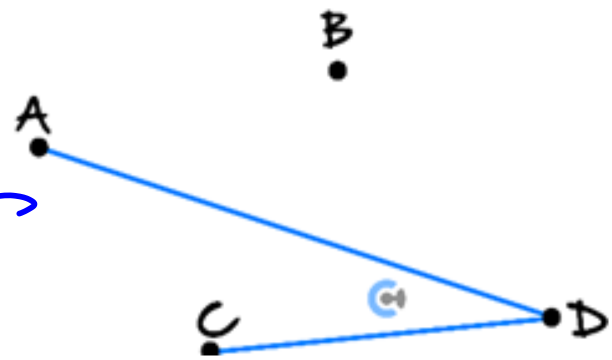
- Always a square matrix.
- Any non-zero value in the leading diagonal will indicate the existence of a loop.

$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$



- A row consisting of all zeros indicates an Isolated Vertex.

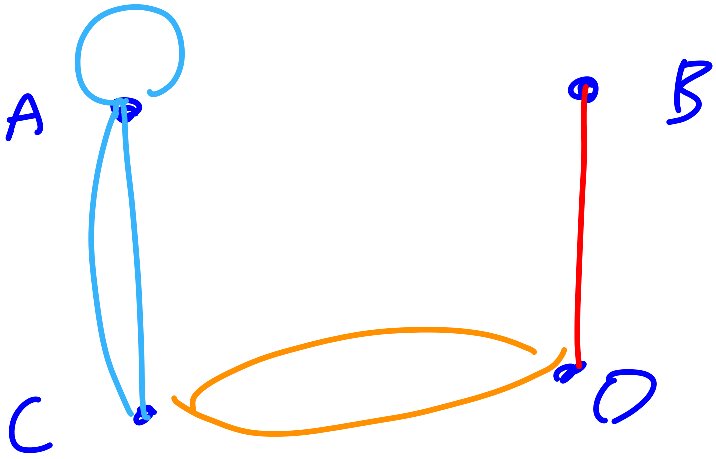
$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$



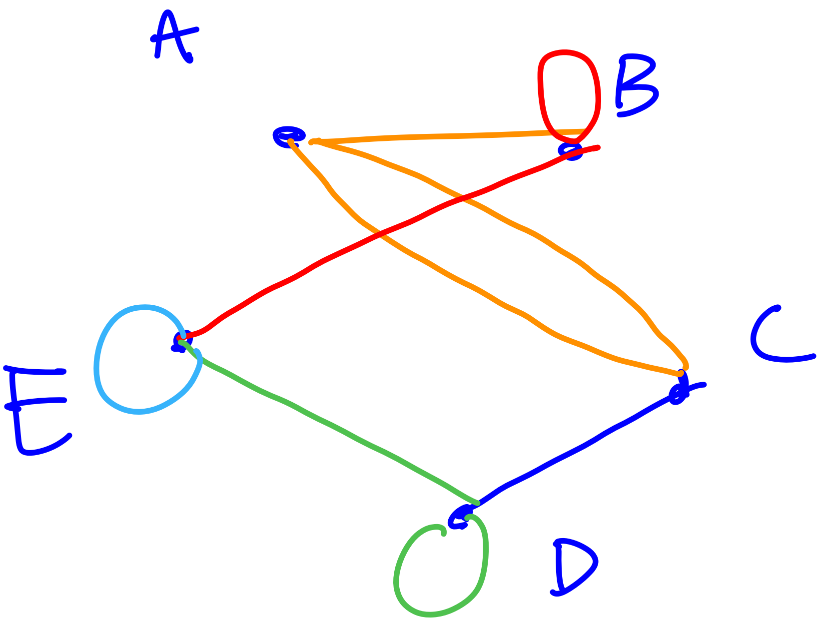
Question 9

Draw graphs to represent the following adjacency matrices.

a. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$



b. $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

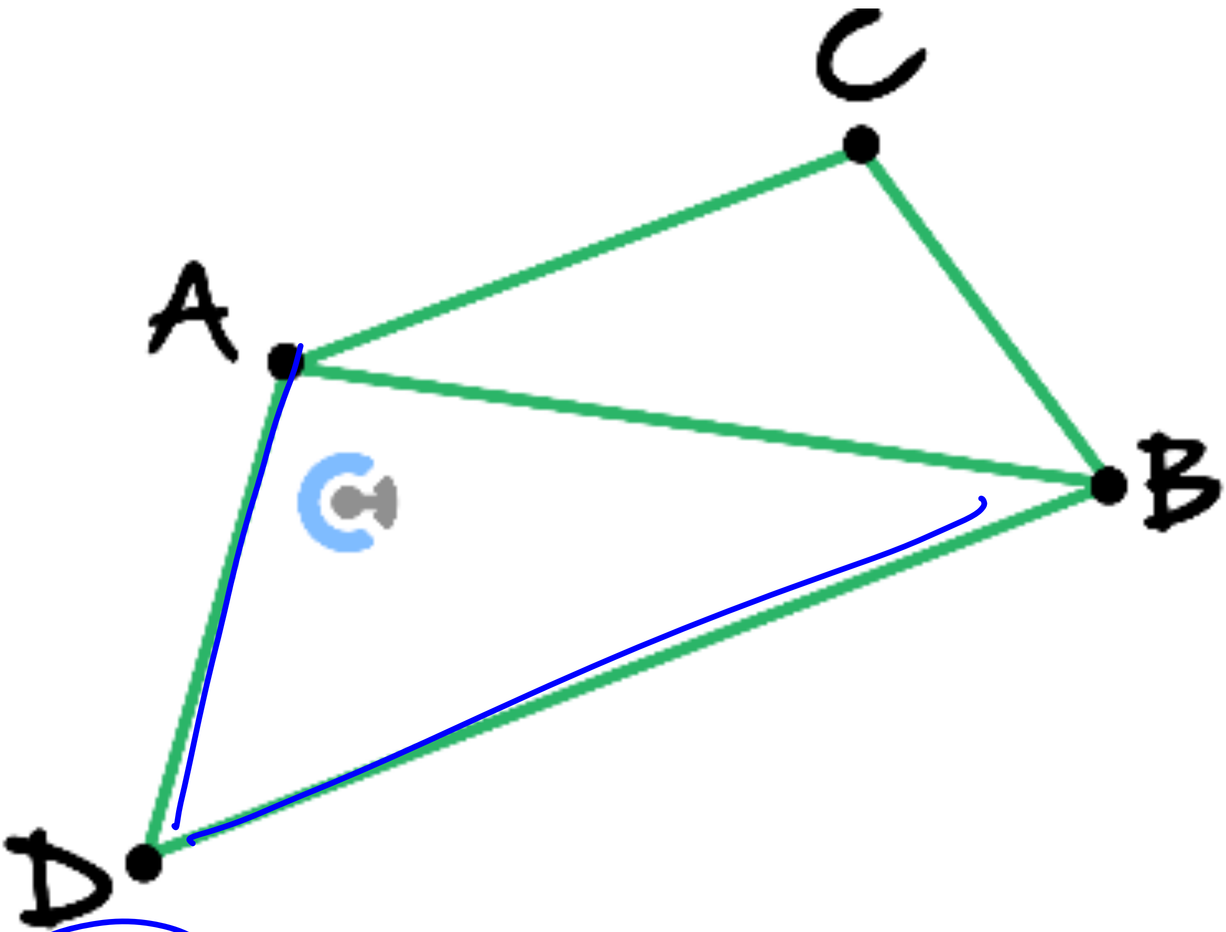


Section C: Types of Graphs

Sub-Section: Simple Graph

This one should be simple. ☺

Simple Graph



- A simple graph is one in which pairs of vertices are connected by one edge at most.

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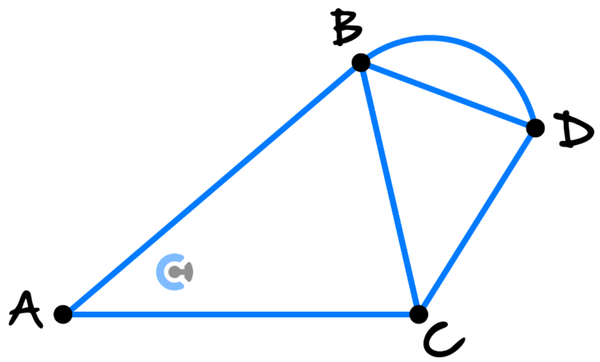
Question 10

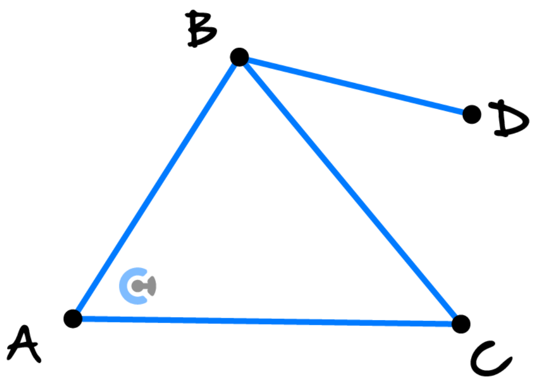
State whether the following graphs are simple graphs or not.

a.



Not simple

b.



Simple.

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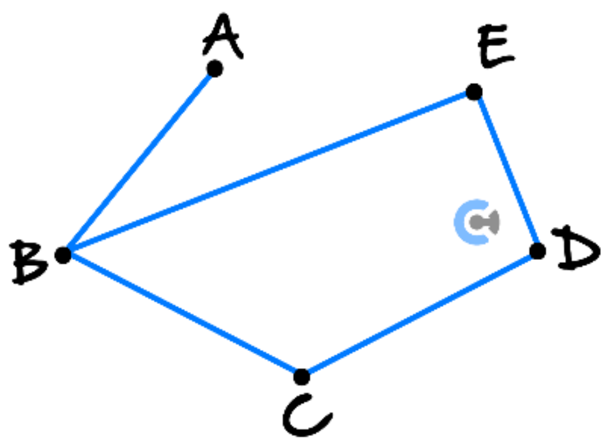
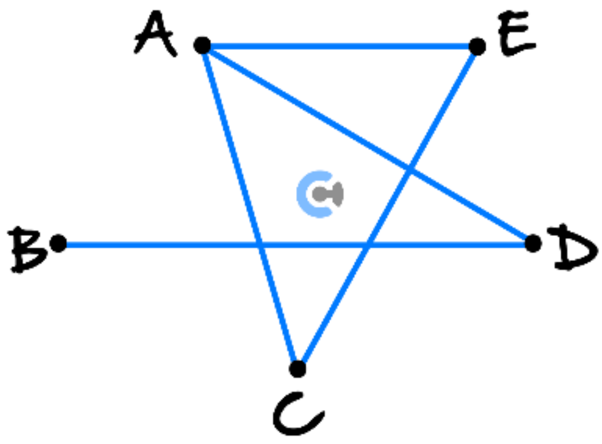
The Complement of a Simple Graph

Simple Graph	The Complement
	

Complement of $G = \overline{G}$

- The complement of a simple graph contains the same set of vertices.
- But it contains opposite set of edges. (Edges the original graph does not have)

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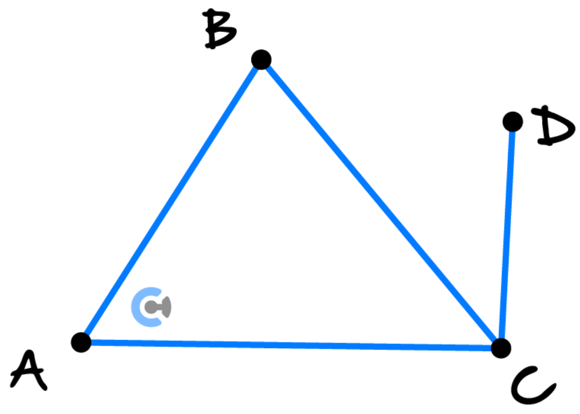
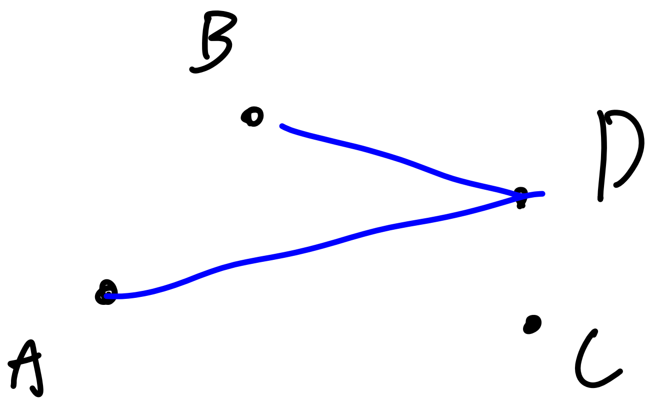
Question 11

Consider a simple graph below.



Sketch the complement of the simple graph above.



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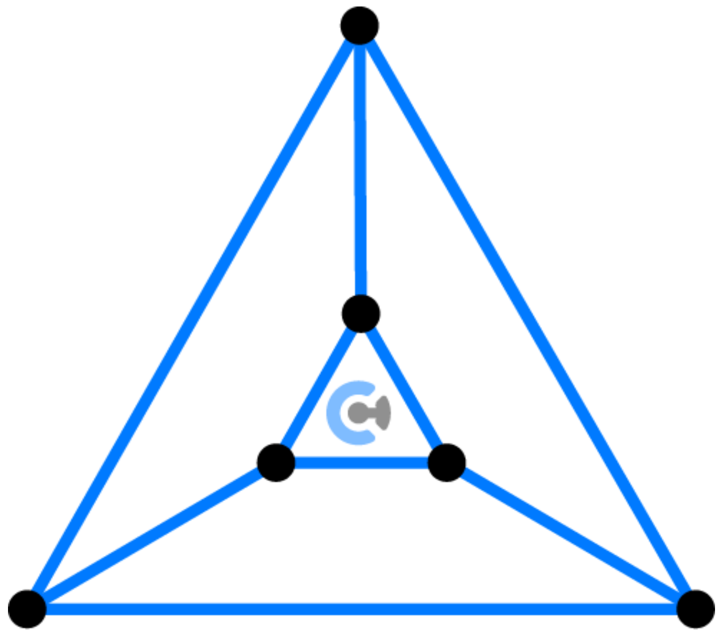
Sub-Section: Regular Graphs

What do we call the graph when all its vertices have the same degree?

Regular Graphs



- Regular graph has all its vertices with the same degree
- If each vertex has a degree r then the graph is "regular of degree r" or "r-regular".

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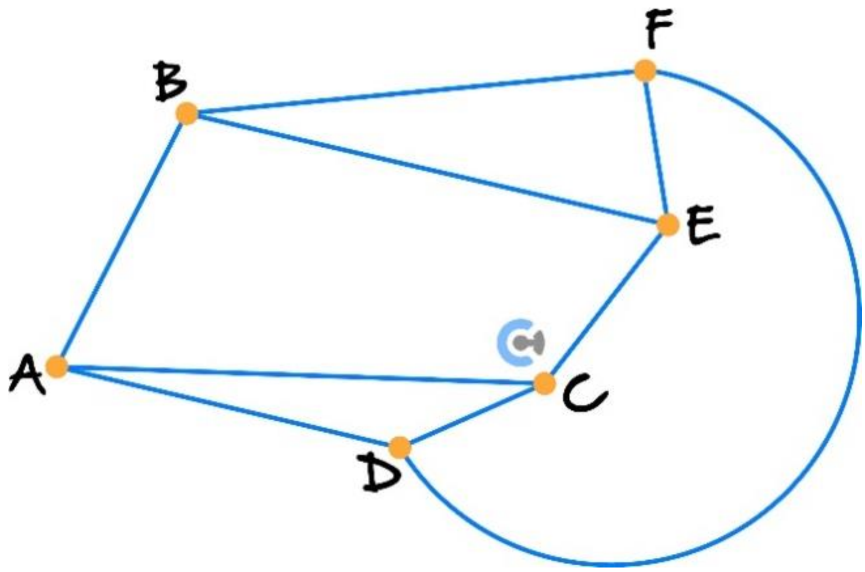
Question 12

Consider the following graph.



a. State the type of graph this is.

3-regular

b. State the number of edges of the graph.

9 edges

c. State the total of all the degrees of all points.

18 degree

d. Hence, what do you notice?

↳ double : total degrees = 2 × edges

Definition

<u>Number of Edges and Degree of all Vertices of a Regular Graph</u>

$$\text{Number of Edges} \times 2 = \text{Total Degree of all Vertices}$$

<u>Discussion:</u> How does that make sense? Think about how many "degrees" each edge generates.

each edge connects to 2 vertices

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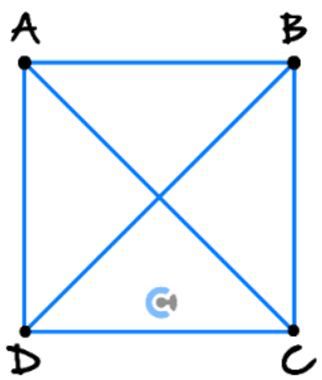
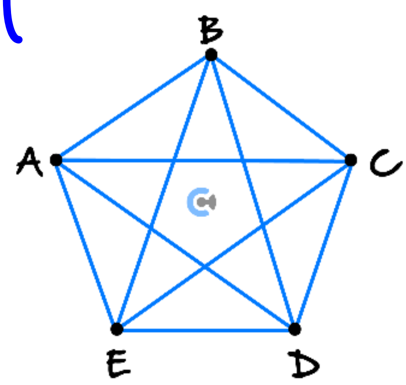
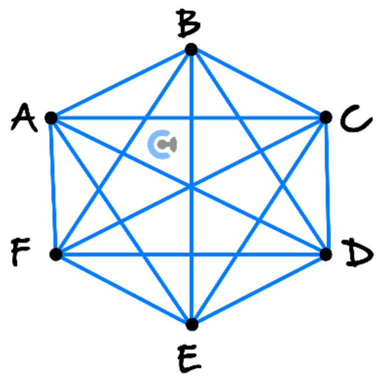
Sub-Section: Complete Graph

Complete Graph (K_n)

degree = $n-1$

K_4	K_5	K_6
		

- A complete graph is a simple graph in which each vertex is connected to every other vertex.
- A complete graph is denoted by K_n, where n is the number of vertices in the graph.
- Complete graph is a type of regular graph.

Discussion: Can a complete graph also be a type of regular graph?

Yes

each vertex same degree

Discussion: What would the simple graph and its complement add up / overlap to?

$G + \overline{G} = K$

<u>Exploration</u>: Number of edges in a complete graph

- Consider n many people handshaking each other.
- It is known that one person needs to handshake everyone else.
- How many handshakes will it take?

Number of Handshakes = $(n-1)!$

- Recalling the arithmetic sum formula: $S_n = \frac{n}{2}(a+l)$, find the total number of handshakes.

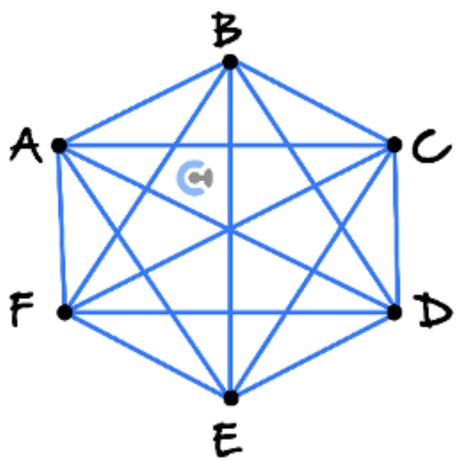
Number of Handshakes = $\frac{n-1}{2}(1+(n-1)) = \frac{n(n-1)}{2}$

- Similarly Consider K_n: Complete graph with n many vertices.
- How many edges (or handshakes) would the graph have in total?

Number of Edges for Complete Graph = $\frac{n\cdot(n-1)}{2}$

<u>Number of Edges for Complete Graph</u>

Definition

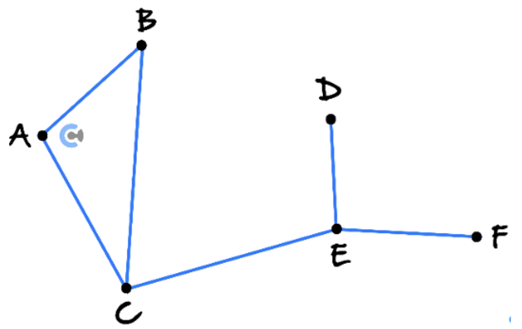
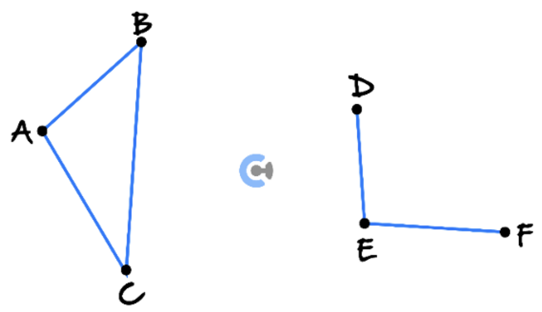


- For K_n:

Number of Edges for Complete Graph $= \frac{n(n-1)}{2}$

Sub-Section: Connected Graphs

Connected Graphs

Connected Graph	Disconnected Graph
	

- A connected graph is a graph where it is possible to reach all vertices by moving along edges.
- A graph which is not connected is called a disconnected graph.

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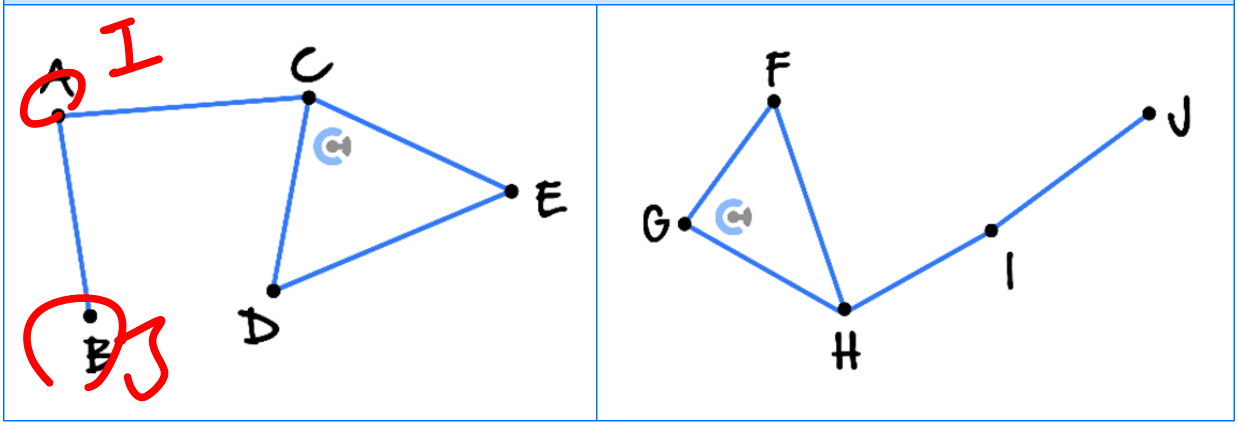
Section D: Isomorphism and Subgraphs

Sub-Section: Isomorphism

Isomorphism

Isomorphic Graph



- Two graphs are **isomorphic** if their vertices and edges differ only by the way in which they are names.
- **Checklist for determining isomorphism:**
  - Are the number of vertices the same in each graph?
  - Are the number of edges the same in each graph?
  - Check that the degrees of each vertex match for both graphs.
  - Label each vertex on both graphs and check if there is a correspondence between the vertices.

we can draw an adjacency matrix

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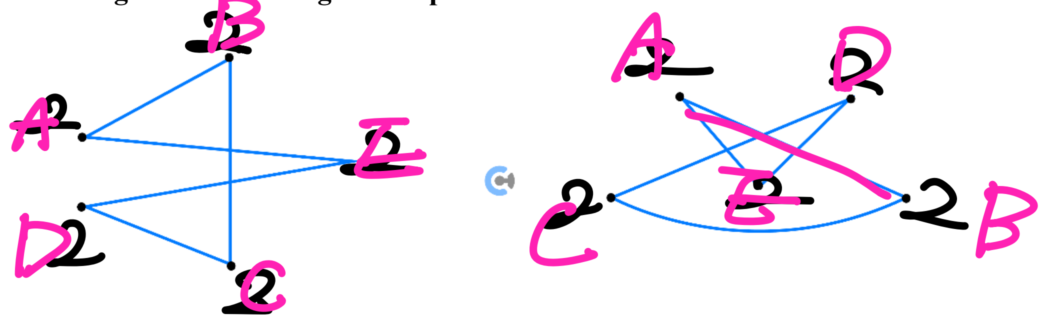
Question 13 Walkthrough – Determining Isomorphism.



Checklist for determining isomorphism:

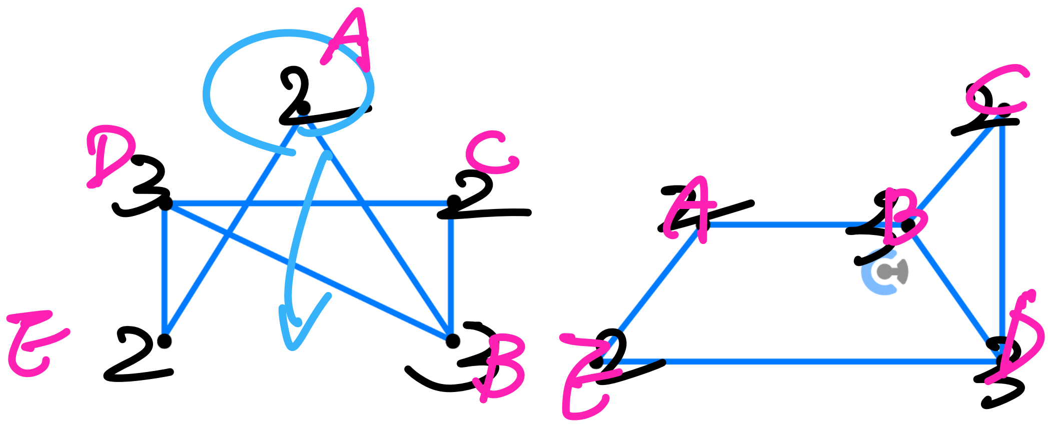
1. Are the number of vertices the same in each graph? ✓
2. Are the number of edges the same in each graph? ✓
3. Check that the degrees of each vertex match for both graphs. ✓
4. Label each vertex on both graphs and check if there is a correspondence between the vertices.

Yes — isomorphic

Question 14

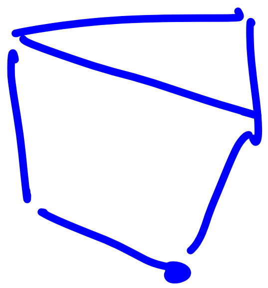
For each of the following pairs of graphs, determine whether the graphs are isomorphic.

a.

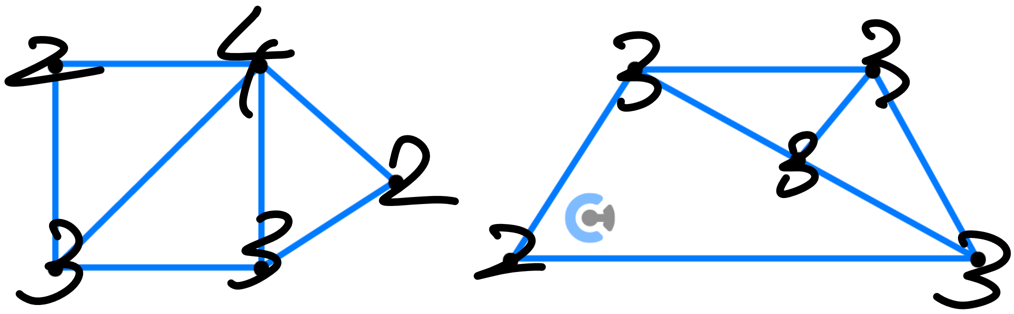


5 vertices

6 edges



b.



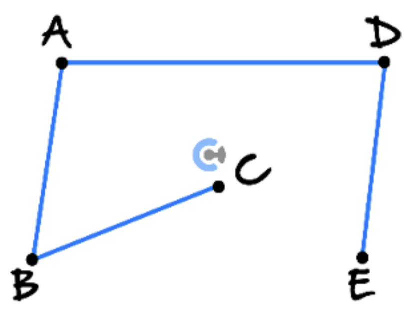
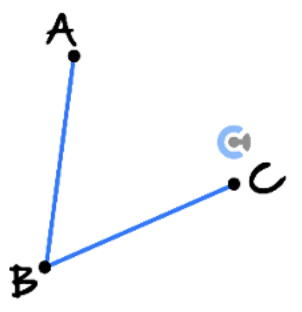
vertices ✓ (5)

edges ✓ (7)

degrees ✗ → not isomorphic

Sub-Section: Subgraphs

Subgraphs

Original Graph	Subgraph
	

- A subgraphs is a graph whose vertices and edges are all contained within the original graph.
- A subgraph can be created by removing edges and vertices from the original graph.

NOTE: There are multiple possible subgraphs.

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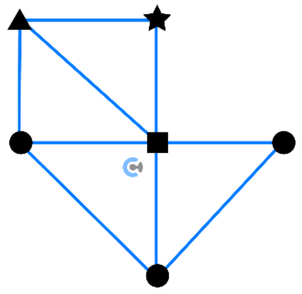
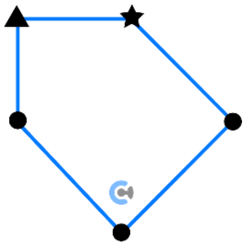
Question 15

Consider this network graph.

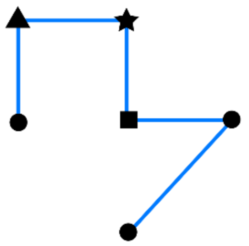


A subgraph of this graph is:

A.



B.



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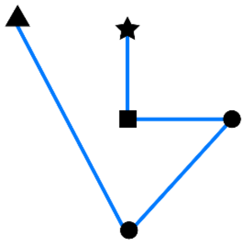

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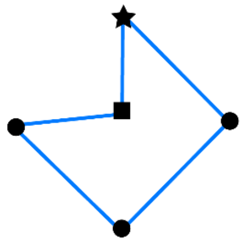

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