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# VCE Specialist Mathematics ½
## Graph Theory I [5.3]
## Workbook

## Outline:



## Learning Objectives:

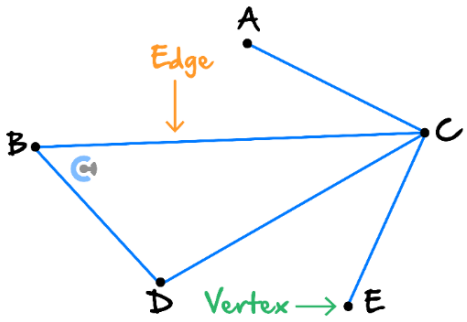
- ☐ **SM12 [5.3.1]** - Graph Theory Fundamentals - Vertices, Edges, Degree, Adjacency Lists, and Matrices
- ☐ **SM12 [5.3.2]** - Types of Graphs
- ☐ **SM12 [5.3.3]** - Isomorphisms and Subgraphs

## Section A: Graphs

### Sub-Section: Vertices and Edges

*What does the graph have?*

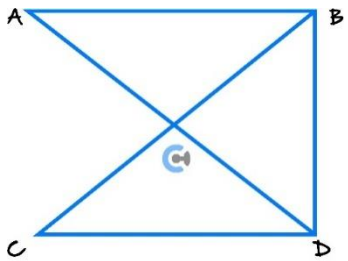
**Vertices and Edges**




- A graph consists of a set of points called _____ and a set of unordered pairs of vertices, called _____.

**Question 1 Walkthrough.**

Consider a graph below.

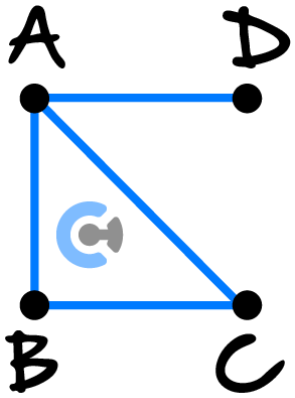



Write down the vertex set and edge set of the given graph.

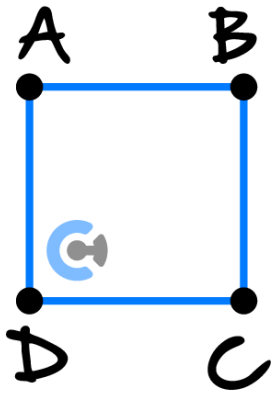
**Question 2**

Write the vertex sets and edge sets for the graphs corresponding to the following pictures.

**a.**




**b.**



**Question 3**

Draw pictures of 2 graphs with the following vertex and edge sets.

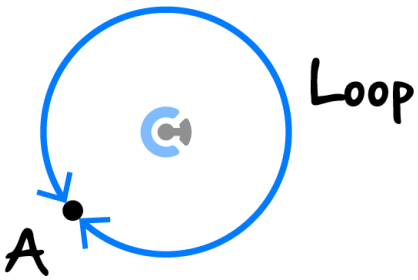
**a.** Vertex set: $\{A, B, C, D\}$
Edge set: $\{AB, BC, BD\}$

**b.** Vertex set: $\{A, B, C, D, E\}$
Edge set: $\{AB, BC, CA, DE\}$

*What if an edge connects A to A?*

Definition

**Loops**



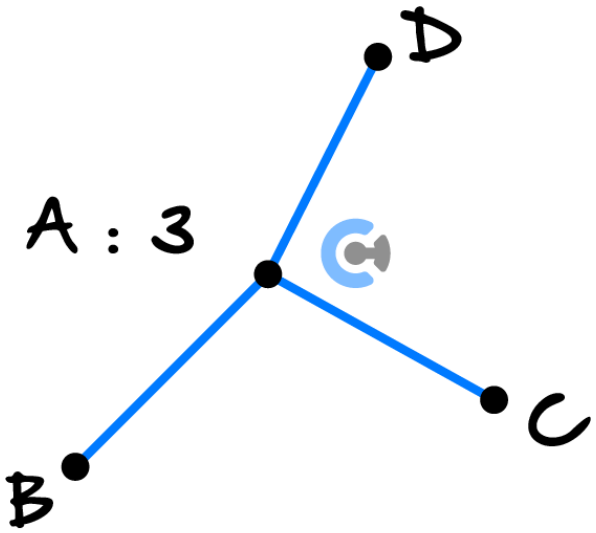

*AA*

- Loop is an edge which connects to the same vertex.

## Sub-Section: Degree of a Vertex

*Let's consider the degree of a vertex!*

### Degree of a Vertex

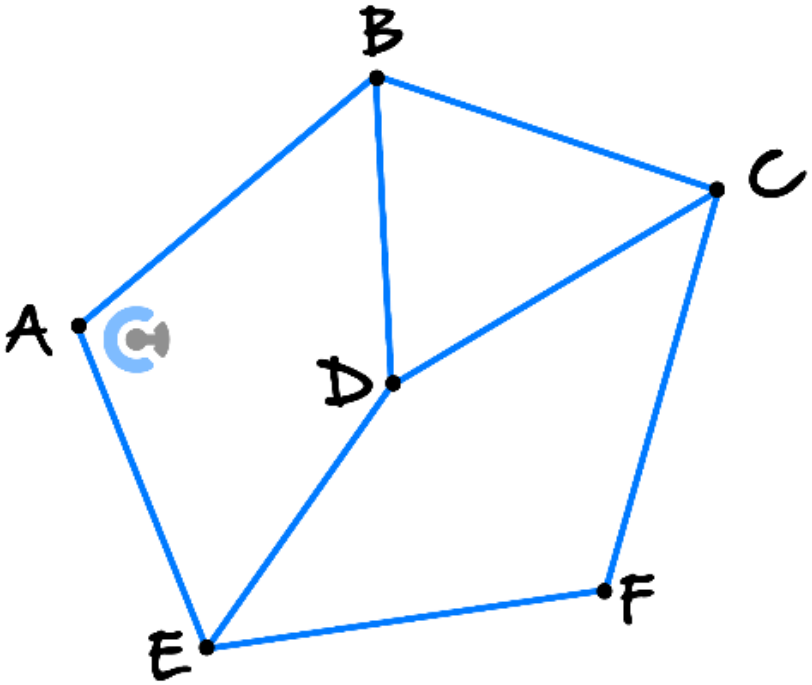



- Degree of a vertex is the _____ connected to the vertex.

### Space for Personal Notes

**Question 4 Walkthrough.**

Fill in the following information for the graph below.

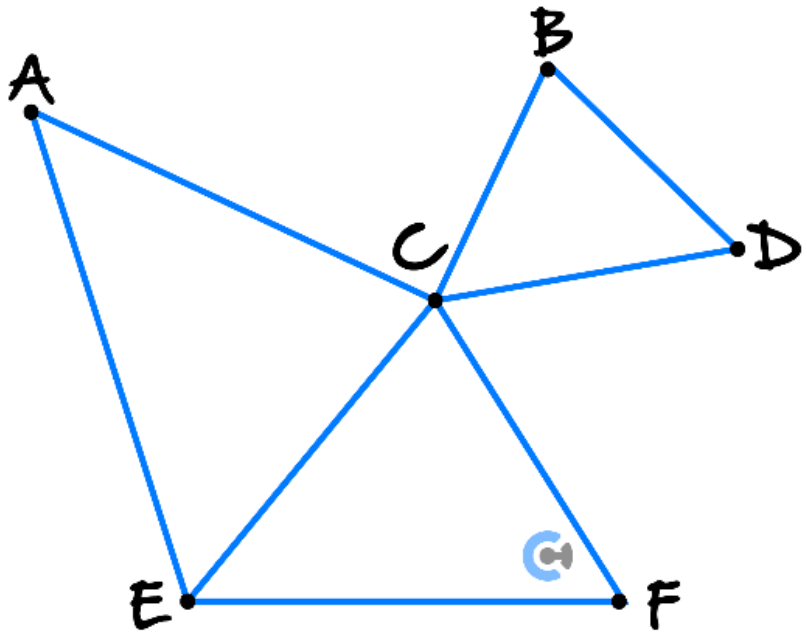



| Vertex | Degree of Vertex |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |
| **Number of Edges: _____** | **Sum of Degrees: _____** |

**Question 5**

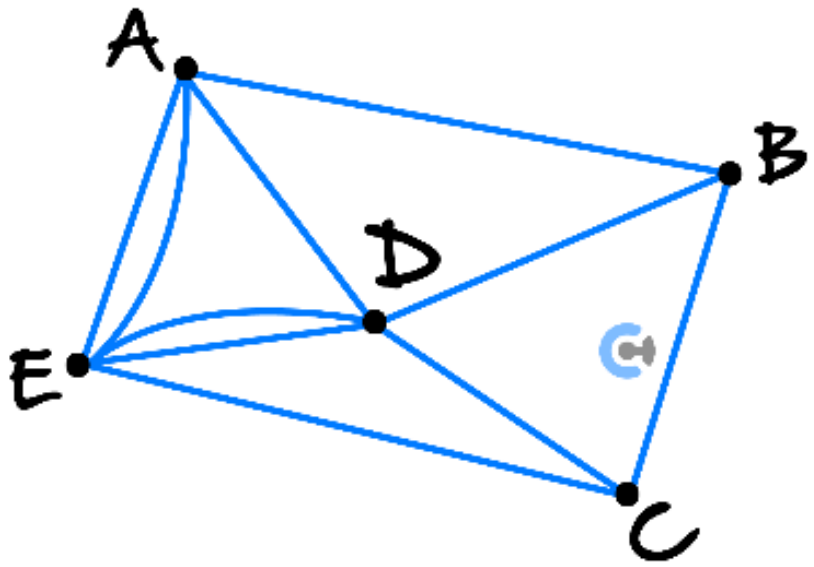
Fill in the following information for the graphs below.

**a.**




| Vertex | Degree of Vertex |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |
| **Number of Edges: _____** | **Sum of Degrees: _____** |

**b.**




| Vertex | Degree of Vertex |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |
| **Number of Edges: _____** | **Sum of Degrees: _____** |

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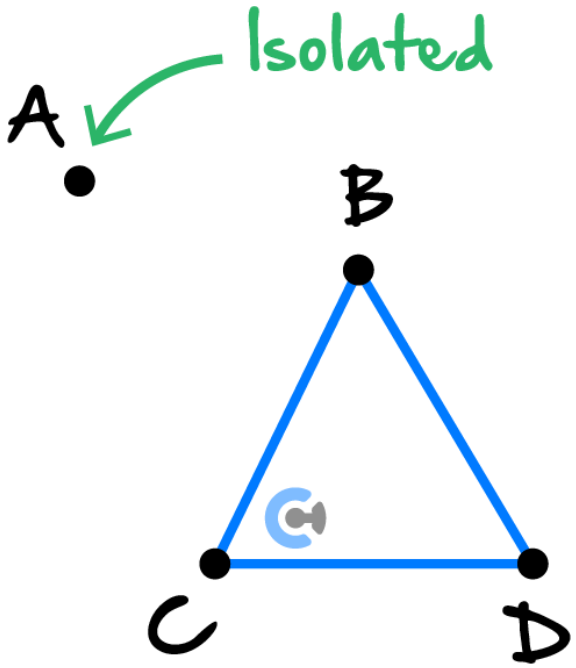
*What about if a vertex is not connected to any other point (including itself)?*

Definition

**Isolated Vertex**




- Isolated vertex has no edges connected to it.
- Its degree is equal to _____.

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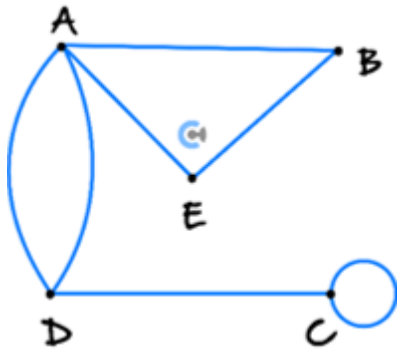
## Section B: Adjacency List and Matrix

### Sub-Section: Adjacency List

**Discussion:** What do we call two points that are connected by an edge?

**Adjacency Lists**

| Graph | Adjacency List |
|---|---|
|   | $A \to (B, D, D, E)$<br>$B \to (A, E)$<br>$C \to (C, D)$<br>$D \to (A, A, C)$<br>$E \to (A, B)$ |

- Adjacency list contains all the vertices a given vertex is connected to.
- If the point is connected multiple times, we write the vertex_____.
- If a point is _____ with itself, we write the vertex to be adjacent to itself.

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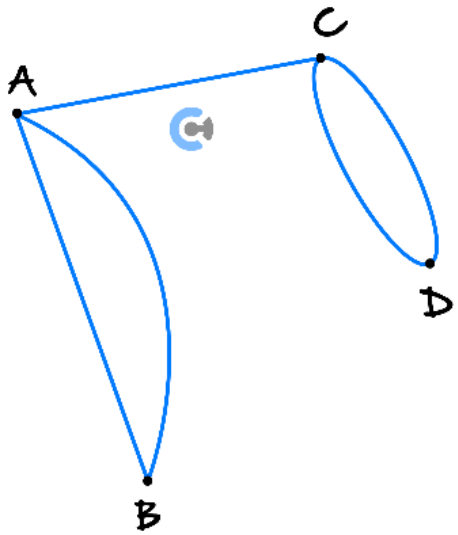
**Question 6**

Create an adjacency list that describes the following graph.




Graph

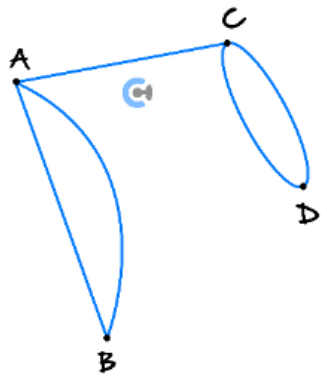
Space for Personal Notes

## Sub-Section: Adjacency Matrix

**Discussion:** Is there a way to convert the adjacency list into a matrix form?

**Question 7 Walkthrough.**

Consider the previous question with its graph and adjacency list.




Graph

$A \to (B, B, C)$

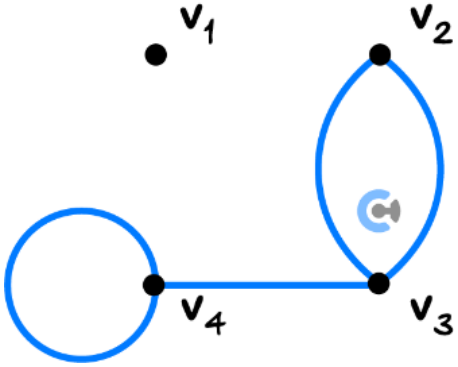
$B \to (A, A)$

$C \to (A, D, D)$

$D \to (C, C)$

Convert the given adjacency list into an adjacency matrix.

## Adjacency Matrix




$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

- A matrix that represents the vertices and edges that connect the vertices of a graph.

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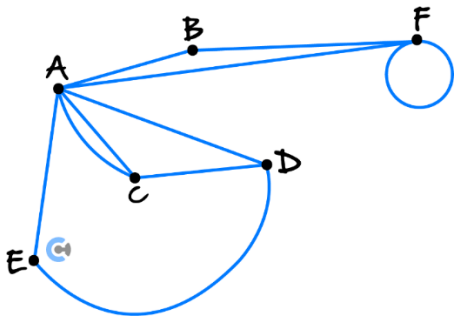
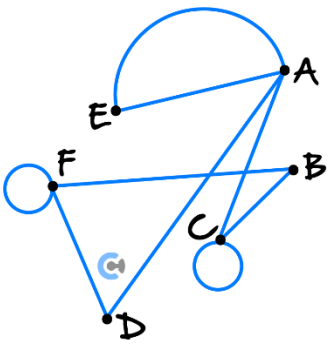
**Question 8**

Construct the adjacency matrix for the given graphs.

**a.**




**b.**



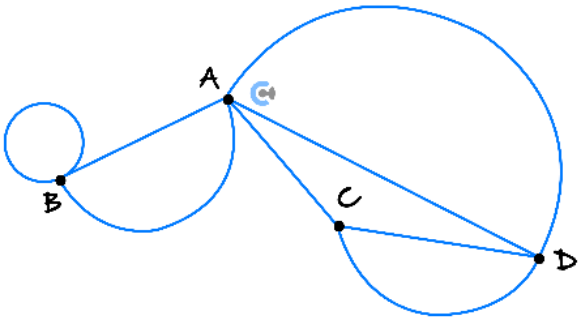
**Discussion:** Could the adjacency matrix be a non-square matrix? (Same number of rows and columns)

**Discussion:** What does it mean when there is a non-zero value in the diagonal element of the adjacent matrix?

## Properties of Adjacency Matrices

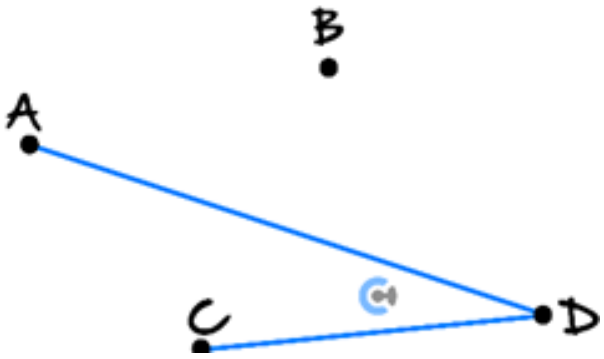
- Always a square matrix.
- Any _____ value in the _____ will indicate the existence of a _____.

$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$




- A row consisting of all zeros indicates an _____.

$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$



**Question 9**

Draw graphs to represent the following adjacency matrices.

**a.** $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

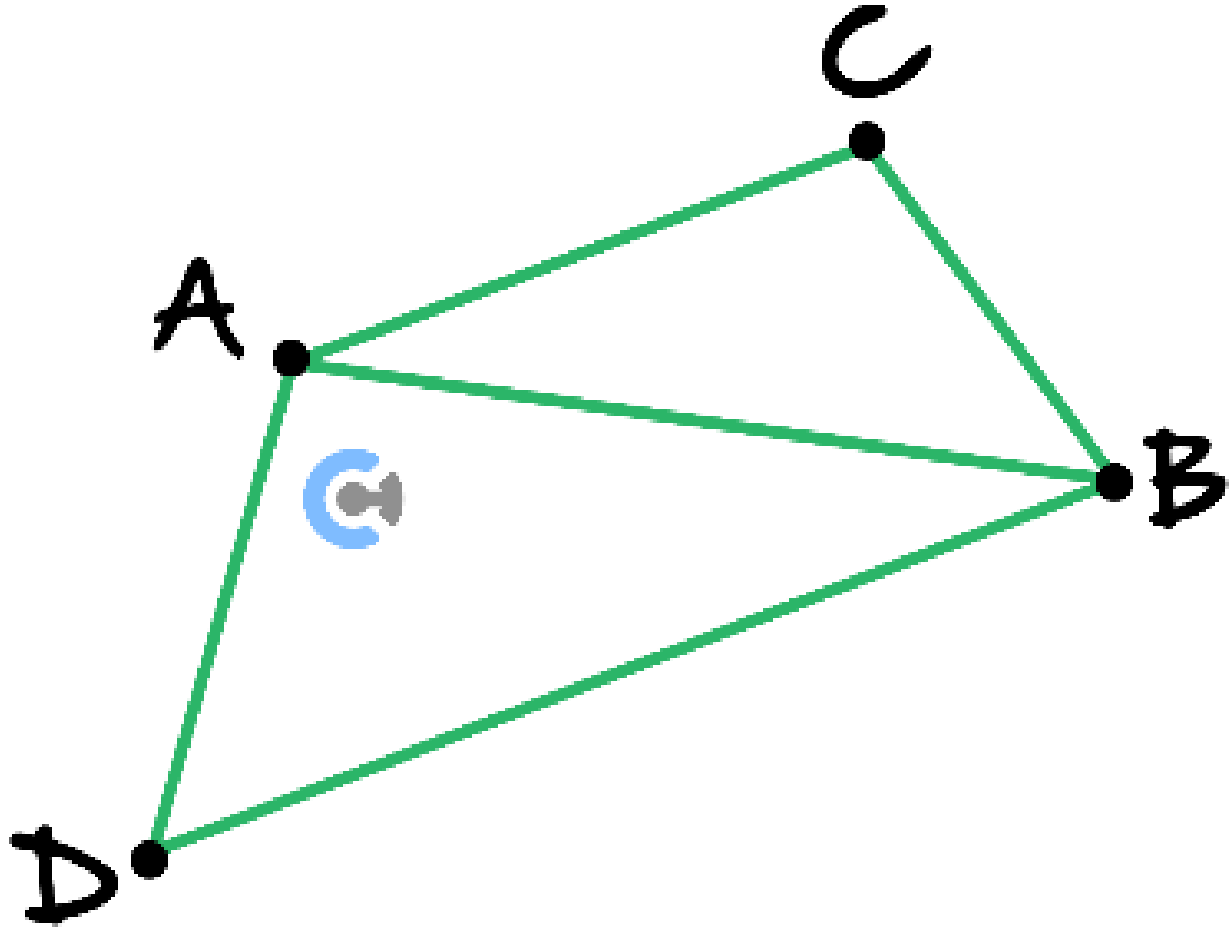
**b.** $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

## Section C: Types of Graphs

### Sub-Section: Simple Graph

*This one should be simple.* ☺

**Simple Graph**




- A simple graph is one in which pairs of vertices are connected by _____.

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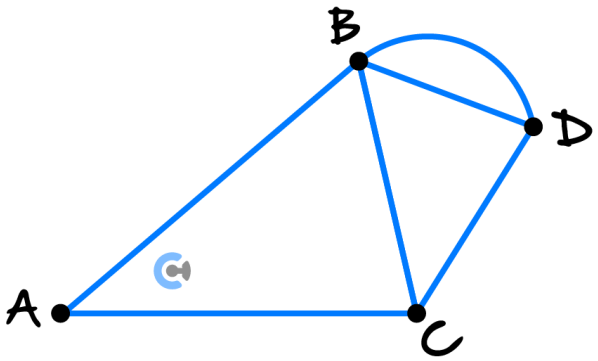
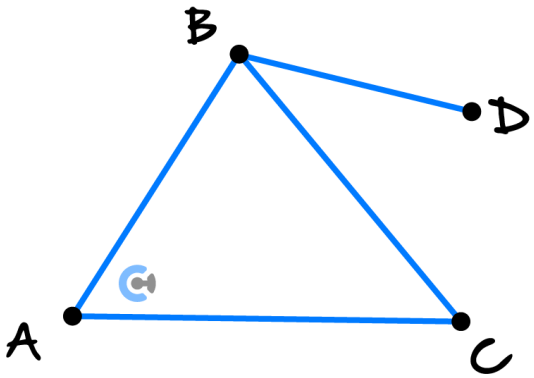
**Question 10**

State whether the following graphs are simple graphs or not.

**a.**




**b.**

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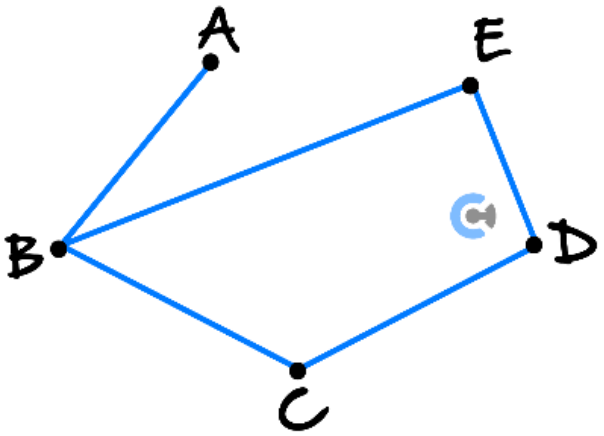
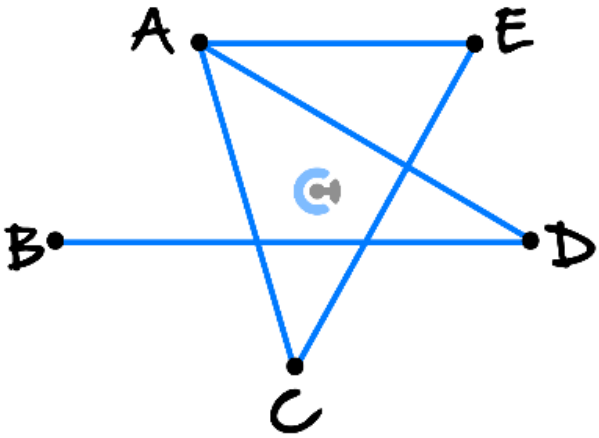
## The Complement of a Simple Graph

| Simple Graph | The Complement |
|---|---|
|  |  |



### Complement of $G = \overline{G}$

- The complement of a simple graph contains the _____ set of vertices.
- But it contains _____ set of edges. (Edges the original graph does not have)

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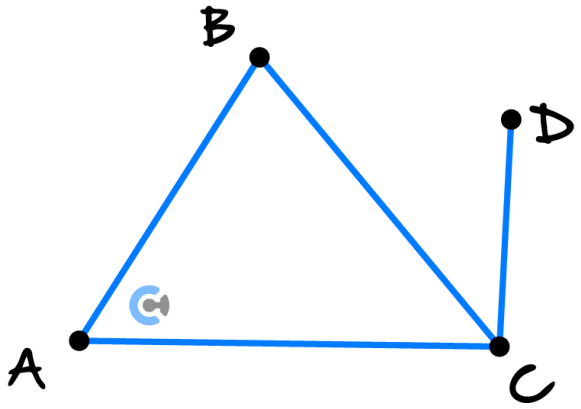
**Question 11**

Consider a simple graph below.




Sketch the complement of the simple graph above.

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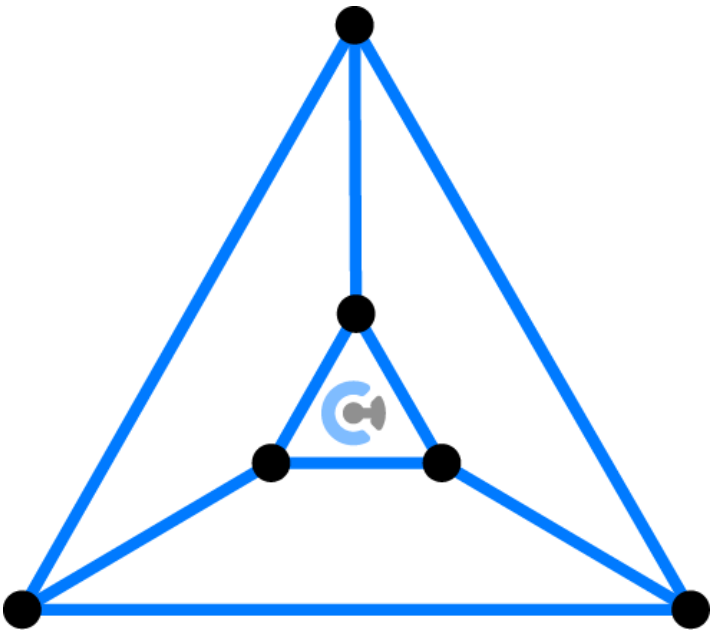
## Sub-Section: Regular Graphs

*What do we call the graph when all its vertices have the same degree?*

### Regular Graphs



- Regular graph has all its vertices with the _____.
- If each vertex has a degree $r$ then the graph is "regular of degree $r$" or "_____".

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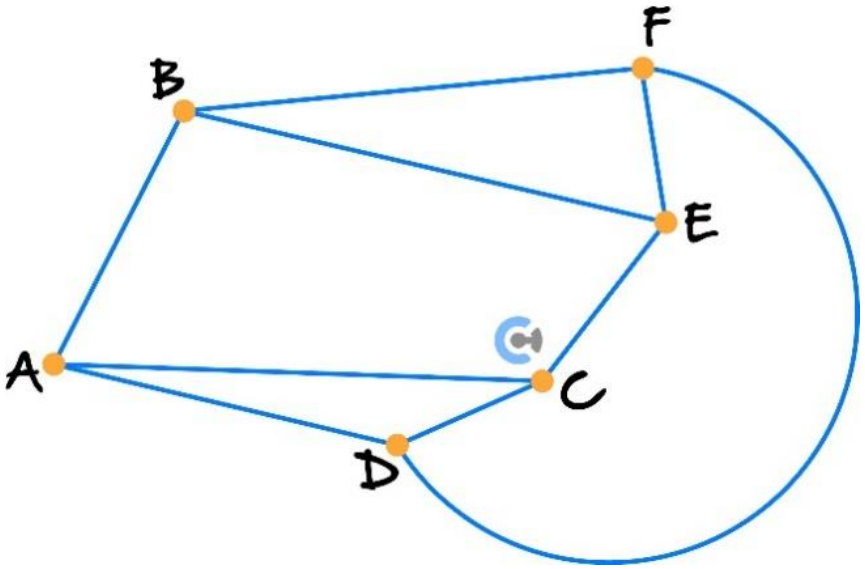
**Question 12**

Consider the following graph.




**a.** State the type of graph this is.

**b.** State the number of edges of the graph.

**c.** State the total of all the degrees of all points.

**d.** Hence, what do you notice?

Number of Edges and Degree of all Vertices of a Regular Graph

$$\text{Number of Edges} \times 2 = \text{Total Degree of all Vertices}$$

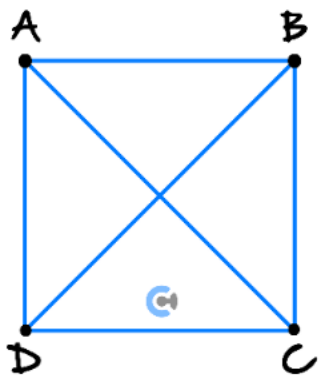
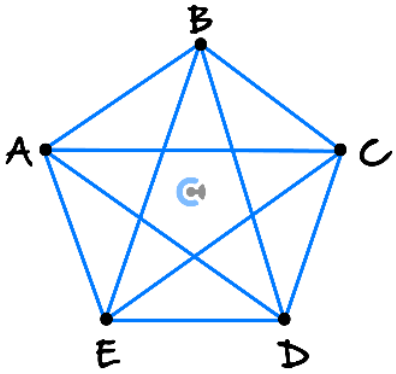
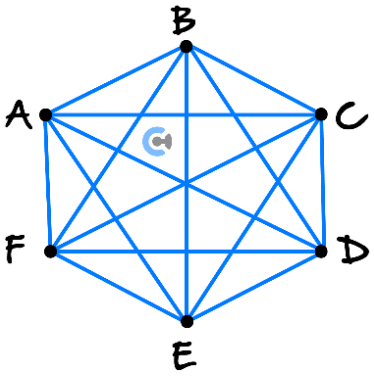
Discussion: How does that make sense? Think about how many "degrees" each edge generates.



Space for Personal Notes

## Sub-Section: Complete Graph

### Complete Graph $(K_n)$

| $K_4$ | $K_5$ | $K_6$ |
|---|---|---|
|  |  |  |







- A complete graph is a simple graph in which each vertex is connected to _____.
- A complete graph is denoted by $K_n$, where $n$ is the number of vertices in the graph.
- Complete graph is a type of regular graph.

**Discussion:** Can a complete graph also be a type of regular graph?

**Discussion:** What would the simple graph and its complement add up / overlap to?

**Exploration: Number of edges in a complete graph**

- Consider $n$ many people handshaking each other.
- It is known that one person needs to handshake everyone else.
- How many handshakes will it take?

Number of Handshakes =

- Recalling the arithmetic sum formula: $S_n = \frac{n}{2}(a + l)$, find the total number of handshakes.

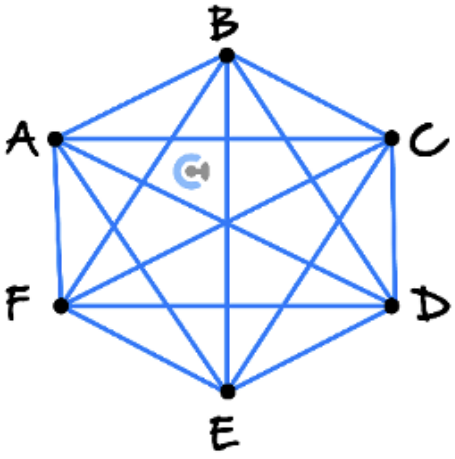
Number of Handshakes =

- Similarly Consider $K_n$: Complete graph with $n$ many vertices.
- How many edges (or handshakes) would the graph have in total?

Number of Edges for Complete Graph =

Definition

**Number of Edges for Complete Graph**

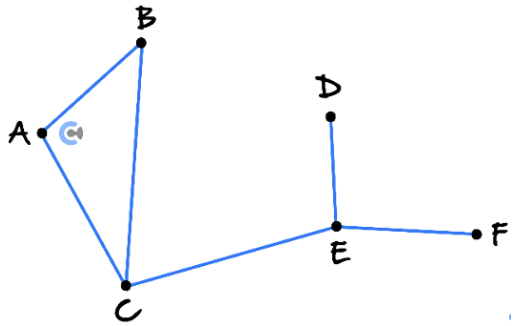
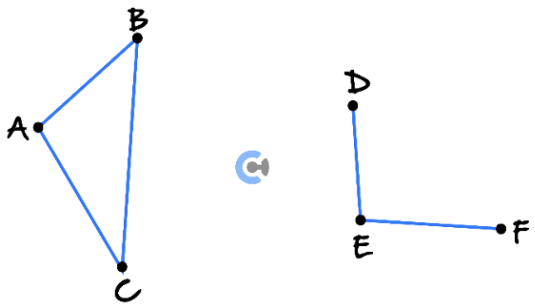



- For $K_n$:

Number of Edges for Complete Graph = $\frac{n(n-1)}{2}$

# Sub-Section: Connected Graphs

## Connected Graphs

| Connected Graph | Disconnected Graph |
| --- | --- |
|  |  |



- A _____ graph is a graph where it is possible to reach all vertices by moving along edges.
- A graph which is not connected is called a _____ graph.

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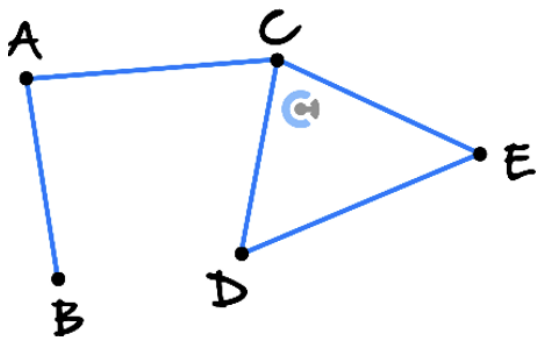
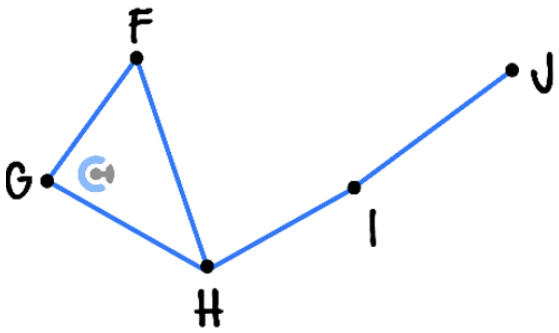
## Section D: Isomorphism and Subgraphs

### Sub-Section: Isomorphism

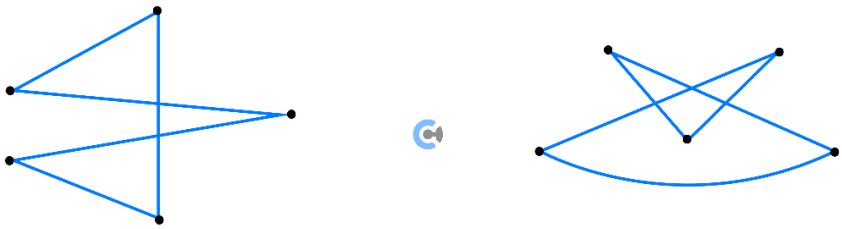
#### Isomorphism

| Isomorphic Graph | |
|---|---|
|   |   |

- Two graphs are **isomorphic** if their vertices and edges differ only by the way in which they are _____.
- **Checklist for determining isomorphism:**
  - Are the number of vertices the same in each graph?
  - Are the number of edges the same in each graph?
  - Check that the degrees of each vertex match for both graphs.
  - Label each vertex on both graphs and check if there is a correspondence between the vertices.

**Space for Personal Notes**

**Question 13 Walkthrough – Determining Isomorphism.**



**Checklist for determining isomorphism:**

**1.** Are the number of vertices the same in each graph?

**2.** Are the number of edges the same in each graph?

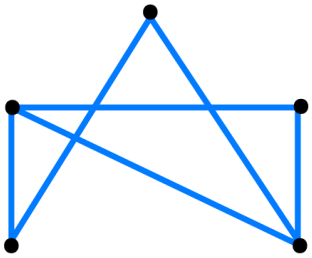
**3.** Check that the degrees of each vertex match for both graphs.

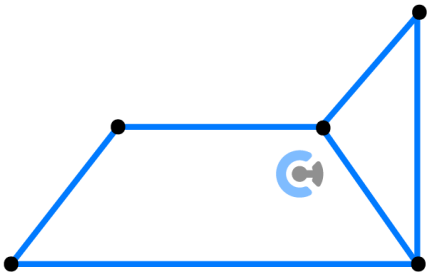
**4.** Label each vertex on both graphs and check if there is a correspondence between the vertices.

**Question 14**

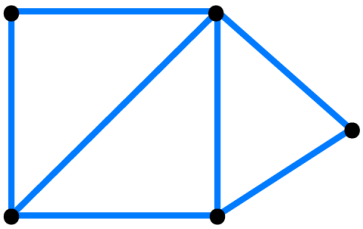
For each of the following pairs of graphs, determine whether the graphs are isomorphic.

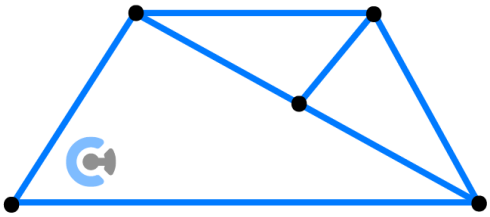
**a.**





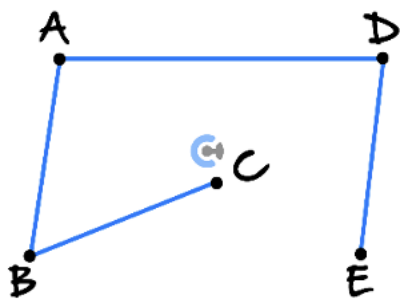
**b.**





## Sub-Section: Subgraphs

### Subgraphs

| Original Graph | Subgraph |
|---|---|
|   |   |

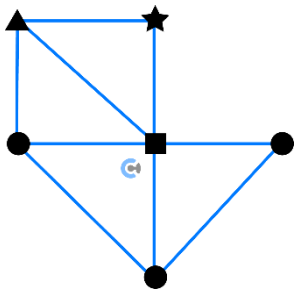
- A _____ is a graph whose vertices and edges are all _____ within the original graph.
- A subgraph can be created by _____ edges and vertices from the original graph.

**NOTE:** There are multiple possible subgraphs.

**Space for Personal Notes**

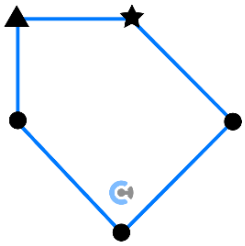
**Question 15**

Consider this network graph.

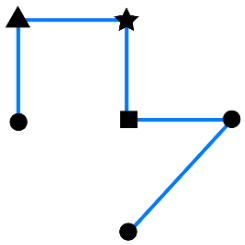


A subgraph of this graph is:

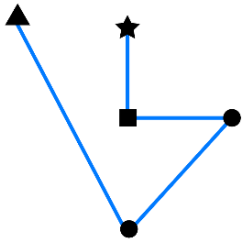
A.



B.

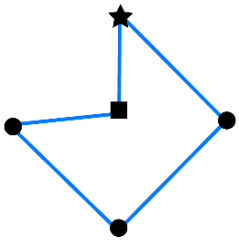


C.



D.

E.



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