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VCE Specialist Mathematics ½

Graph Theory I [5.3]

Homework

Admin Info & Homework Outline:

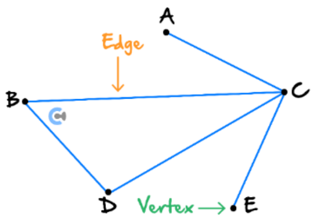
Student Name	
Questions You Need Help For	
Recap	Pg 2 - Pg 4
Basic (Compulsory) Questions	Pg 5 - Pg 13
Problem Solving Questions	Pg 14 - Pg 22
Supplementary Questions	Pg 23 - Pg 33

Section A: Recap

Cheat Sheet

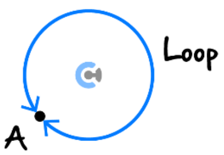
[5.3.1] - Graph theory fundamentals - vertices, edges, degree, adjacency lists and matrices

- Vertices and Edges



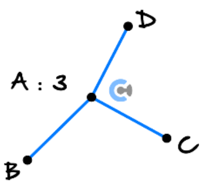
  - A graph consists of a set of points called vertices and set of unordered pairs of vertices, called edges.

- Loops



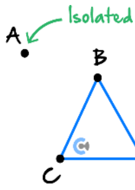
  - Loop is an edge which connects to the same vertex.

- Degree of a Vertex



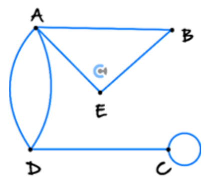
  - Degree of a vertex is the number of edges connected to the vertex.

- Isolated Vertex



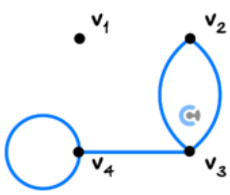
  - Isolated vertex has no edges connected to it.
  - Its degree is equal to zero.

- Adjacency Lists

Graph	Adjacency List
	$A \rightarrow (B, D, D, E)$ $B \rightarrow (A, E)$ $C \rightarrow (C, D)$ $D \rightarrow (A, A, C)$ $E \rightarrow (A, B)$

  - Adjacency list contains all the vertices a given vertex is connected to.
  - If the point is connected multiple times, we write the vertex multiple times.
  - If a point is looped with itself, we write the vertex to be adjacent to itself.

- Adjacency Matrix



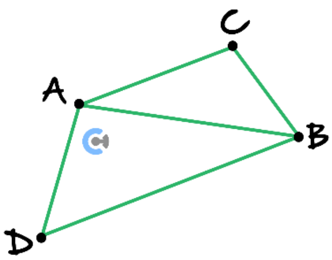
$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

  - A matrix that represents the vertices and edges that connect the vertices of a graph.

Cheat Sheet

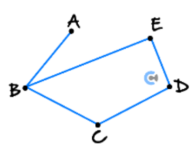
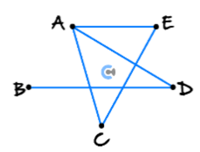
[5.3.2] - Types of graphs

- **Simple Graph**



- A simple graph is one in which pairs of vertices are connected by one edge at most.

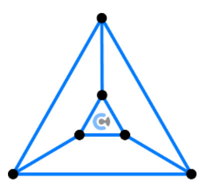
- **The Complement of a Simple Graph**

Simple Graph	The Complement
	

$$\textit{Complement of } G = \overline{G}$$

- The complement of a simple graph contains the same set of vertices.
- But it contains complement set of edges. (Edges the original graph does not have.)

- **Regular Graphs**



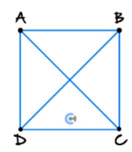
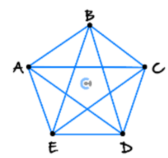
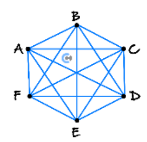
- Regular graph has all its vertices with the same degree.
- If each vertex has a degree r then the graph is "regular of degree r" or "r-regular".

- **Number of Edges and Degree of All Vertices of a Regular Graph**

$$\textit{Number of Edges} \times 2 = \textit{Total Degree of all Vertices}$$

- The above result is commonly known as the Handshake Lemma.

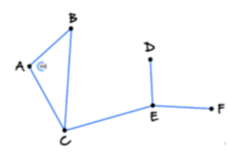
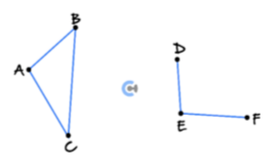
- **Complete Graph** (K_n)

K_4	K_5	K_6
		

- A complete graph is a simple graph in which each vertex is connected to every other vertex.
- A complete graph is denoted by K_n, where n is the number of vertices in the graph.
- Complete graph is a type of regular graph.
- For K_n:

$$\textit{Number of Edges for Complete Graph} = \frac{n(n-1)}{2}$$

- **Connected Graphs**

Connected Graph	Disconnected Graph
	

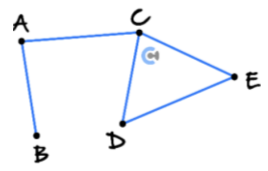
- A connected graph is a graph where it is possible to reach all vertices by moving along edges.
- A graph which is not connected is called a disconnected graph.

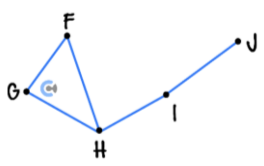
Cheat Sheet

[5.3.3] – Isomorphisms and subgraphs.

- Isomorphism

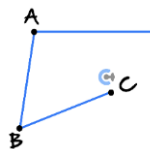
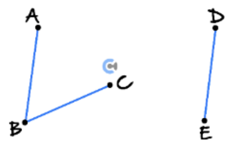
Isomorphic Graph





- Two graphs are **isomorphic** if their vertices and edges differ only by the way in which they are named.
- Checklist for Determining Isomorphism
  - Are the number of vertices the same in each graph?
  - Are the number of edges the same in each graph?
  - Check that the degrees of each vertex match for both graphs.
  - Label each vertex on both graphs and check if there is a correspondence between the vertices.

- Subgraphs

Original Graph	Subgraph
	

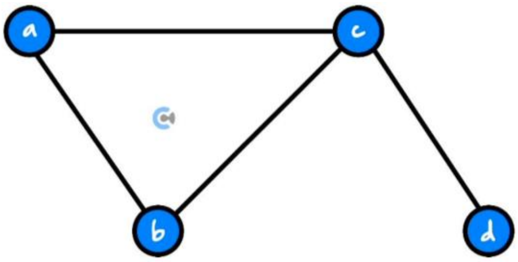
- A subgraph is a graph whose vertices and edges are all contained within the original graph.
- A subgraph can be created by removing edges and vertices from the original graph.

Section B: Basic (Compulsory) Questions

Sub-Section [5.3.1]: Graph Theory Fundamentals – Vertices, Edges, Degree, Adjacency Lists and Matrices

Question 1

Write down the vertex set and edge set of the given graph.



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Question 2

Draw pictures of 2 graphs with the following vertex and edge sets.

a. Vertex set: $\{A, B, C, D\}$

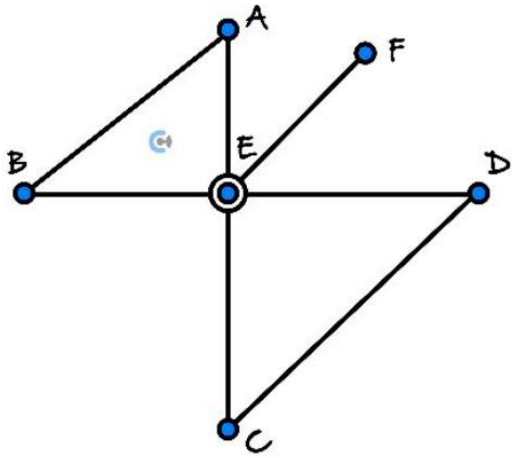
Edge set: $\{AB, AC, AD, BD\}$

b. Vertex set: $\{A, B, C, D, E\}$

Edge set: $\{AB, BC, DE\}$

Question 3

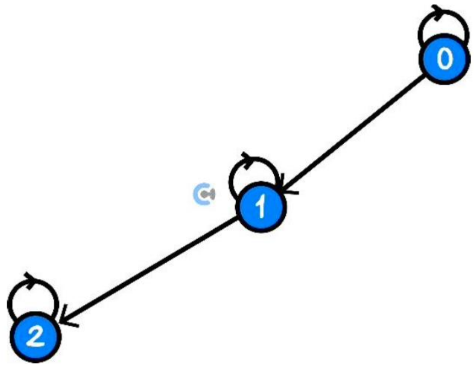
Fill in the following information for the graphs below.



Vertex	**Degree of Vertex**
Number of Edges:	Sum of Degrees:

Question 4

Create an adjacency list that describes the following graph.



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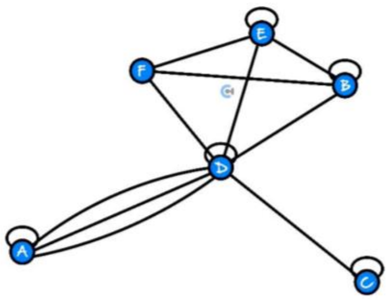
Question 5

a. Construct the adjacency matrix for the given graphs.



b. Draw graphs to represent the following adjacency matrices.

Adjacency Matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

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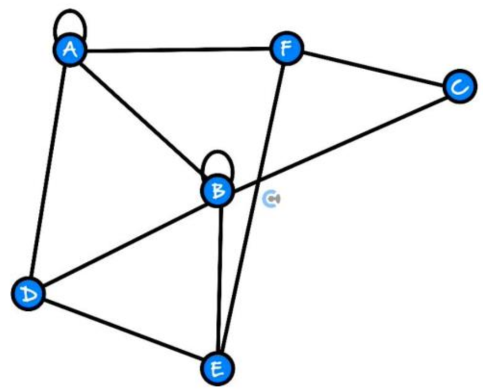
Sub-Section [5.3.2]: Types of Graphs

Question 6

State whether the following graphs are simple graphs or not.



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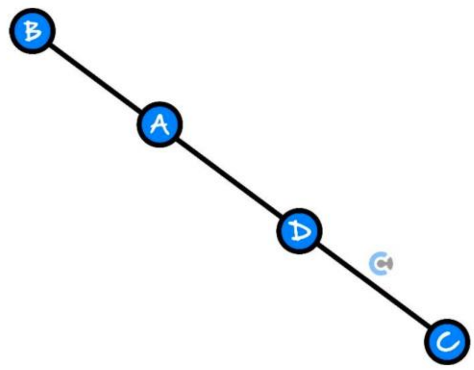

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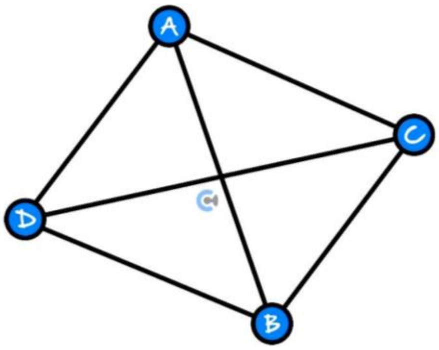
Question 7

State whether the following graphs are complete graphs or not.



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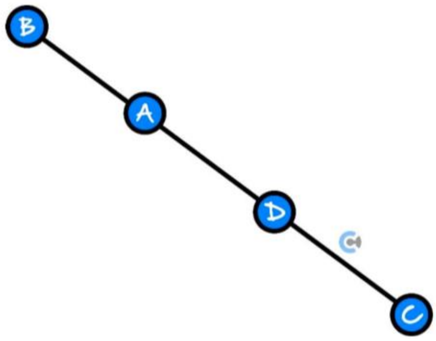

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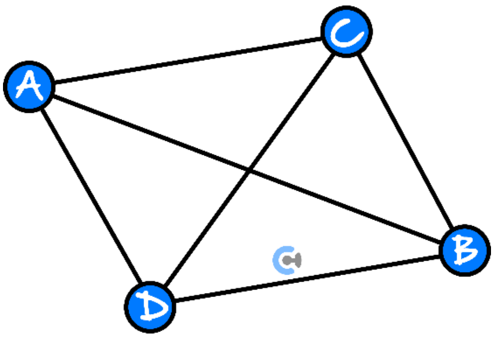
Question 8

State whether the following graphs are regular graphs or not.



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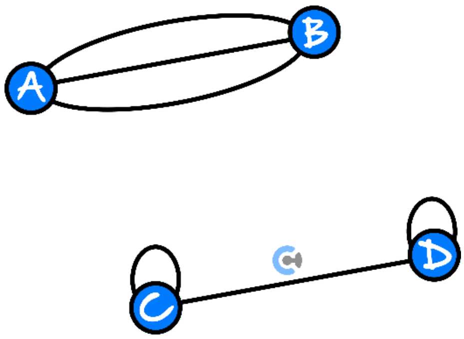

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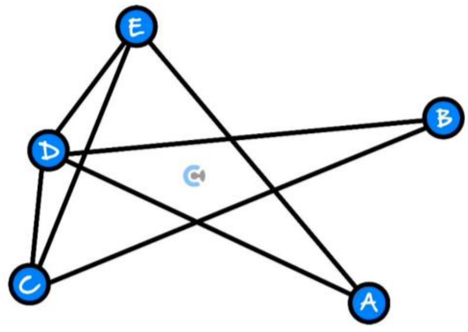
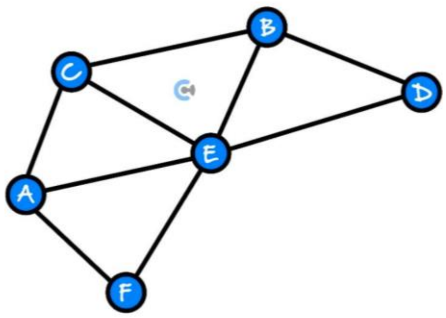
Question 9

Consider simple graphs below. Sketch the complement of each simple graph.





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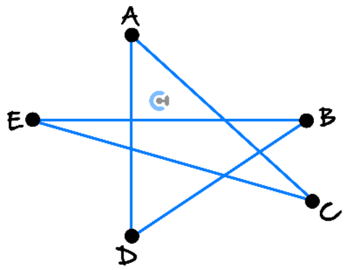
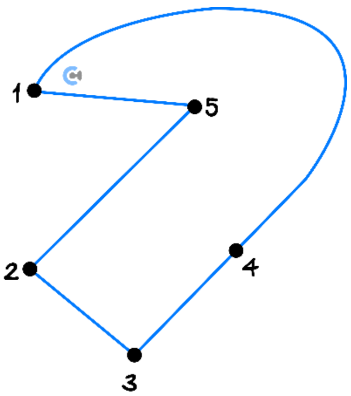
Sub-Section [5.3.3]: Isomorphisms and Subgraphs

Question 10

State whether the following graphs are isomorphic or not.

a.





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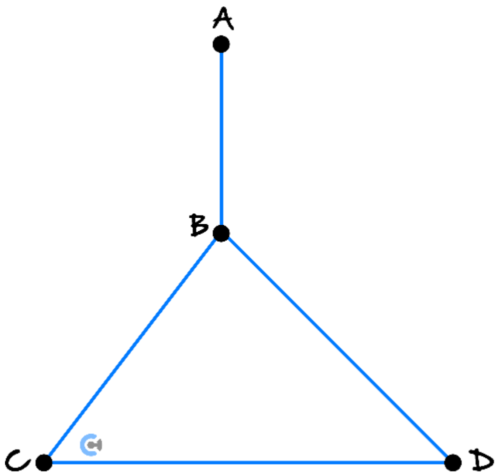
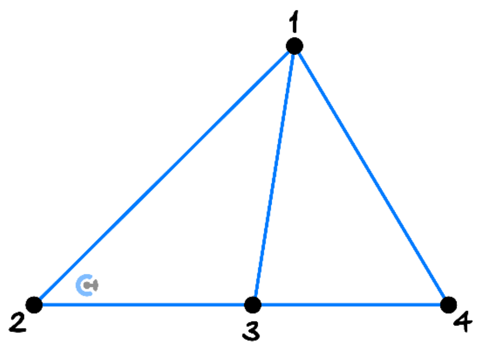
b.





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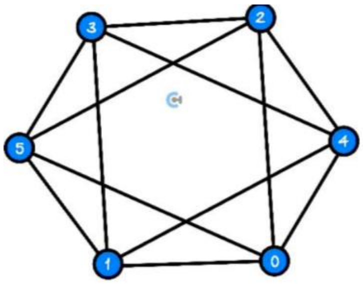
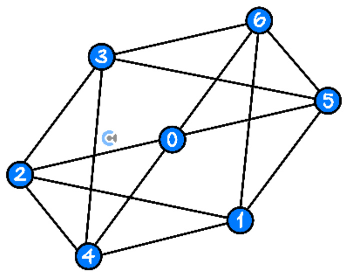
Question 11

Consider this network graph below.

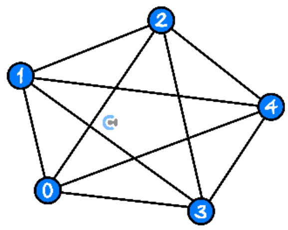


A subgraph of this graph is:

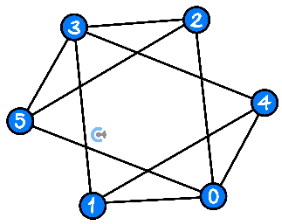
A.



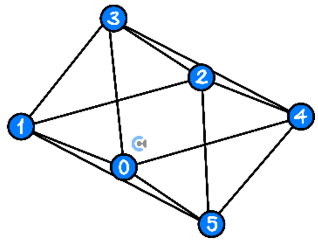
C.



B.



D.



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Section C: Problem Solving Questions

Question 12

A town has 4 intersections labelled A, B, C and D. Roads exist between:

- A and B.
- A and C.
- B and C.

a. Draw the graph for this road network.

b. Draw the complement graph (i.e., connect all pairs that were not connected).

c. What does the complement tell you about missing road connections in the town?

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Space for Personal Notes

Question 13

A graph shows connections between subjects that share common students:

- Maths ↔ Physics
- Physics ↔ Chemistry
- Chemistry ↔ Biology
- Biology ↔ English

a. Draw the graph.

b. Identify a subgraph that contains only science subjects.

c. What is the degree of Physics?

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Question 14

Imagine a mini social network with five students: Alex, Bella, Chris, Dana, and Eli. The following pairs follow each other:

- Alex ↔ Bella
- Alex ↔ Chris
- Bella ↔ Dana
- Dana ↔ Eli

a. Draw a simple graph to represent the network.

b. List the degree of each student.

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c. Write the adjacency list and adjacency matrix.

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d. Is this graph regular? Why or why not?

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Question 15

Let G be a simple graph where every vertex has odd degree.

Prove that the number of vertices with odd degree must be even.

Hint: Use the Handshake Lemma.

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Question 16

Let K_n be a complete graph on n vertices.

a. Prove that each vertex in K_n has degree $n-1$.

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b. How many edges does K_n have? Prove your answer.

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Question 17

A small friendship network of 6 students has the following number of friends:

- A has 2 friends.
- B has 3 friends.
- C has 2 friends.
- D has 1 friend.
- E has 2 friends.
- F has an unknown number of friends.

a. Use the Handshake Lemma to find how many friends F may have.

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b. How many total friendship pairs (edges) are there?

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Question 18

In a classroom, students are asked to draw lines to represent if they've ever worked with another student on a group project. The resulting graph has 10 vertices and 21 edges.

a. What is the sum of the degrees of all students?

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b. Can every student have degree 5 in this graph? Explain your answer.

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c. What is the minimum possible degree of a vertex in this graph?

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Sub-Section: The Tech-Free "Final Boss"

Question 19

Two schools are studying student friendship patterns. In each school, students are represented as vertices, and a friendship between two students is an edge. The graphs are simple.

School A's friendship graph has:

- 6 students.
- The degrees are: $3, 2, 3, 1, 2, 1$.

School B's friendship graph is given as an adjacency matrix:

$$\begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & 0 & 1 & 1 & 0 & 0 & 0 \\ B & 1 & 0 & 1 & 0 & 1 & 0 \\ C & 1 & 1 & 0 & 0 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 1 & 0 \\ E & 0 & 1 & 0 & 1 & 0 & 1 \\ F & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

a. Use the Handshake Lemma to verify the number of edges in each school's graph.

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b. Are the two graphs necessarily be isomorphic? Justify your answer.

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c. Represent School B's graph as an adjacency list.

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d. Identify and draw a subgraph in School B that forms a complete graph.

e. Draw the complement of School $B's$ graph, and what does it represent in the friendship context?

Space for Personal Notes

Section D: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 20

In a table tennis tournament, each player must play every other player exactly once.

a. If there are 4 players, draw the graph representing all matches.

b. What type of graph is this? (eg., Simple, complete, regular.)

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c. If 4 players each play 2 matches, can the graph be regular but not complete? Draw and explain.

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Space for Personal Notes

Question 21

Let G be a **3**-regular graph (each vertex has degree 3). Let H be a subgraph of G.

Is H necessarily regular? If not, provide a reason (or a counterexample).

Question 22

At a party, each person shakes hands with some others. There are 8 people at the party, and together they exchanged 20 handshakes.

a. What is the sum of the degrees of all the people (vertices)?

b. Is it possible that everyone shook hands with exactly 5 people? Why or why not?

Space for Personal Notes

Question 23

A graph has 7 vertices. The degrees of six of them are: 3, 2, 4, 5, 3, 2.

a. Can you find the smallest possible degree of the 7^{th} vertex using the Handshake Lemma?

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b. Is it possible for all vertices to have odd degree in this graph? Why?

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Space for Personal Notes

Question 24

A group of 7 students- A, B, C, D, E, F and G- are working together on various class projects. Your task is to draw a simple graph based on the following conditions:

1. The graph has 7 vertices, one for each student.

2. Student A has worked with exactly 2 others.

3. Students B and C have the same number of connections (same degree).

4. Students D and E are only connected to each other (and no one else).

5. Students F and G are each connected to 3 other students, but not to each other.

6. The graph has a total of 8 edges.

a. Draw the graph satisfying all the conditions. Label all vertices clearly.

b. Write the degree of each student.

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c. Create an adjacency list for the graph.

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d. Identify any subgraph(s) that form complete graph(s) with at least 3 vertices.

e. Is the graph regular? Why or why not?

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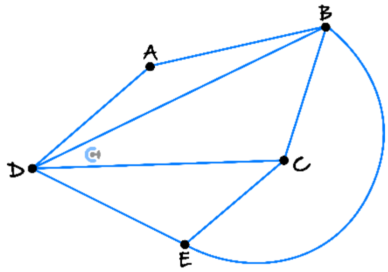
Space for Personal Notes

Sub-Section: Exam 2 (Tech-Active)

Question 25

Which vertices in the network below have the largest degree?



A. A and D.

B. B and D.

C. C and B.

D. D and E.

Question 26

If the sum of the degrees in a graph is 8, how many edges does the graph have?

A. 2

B. 4

C. 8

D. 16

Space for Personal Notes

Question 27

Let M be a complete graph. Let N be its complement.

Prove that N has no edges.

Hint: Use the definition of complement and complete graph.

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Question 28

Let G be a simple graph with n vertices.

Prove that the maximum number of edges in G is $n(n-1)/2$.

Hint: Think about complete graphs again.

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Space for Personal Notes

Question 29

A town is designing a road system between 7 intersections. They want the road network to be:

- A regular graph of degree 4.
- Have no loops or multiple edges.

Then, the town planner mistakenly adds a road that connects an intersection to itself (a loop) and creates a multiple edge between two intersections.

a. How many edges does the graph planned to have if it's 4-regular (A regular graph of degree 4.) on 7 vertices?

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b. After adding the loop and multiple edge, does the Handshake Lemma still apply? Why or why not?

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c. Is the resulting graph still simple? Explain.

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Space for Personal Notes

Question 30

A company creates a communication graph where:

- Each department is a vertex.
- A connection (edge) indicates direct communication between departments.
- The graph has 8 vertices and is not complete, but each department is connected to at least 4 others.
- One department (vertex v) is only connected to departments that themselves all connect with each other.

a. Prove that the subgraph formed by v's neighbours is a complete graph.

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b. Can the overall graph be regular? Why or why not?

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c. What is the minimum possible number of edges in this graph?

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d. If you take the complement of the graph, what does a connected component represent in this context?

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Question 31

In a university, each student joins a club. A graph is formed where:

- Each vertex represents a student.
- An edge connects two students if they are in the same club.
- The graph has 10 vertices, and the degrees of the vertices are: 3,4,3,4,5,4,3,4,5, x.
- The complement of the graph represents students who do not share any club.

a. Use the Handshake Lemma to determine the possible value(s) for x.

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b. What is the total number of clubs (edges) if students only form pairs?

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c. Identify whether this graph could be regular. Justify.

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d. One club has 4 members and they are all connected to each other. Identify the subgraph type formed and explain why.

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e. Suppose another university has the same degree sequence but different connections. Can the two graphs still be isomorphic? Why?

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