5.2

CONTOUREDUCATION

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VCE Specialist Mathematics ½
Combinatorics II [5.2]
Workbook

Outline:

Learning Objectives:

- [ ] **SM12 [5.2.1]** – Find Number of Permutations and Combinations
- [ ] **SM12 [5.2.2]** – Find Number of Permutations and Combinations with Restrictions/Composite

Section A: Combinatorics I Recap

Small recap of last week!

Contour Check

- [ ] **Learning Objective: [5.1.1] – Find number of permutations and combinations**

Key Takeaways

- [ ] Box Diagram for Arrangements
  - **Definition:** We can use it to write down number of arrangements for each position represented by each box.
- [ ] Arrangement

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

  - Generally:

$$\text{Ways to arrange/order } n \text{ many things for } r \text{ spots} = \frac{n!}{(n-r)!}$$

  - We call this ${}^nP_r$.

$${}^nP_r = \frac{n!}{(n-r)!}$$

- [ ] Selection
  - Generally:

$$\text{Ways to select } r \text{ things from } n \text{ many things} = \frac{{}^nP_r}{r!}$$

  - We call this ${}^nC_r$.

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

  - Where r = number of selection spots.

Learning Objective: [5.1.2] – Find number of permutations and combinations with restrictions/composite

Key Takeaways

- **Composite Arrangements**
  - **Definition:** Occurs when an arrangement happens within another arrangement.
  - Steps:
    - Consider each group as one object and find the arrangements.
    - Consider the arrangements within the groups and multiply.
- **Arrangements with Restrictions**
  - The general principle to deal with restrictions is to:
    - Use a box diagram.
    - Fill in the number of options for the slot that has the restriction first.

Definition

General Arrangement

- Generally,

$$\text{Ways to arrange/order } n \text{ many things for } r \text{ spots} = \frac{n!}{(n-r)!}$$

- We call this ${}^nP_r$,

$$ {}^nP_r = \frac{n!}{(n-r)!}$$

Space for Personal Notes

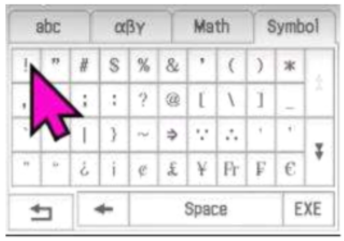
Calculator Commands: Factorial on Technology

- **Mathematica**
  - Exclamation Mark

    $x!$

- **TI-Nspire**
  - Menu 51

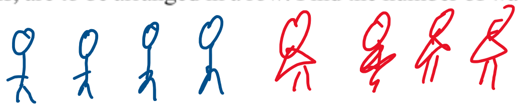
- **Casio Classpad**



Question 1 Tech-Active.

Eight people, consisting of 4 boys and 4 girls, are to be arranged in a row. Find the number of ways this can be done if:

a. There are no restrictions.



$\boxed{8}\boxed{7}\boxed{6}\boxed{.}\boxed{.}\boxed{.}\boxed{\ }\boxed{\ } = 8! = 40320$

b. The boys and girls are to alternate.

$\boxed{4}\boxed{4}\boxed{3}\boxed{3}\boxed{2}\boxed{2}\boxed{1}\boxed{1}$ $\quad 4! \times 4! \times 2$

B G B G B G B G $\quad = 1152$

G B G B

c. There are no restrictions on seating arrangements, but the people are arranged in a circle instead of a row.

$\boxed{8}\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = \frac{8!}{8} = 5040$

A B C D E F G H

ABCDEFGH
BCDEFGHA

d. The end seats must be occupied by a girl.

$4 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3$

g g

$= 8640$

e. A brother and sister in the group must not sit together.

$\boxed{2}\boxed{1} = 2!$

$\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1}$ $7!$

Not sitting together = Total − sitting together

$7! \times 2! = 10080$

$= 40320 - 10080 = 30240$

f. The girls must sit together.

1 2 3 4

$\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 4!$

5

$\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 5!$

$= 4! \times 5! = 2880$

And now selections!

General Selection

- Generally,

$$\text{Ways to select } r \text{ things from } n \text{ many things} = \frac{^{n}P_{r}}{r!}$$

- We call this $^{n}C_{r}$,

$$^{n}C_{r} = \frac{^{n}P_{r}}{r!} = \frac{n!}{r!\,(n-r)!}$$

- Where $r =$ number of selection spots.

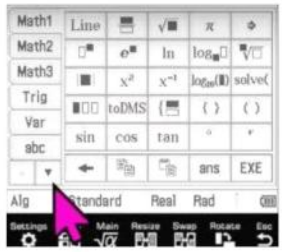
Calculator Commands: Combinations of Technology

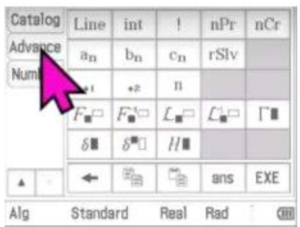
- **Mathematica**
  - Binomial $[n, r]$

- **TI-Nspire**
  - Menu 53

$^{n}C_{r}\ (n, r)$

- **Casio Classpad**





$^{n}C_{r}\ (n, r)$

Question 2 Tech-Active.

A panel of 8 is to be selected from a group of 8 men and 10 women. What's the number of possible selections such that:

a. There are 5 men and 3 women on the panel?

$\cancel{{}^{18}C_8}$

$$\underbrace{{}^{8}C_5}_{\text{Men}} \times \underbrace{{}^{10}C_3}_{\text{Women}} = 6720 \text{ ways}$$

b. There are at least 6 men on the panel?

$$6M, 2W + 7M, 1W + 8M$$

$$^{8}C_6 \times {}^{10}C_2 + {}^{8}C_7 \times {}^{10}C_1 + {}^{8}C_8$$

$$= 1341$$

c. A particular man and woman both be included?

$$^{16}C_6 = 8008$$

d. Two particular men, Angad and Amitav, cannot both be included?

Not included = Total − included

$^{14}C_8 - {}^{16}C_6$

$= 35750$

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Section B: Advanced Selections

Sub-Section: Pascal's Triangle and Selections

Exploration: Using Pascal's triangle to find ${}^nC_r$

- Once a Pascal's triangle is constructed (add two numbers on the top for the number below).
- We can use Pascal's triangle to get the combination ${}^nC_r$.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
\equiv n =
\begin{array}{ccccccccc}
 & & & & {}_0C_0 & & & & \\
 & & & {}_1C_0 & & {}_1C_1 & & & \\
 & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & \\
 & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & \\
{}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4
\end{array}
$$

$r =$

- Try to add another row to Pascal's triangle above and find ${}^5C_3$.

$${}^5C_3 = \underline{\ 10\ }$$

Pascals Triangle and ${}^nC_r$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
\equiv
\begin{array}{ccccccccc}
 & & & & {}_0C_0 & & & & \\
 & & & {}_1C_0 & & {}_1C_1 & & & \\
 & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & \\
 & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & \\
{}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4
\end{array}
$$

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Question 3 Walkthrough.

Using Pascal's triangle, find ${}^5C_2$.

$n =$

0: 1
1: 1 1
2: 1 2 1
3: 1 3 3 1
4: 1 4 6 4 1
5: 1 5 10 10 5 1

0 1 2 3 4 5

$r =$

${}^5C_2 = 10$

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Question 4

Using Pascal's triangle, find $^6C_4$.

n

0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1
	0	1	2	3	4		

$^6C_4 = 15$

✓

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Sub-Section: Symmetrical Property of Selections

Exploration: Symmetrical Property

- Consider the number of ways we can select 5 pets from a group of 12 available pets.
  - So, we're asking, "Which 5 pets do we select if order didn't matter?", i.e., ${}^{12}C_5$.
  - 12 - 5
  - BUT that's the same as asking "Which 7 pets did we NOT select if order didn't matter?" i.e., ${}^{12}C_7$, which should intuitively evaluate to the same result - and mathematically does as well!

$$ {}^{n}C_r = {}^{n}C_{n-r} $$

Symmetrical Property

$$ {}^{n}C_r = {}^{n}C_{n-r} $$

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Question 5 Walkthrough.

It is known that ${}^{100}C_3 = a$. Find ${}^{100}C_{97}$.

${}^{100}C_{97} = a$

$n=100$ $r=3$

${}^{n}C_{r} = {}^{n}C_{n-r}$

${}^{100}C_{3} = {}^{100}C_{100-3}$

$= {}^{100}C_{97}$

Space for Personal Notes

Question 6

It is known that $^{45}C_3 = 14190$. Find $^{45}C_{42}$.

14190

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Sub-Section: Selections of Any Size

Exploration: Selections of Any Size

- Consider a situation where we want to count all the combinations of any size r from a group of n objects.
- We can use the following shortcut to evaluate this quickly:

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = \underline{\quad 2^n \quad}$$

Selection of Any Size

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$$

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Question 7 Walkthrough.

Yilou is picking a subject to study today out of her 6 VCE subjects. Given that she can study any number of subjects (including nothing at all), how many different selections of subjects can she study?

0 subjects + 1 subjects + 2... + 3.. + 4...+5..+6...

$$\underbrace{{}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6}_{n=6}$$

$$= 2^6 = 64$$

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Question 8

Consider Daniel is making a guest list for his party. He has 7 friends to choose from and he must invite at least one friend. How many possible guest lists exist?

$$\underbrace{{}^7C_0 + {}^7C_1 + {}^7C_2 + \ldots + {}^7C_7}_{n=7} = 2^7$$

Not allowed $\Rightarrow$ Ans $= 2^7 - {}^7C_0 = 2^7 - 1$

$= 127$

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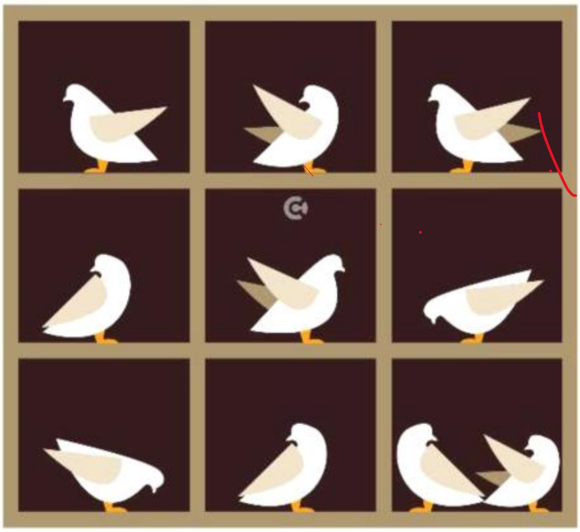
Section C: Pigeonhole Principle

Sub-Section: Introduction to Pigeonhole Principle

Context: Pigeonhole Principle

- Suppose you have been given 10 pigeons to arrange into a 3×3 grid of pigeonholes.



- Is it inevitable to have a hole with two pigeons in them?

Yes

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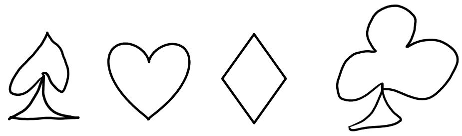
Okay, but what does that mean?

Discussion: If you pick five cards from a standard deck of 52 cards, can they all be of different suits?

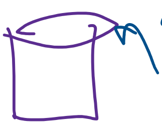
Pigeon: 5

Pigeonholes: 4 suits

NO.



Discussion: If you have 10 black socks and 10 white socks, and you are picking socks randomly, how many socks do I need to pick to find a matching pair?



Pigeon: 3 socks

Pigeonholes: 2

Discussion: Let's suppose that humans cannot have more than 1 million hairs on their heads. Could we say that there are at least two people in Australia with the same number of hairs on their heads?

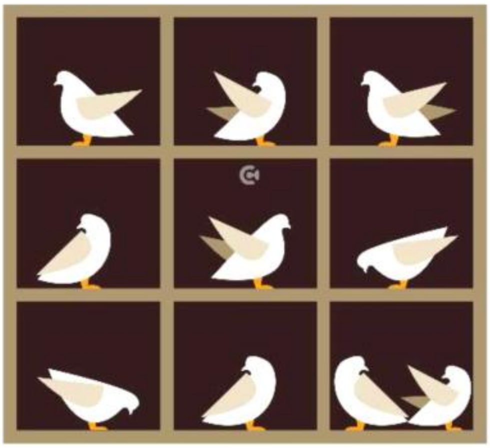
25 million

Pigeon: 25 million

Pigeonholes: 1 million

Sub-Section: Pigeonhole Principle

Pigeonhole Principle



"If $n + 1$ pigeons are placed into n holes, then some hole contains at least two pigeons."

- If $n+1$ or more objects are placed into n places, then someplace contains at least two objects.

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Question 9 Walkthrough.

Consider the question below.

Suppose that 4 Contour Specialist Maths students are spread across the three campuses: Box Hill, Melbourne CBD and Glen Waverly. Then, one class must have how many minimum number of students?

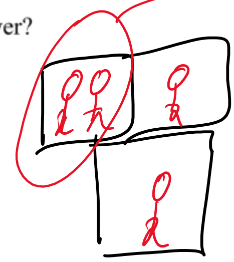
a. What would be considered the number of "pigeons" in this question?

Pigeons : 4 students

b. What would be considered the number of "pigeon holes" in this question?

Pigeon holes : 3 campuses

c. Hence, what would be the answer?



2 students

Space for Personal Notes

Question 10

Consider the question below.

Suppose that 20 Contour Specialist Maths students are assigned a number from 1 to 19 inclusive. Is it possible for each student to be assigned a different number?

a. What would be considered the number of "pigeons" in this question?

Pigeon : 20 students

b. What would be considered the number of "pigeon holes" in this question?

Pigeon holes : 19 numbers

c. Hence, what would be the answer?

NO!

Can't fit 20 pigeons into 19 pigeonholes

NOTE: It is important to understand what "pigeons" are and "pigeonholes" are.

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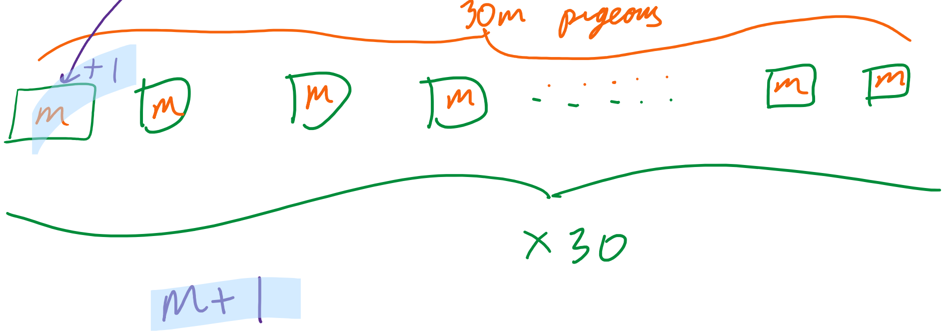
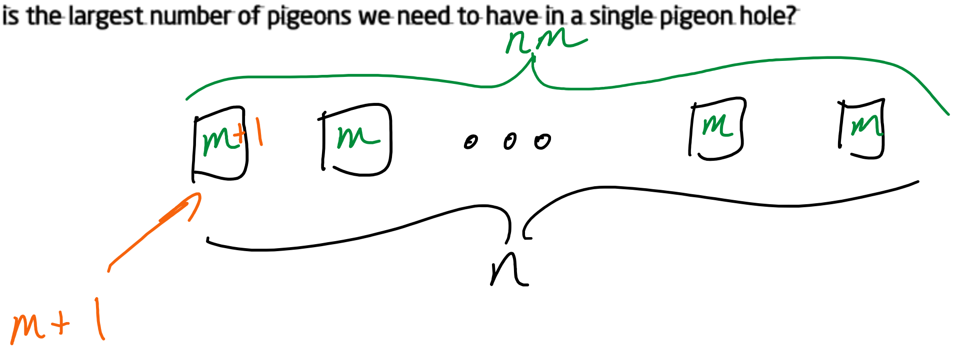
Sub-Section: Generalised Pigeonhole Principle

Discussion: What happens if we spread around $30 \cdot m + 1$ many pigeons into 30 pigeon holes? What is the largest number of pigeons we need to have in a single pigeon hole?



Discussion: What happens if we spread around $n \cdot m + 1$ many pigeons into n pigeon holes? What is the largest number of pigeons we need to have in a single pigeon hole?



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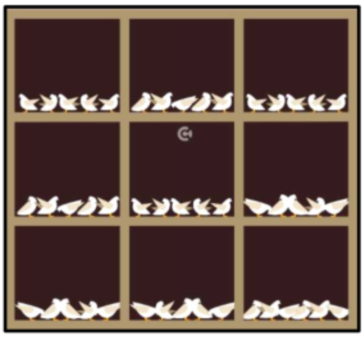
<u>Generalised Pigeonhole Principle</u>



"If $mn + 1$ pigeons are placed into n holes, then some hole contains at least $m + 1$ pigeons."

Space for Personal Notes

Question 11 Walkthrough.

Consider the question below.

Pranit asks each student in his 16 people class to write a different number between 1 and 3 inclusive.

One number must have a minimum of how many students choosing it?

a. What would be considered the number of "pigeons" in this question?

Pigeons : 16 people

$mn + 1$

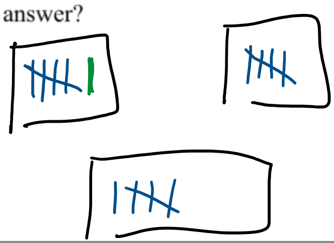
b. What would be considered the number of "pigeon holes" in this question?

Pigeon holes : 1, 2, 3

$n = 3$

c. Hence, what would be the answer?

$n = 3 \rightarrow mn + 1 = 16$



6 times

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Your turn!

Question 12

Consider the question below.

In the month of January, Krish takes one type of public transport (bus, train and tram) each day. What is the minimum number of days Krish must have spent on one type of transport?

a. What would be considered the number of "pigeons" in this question?

Pigeon : 31 days

$mn+1 = 31$

b. What would be considered the number of "pigeon holes" in this question?

Pigeon holes : Bus, train, tram

$n = 3$

c. Hence, what would be the answer?

11 days.

$\boxed{10+1}$

$\boxed{10}$ $\boxed{10}$

Section D: The Inclusion-Exclusion Principle

Sub-Section: Revision of Set Theory

Operation of Sets

- Negation: Everything but

$$\neg A \text{ or } A'$$

- Intersection

$$A \cap B$$

- Union

$$A \cup B$$

- Difference

$$A \setminus B$$

- Product

$$A \times B = \{(a, b), a \in A, b \in B\}$$

Discussion: How do we state the size of the set: The number of elements in the set?

$A = \{1, 2, 3\}$

$n(A) = 3$

$|A| = 3$

Size of the Set

$$|A| = \textit{Number of elements in set } A$$

Question 13

Consider the three sets of numbers $A = \{2, 3\}$ and $C = \{3, 4, 5\}$.

a. Find $|A|$.

$|A| = 2$

b. Find $A \cup C$.

$A \cup C = \{2, 3, 4, 5\}$

c. Find $|A \cup C|$.

$|A \cup C| = 4$

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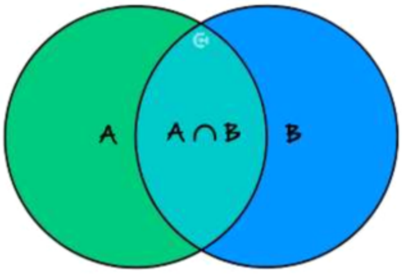
Sub-Section: Inclusion-Exclusion Principle for Two Sets

How do we find the number of elements in the union of two sets?

Inclusion-Exclusion Principle for Two Sets



$$|\boldsymbol{A} \cup \boldsymbol{B}| = |\boldsymbol{A}| + |\boldsymbol{B}| - |\boldsymbol{A} \cap \boldsymbol{B}|$$

- If A and B are two finite sets of objects, then $|A \cup B| = |A| + |B| - |A \cap B|$.

NOTE: This is same idea as $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

Space for Personal Notes

Question 14 Walkthrough. 150

How many integers from 1 to 150 inclusive are divisible by 2 or 5?

$\div 2 \quad \vee \div 5$

$$|\div 2 \text{ or } \div 5| = |\div 2| + |\div 5| - |\div 2 \cap \div 5|$$

$$= |\div 2| + |\div 5| - |\div 10|$$

$$= 75 + 30 - 15$$

$$= 90$$

Space for Personal Notes

Question 15

There are 70 students at a secondary school and each of them must study at least one of Biology or Physics. There are 50 students who study Biology and 35 who study Physics. How many students study both subjects?

$$|\text{Bio or Physics}| = |\text{Bio}| + |\text{Physics}| - |\text{Bio and Physics}|$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$70 = 50 + 35 - |\text{Bio and Physics}|$$

$$|\text{Bio and Physics}| = 15$$

Discussion: What would happen if A and B did not overlap? ($A \cap B = \emptyset$)

$$|A \cup B| = |A| + |B| - \underbrace{\cancel{|A \cap B|}}_{=0}$$

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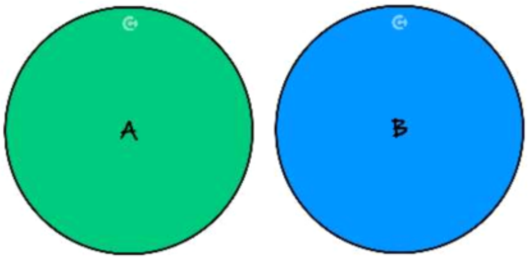
Addition Principle with No Intersection



$$|A \cup B| = |A| + |B|$$

- If A and B are two finite sets of objects such that $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

Space for Personal Notes

Question 16

How many numbers from 1 to 100 start with the number 3 or 5?

|start w/ 3 or 5| = |start w/ 3| + |start w/ 5|

1. 30 → 39
2. 3

1. 50 → 59
2. 5

= 11 + 11

= 22

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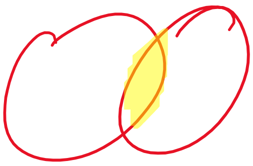
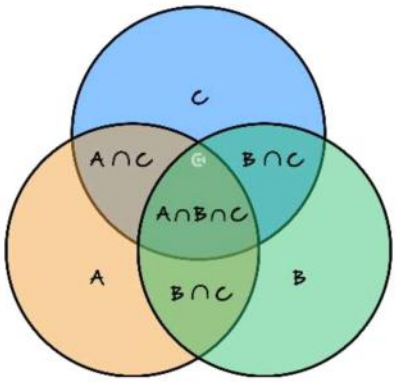
Sub-Section: Inclusion-Exclusion Principle for Three Sets

What about 3 set?

Inclusion-Exclusion Principle for Three Sets



$|A \cup B| = |A| + |B| - |A \cap B|$



$$|\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}|$$
$$= |\boldsymbol{A}| + |\boldsymbol{B}| + |\boldsymbol{C}| - |\boldsymbol{A} \cap \boldsymbol{B}| - |\boldsymbol{A} \cap \boldsymbol{C}| - |\boldsymbol{B} \cap \boldsymbol{C}| + |\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C}|$$

- If A, B and C are three finite sets of objects.

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Question 17 Walkthrough.

Total $= 280$

How many integers from 1 to 280 inclusive are divisible by 2, 5 or 7?

$$|\div 2 \cup \div 5 \cup \div 7| = |\div 2| + |\div 5| + |\div 7| - |\overbrace{\div 2 \cap \div 5}^{\div 10}| - |\overbrace{\div 2 \cap \div 7}^{\div 14}|$$

$$- |\underbrace{\div 5 \cap \div 7}_{\div 35}| + |\underbrace{\div 2 \cap \div 5 \cap \div 7}_{\div 70}|$$

$$= 140 + 56 + 40 - 28 - 20 - 8 + 4$$

$$= 184.$$

Space for Personal Notes

Question 18

There are 90 students at a secondary school and each of them must study at least one of Biology, Physics or Chemistry. There are 28 students who study Biology, 42 who study Physics and 40 who study Chemistry. Moreover, 9 study Biology and Physics, 8 study Biology and Chemistry and 7 study Physics and Chemistry. How many students study all three subjects?

Let B = bio
P = physics
C = chem

$$|B \cup P \cup C| = |B| + |P| + |C| - |B \cap P| - |B \cap C| - |P \cap C| + |B \cap P \cap C|$$

$$90 = 28 + 42 + 40 - 9 - 8 - 7 + |B \cap P \cap C|$$

$$|B \cap P \cap C| = 4$$

Space for Personal Notes

Contour Check

Learning Objective: [5.2.1] – Find number of permutations and combinations

Key Takeaways

- Box Diagram for Arrangements:
  - **Definition:** We can use it to write down number of $\boxed{\text{arrangements}}$ for each position represented by each $\boxed{\text{box}}$.
- Arrangement:
  - Generally:

$$\text{Ways to arrange/order } n \text{ many things for } r \text{ spots} = \boxed{\frac{n!}{(n-r)!}}$$

  - We call this ${}^nP_r$,

$$^nP_r = \boxed{\frac{n!}{(n-r)!}}$$

- Selection:
  - Generally:

$$\text{Ways to select } r \text{ things from } n \text{ many things} = \frac{^nP_r}{r!}$$

  - We call this ${}^nC_r$,

$$^nC_r = \boxed{\frac{^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}}$$

  - Where r = number of selection spots.

Learning Objective: [5.2.2] – Find number of permutations and combinations with restrictions/composite

Key Takeaways

- **Composite Arrangements:**
  - **Definition:** Occurs when an arrangement happens within another arrangement.
  - **Steps:**
    - Consider each group as __ $\boxed{\text{one}}$ __ object and find the arrangements.
    - Consider the __ $\boxed{\text{arrangements}}$ __ within the groups and multiply.
- **Arrangements with Restrictions:**
  - The general principle to deal with restrictions is to:
    - Use a box diagram.
    - Fill in the number of options for the slot that has the restriction __ $\boxed{\text{first}}$ __.

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