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VCE Specialist Mathematics ½
Combinatorics II [5.2]
Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Recap	Pg 2-Pg 4
Basics (Compulsory)	Pg 5-Pg 9
Problem Solving (Compulsory)	Pg 10-Pg 16
Exam 1	Pg 17-Pg 20
Exam 2	Pg 21-Pg 26

Section A: Recap

Contour Check

Learning Objective: [5.2.1] – Selections of any size and Pascal's triangle

Key Takeaways

- Pascal's Triangle and ${}^nC_r$:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \equiv \begin{array}{ccccccccc} & & & & {}_0C_0 & & & & \\ & & & {}_1C_0 & & {}_1C_1 & & & \\ & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & \\ & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & \\ {}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4 \end{array}$$

  - A new entry in Pascal's triangle is found by adding the two entries above it from the previous row.

- Symmetrical Property:

$$ {}^nC_r = {}^nC_{n-r} $$

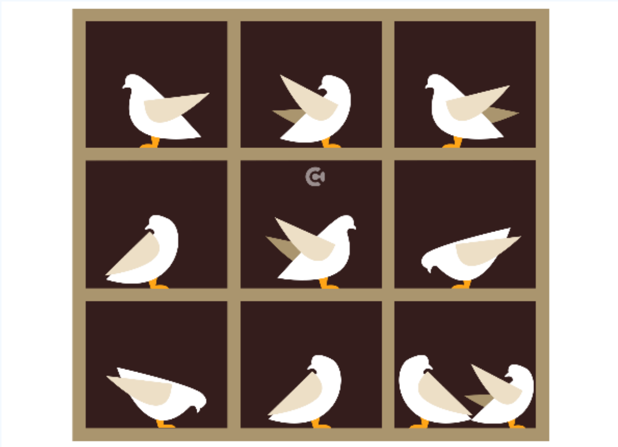
- Selection of any Size:

$$ {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n $$

Learning Objective: [5.2.2] – Pigeonhole principle

Key Takeaways

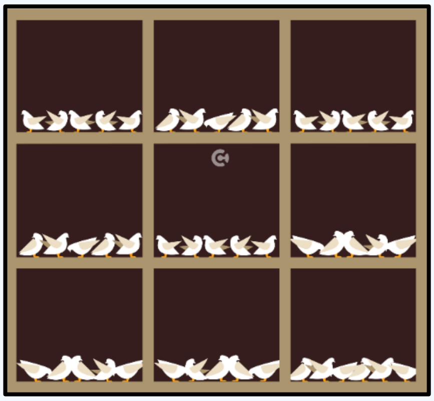
- Pigeonhole Principle:



"If $n+1$ pigeons are placed into n holes, then some hole contains at least two pigeons."

- If $n+1$ or more objects are placed into n places, then someplace contains at least two objects.

- Generalised Pigeonhole Principle:

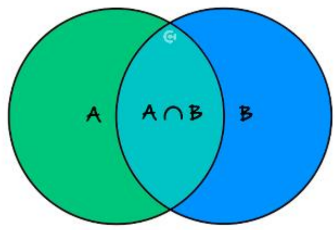


"If $mn+1$ pigeons are placed into n holes, then some hole contains at least $m+1$ pigeons."

Learning Objective: [5.2.3] - Inclusion-Exclusion principle

Key Takeaways

- Inclusion-Exclusion Principle for Two Sets:



$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Inclusion-Exclusion Principle for Three Sets:
  - If A, B, and C are three finite sets of objects:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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Section B: Basics (Compulsory)

NOTE: The notation $\binom{n}{r}$ is equivalent to ${}^nC_r$.

Question 1

a. Evaluate $\binom{7}{3}$.

$\binom{7}{3} = 35$

b. State and use the symmetrical property of combinations to show that $\binom{9}{2} = \binom{9}{7}$.

The symmetrical property says $\binom{n}{r} = \binom{n}{(n-r)}$.

So,

$$\binom{9}{2} = \binom{9}{(9-2)} = \binom{9}{7}$$

c. Find the value of the following.

$$\binom{8}{0}+\binom{8}{1}+\binom{8}{2}+\cdots+\binom{8}{8}$$

The identity says:

$$\sum_{r=0}^{n}\binom{n}{r}=2^n \Rightarrow \sum_{r=0}^{8}\binom{8}{r}=2^8=256$$

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Question 2

a. There are 6 pigeonholes and 13 pigeons. Explain why some pigeonholes must contain at least 3 pigeons.

Using the generalised pigeonhole principle:
$\left\lceil \frac{13}{6} \right\rceil = 3 \Rightarrow$ At least one pigeonhole has 3 or more pigeons.

b. If $5n + 1$ pigeons are placed into n holes, explain why some holes contain at least 6 pigeons.

Using the general pigeonhole principle: If $kn + 1$ pigeons are placed into n holes, some hole contains at least $k + 1$ pigeons. Here, $5n + 1$ pigeons implies at least one hole has $5 + 1 = 6$ pigeons.

c. There are 28 socks in a drawer, each sock is either red, blue, or green. What is the minimum number of socks you must pull out to guarantee that you have at least 9 socks of the same colour?

Apply the pigeonhole principle in reverse.
To avoid 9 of any one colour, you could have at most 8 red, 8 blue, and 8 green:

$$8 + 8 + 8 = 24$$

So, pulling out 25 socks guarantees at least 9 of one colour.

Question 3

a. In a group of students:

- 23 play netball.
- 17 play soccer.
- 9 play both sports.

How many students play at least one of the two sports?

Let A = netball, B = soccer.
Then,

$$|A \cup B| = |A| + |B| - |A \cap B| = 23 + 17 - 9 = 31$$

b. In a school of 60 students:

- 25 study French.
- 30 study Japanese.
- 18 study Indonesian.
- 12 study both French and Japanese.
- 10 study both French and Indonesian.
- 8 study both Japanese and Indonesian.
- 5 study all three languages.

How many students study at least one language?

Let F, J, I be the sets of French, Japanese, Indonesian.

$$|F \cup J \cup I| = |F| + |J| + |I| - |F \cap J| - |F \cap I| - |J \cap I| + |F \cap J \cap I|$$
$$= 25 + 30 + 18 - 12 - 10 - 8 + 5 = 48$$

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Section C: Problem Solving (Compulsory)

Question 4

At a party, every guest leaves their shoes at the entrance. The shoes are mixed randomly in a pile. There are 10 different types of shoes (e.g., boots, sneakers, sandals, etc.), and each guest owns exactly one pair from a type. Suppose there are 41 individual shoes (not pairs) in the pile.

Prove that at least five shoes of the same type must be present in the pile.

Apply the pigeonhole principle with 10 types (pigeonholes) and 41 shoes (pigeons). The most you can have without reaching 5 of any type is:

$4 \times 10 = 40$ shoes

The 41^{st} shoe must cause at least one type to reach 5. Therefore, at least one type appears at least 5 times.

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Question 5

A tech company surveys 120 employees on their skillsets:

- 72 know Python.
- 65 know Java.
- 58 know C + +.
- 38 know both Python and Java.
- 31 know both Python and C + +.
- 28 know both Java and C + +.
- 18 know all three.

How many employees know at least one of the three programming languages?

Let P, J, and C be the sets of people who know Python, Java, and C + + respectively.

$$|P \cup J \cup C| = |P| + |J| + |C| - |P \cap J| - |P \cap C| - |J \cap C| + |P \cap J \cap C|$$
$$= 72 + 65 + 58 - 38 - 31 - 28 + 18$$
$$= 195 - 97 + 18 = 116$$

116 employees know at least one language.

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Question 6

A school has 1000 lockers, each secured by a 3-digit code using digits 0-9. Each code must contain at most two distinct digits (e.g., $111, 112, 212$ are valid; 123 is not).

If there are 350 students, show that at least two students must have the same locker code.

We count the total number of 3-digit codes with at most two distinct digits.
Case 1: All digits the same: There are 10 such codes: $000, 111, \ldots, 999$.
Case 2: Two distinct digits: - Choose 2 digits from 10: $\binom{10}{2} = 45$ - Each digit pair can form codes with 2 digits in different arrangements: There are $3! = 6$ such codes per pair (e.g., for 1 and 2: $112, 121, 211, 221, 212, 122$).
So, total number of valid codes:

$$10 + 45 \times 6 = 280$$

Since $350 > 280$, by the pigeonhole principle, at least two students have the same code.

Question 7

A drawer contains 18 pairs of gloves in 4 colours: red, blue, green, and black. All gloves are mixed together (left and right gloves are not paired).

What is the minimum number of gloves that must be pulled out to guarantee at least one matching pair of the same colour and hand (i.e., left and right glove of the same colour)?

Each colour has 2 gloves of each hand. Worst case: all left gloves (or all right), and all different colours.
There are 4 colours $\times$ 2 hands = 8 distinct types. To avoid a matching pair: you could pull 8 gloves, all different types.
Pulling a 9^{th} glove ensures at least one left and one right glove of the same colour.
Answer: 9

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Question 8

In a group of 90 people:

- 45 like tea.
- 50 like coffee.
- 20 like both.

How many people like neither tea nor coffee?

Let T = tea, C = coffee. Using inclusion-exclusion:
$|T \cup C| = |T| + |C| - |T \cap C| = 45 + 50 - 20 = 75$
Then the number who like neither is:
$$90 - 75 = 15$$

Question 9

A bowl contains 30 fruit pieces: bananas, oranges, and apples.

What is the minimum number of fruit you must pick to ensure that you have at least 11 pieces of the same type?

There are 3 types of fruit. To avoid 11 of any type, the maximum you can pick is $10 + 10 + 10 = 30$.
Picking 31 fruit ensures at least one type appears 11 times.

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Question 10

A list of 13 integers is selected from the set $\{1, 2, 3, \ldots, 24\}$ (without replacement).

Prove that at least two of the selected integers differ by a multiple of 12.

We divide the numbers 1 to 24 into 12 pairs where each pair consists of numbers that differ by 12:

$$(1, 13), (2, 14), (3, 15), \ldots, (12, 24)$$

Each pair contains numbers whose difference is exactly 12.
These 12 pairs act as pigeonholes. If we choose 13 numbers, then by the pigeonhole principle, at least two numbers must come from the same pair.
Therefore, at least two of the selected numbers differ by exactly 12, which is a multiple of 12.

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Question 11

In a school of 100 students:

- 45 play netball.
- 52 play volleyball.
- 40 play basketball.
- 20 play both netball and volleyball.
- 25 play both netball and basketball.
- 18 play both volleyball and basketball.
- 10 play all three sports.

How many students play none of the three sports?

Let N, V, and B be the sets of students who play netball, volleyball, and basketball. Using inclusion-exclusion:

$$|N \cup V \cup B| = |N| + |V| + |B| - |N \cap V| - |N \cap B| - |V \cap B| + |N \cap V \cap B| = 45 + 52 + 40 - 20 - 25 - 18 + 10 = 84$$

So, the number who play none is:

$$100 - 84 = 16$$

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Question 12 Tech-Active.

Consider all the arrangements of the letters in the word CALCULATE.

a. How many distinct arrangements of the letters are there?

CALCULATE has 9 letters, with: - A appears twice, - C appears twice, - L appears twice.
So total arrangements:

$$\frac{9!}{2! \times 2! \times 2!} = 45360$$

b. Calculate the number of arrangements where at least one repeated pair (AA, CC, or LL) occurs adjacently.

Each repeated adjacent pair is treated as a block:

$$\text{AA: } \frac{8!}{2! \times 2!} = 10080$$

$$\text{CC: } \frac{8!}{2! \times 2!} = 10080$$

$$\text{LL: } \frac{8!}{2! \times 2!} = 10080$$

Double counts:

AA and CC: $\frac{7!}{2!} = 2520$

AA and LL: $\frac{7!}{2!} = 2520$

CC and LL: $\frac{7!}{2!} = 2520$

All three adjacent:

AA, CC, LL all adjacent: $6! = 720$

Using inclusion-exclusion:

$$|A \cup C \cup L| = 3 \times 10080 - 3 \times 2520 + 720$$
$$= 30240 - 7560 + 720 = 23400$$

c. Hence, how many of these arrangemen

$$45360 - 23400 = 21960$$

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Section D: Exam 1

Question 13

A teacher is giving away coloured pens to students as part of a game. The students can choose from 5 unique colours of pens.

a. Each student writes down **at least one** of the colours of pen that they want to win.

How many possible selections of colours are there?

$2^5 - 1 = 32 - 1 = 31.$

b. A student will win the colour of pen that they wrote down as long as nobody else wrote down the exact same colour of pen.

What is the least number of students that must be in the class, to guarantee that nobody will win?

32 students by the pigeonhole principle.

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Question 14

In a group of 40 students:

- 20 like tennis.
- 18 like basketball.
- 15 like soccer.
- 8 like both tennis and basketball.
- 6 like both tennis and soccer.
- 5 like both basketball and soccer.
- 3 like all three sports.

a. How many students like at least one of the three sports?

We apply the inclusion-exclusion principle:

$$|T \cup B \cup S| = |T| + |B| + |S| - |T \cap B| - |T \cap S| - |B \cap S| + |T \cap B \cap S|$$
$$= 20 + 18 + 15 - 8 - 6 - 5 + 3$$
$$= 53 - 19 + 3 = 37$$

b. How many students like at least two of the three sports?

We subtract the overlap of all three from each pairwise total to get those who like exactly two:

Tennis & Basketball only = $8 - 3 = 5$
Tennis & Soccer only = $6 - 3 = 3$
Basketball & Soccer only = $5 - 3 = 2$

Then add those who like all three:

$$5 + 3 + 2 + 3 = 13$$

Question 15

Consider all 8 letters of the word TANZANIA.

a. How many different ways can all the letters be arranged if the three A's are kept together?

Treat AAA as one block. Remaining letters: T, N, Z, N, I
→ 6 total elements (AAA, T, N, Z, N, I)
So:

$$\frac{6!}{2!} = \frac{720}{2} = 360$$

b. How many arrangements are there where consonants (T, N, Z) and vowels (A, I) alternate, beginning with a consonant?

There are 4 consonants: T, N, N, Z, Vowels: A, A, A, I
Pattern: C-V-C-V-C-V-C-V (starts with consonant) - 4 of each required, so okay.
Consonant arrangements: $\frac{4!}{2!} = 12$
Vowel arrangements: $\frac{4!}{3!} = 4$
Total: $12 \times 4 = 48$

From the 8 letters of TANZANIA, 4 are selected.

c. How many different selections include exactly one N and one A?

There are 2 N's and 3 A's. Our choice of 1 of each is fixed and the letters are not considered unique.
Remaining 2 letters must be from 3 other letters: T, Z, I
→ $\binom{3}{2} = 3$
Therefore, only 3 possible selections.

d. How many different selections include exactly one N?

0 A: Nxxx = 1 way.
1 A: NAxx = $\binom{3}{2}$ = 3 ways.
2 A: NAAx = $\binom{3}{1}$ = 3 ways.
3 A: NAAA = 1 way.
Therefore, a total of 8 ways.

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Section E: Exam 2

Question 16

Maya owns three types of collectables: 6 unique wooden carvings, 4 different sea shells, and 3 handmade pottery birds.

a. She allows her niece, Lily, to choose 5 items to play with. Lily must select at least one item of each type. How many different selections can Lily make?

We must count the number of combinations of 5 items including at least one of each type: W (wood), S (shell), B (bird).
We now calculate the number of combinations for each valid triple (w, s, b) and sum:

$$(3,1,1): \binom{6}{3}\binom{4}{1}\binom{3}{1} = 20 \times 4 \times 3 = 240$$
$$(2,2,1): \binom{6}{2}\binom{4}{2}\binom{3}{1} = 15 \times 6 \times 3 = 270$$
$$(2,1,2): \binom{6}{2}\binom{4}{1}\binom{3}{2} = 15 \times 4 \times 3 = 180$$
$$(1,3,1): \binom{6}{1}\binom{4}{3}\binom{3}{1} = 6 \times 4 \times 3 = 72$$
$$(1,2,2): \binom{6}{1}\binom{4}{2}\binom{3}{2} = 6 \times 6 \times 3 = 108$$
$$(1,1,3): \binom{6}{1}\binom{4}{1}\binom{3}{3} = 6 \times 4 \times 1 = 24$$

$$\text{Total} = 240 + 270 + 180 + 72 + 108 + 24 = \boxed{894}$$

Maya arranges 10 of the 13 items in a row on a shelf.

b. How many arrangements are possible if she places a bird at each end and no birds appear anywhere else?

Choose 2 birds for ends and arrange: $\binom{3}{2} \times 2! = 3 \times 2 = 6$

Remaining 8 items must be chosen from 6 wood + 4 shells = 10 items: choose 8 from these.

$$\binom{10}{8} \times 8! = 45 \times 40320 = 1,814,400$$

Total arrangements: $6 \times 1,814,400 = \boxed{10,886,400}$

c. How many arrangements are possible if she places a bird at each end, and wooden carvings and sea shells are alternated in the positions between?

There are 8 positions between the birds. Alternating: 4 wood, 4 shells (must be exact).
The 4 shells can be positioned in $4! = 24$ ways.
The 4 wood carvings can be positioned in $6 \times 5 \times 4 \times 3 = 360$ ways.
Then alternate whether wood or shell is first. So 2 ways.
Can choose 2 birds and arrange them at ends in $\binom{3}{2} \times 2 = 6$ ways.
Total: $6 \times 24 \times 360 \times 2 = \boxed{103,680}$ ways.

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Question 17

A student committee has members from three year levels: Year 10, Year 11, and Year 12. There are 7 students from Year 10, 2 from Year 11, and 2 from Year 12.

Five members are selected to form a subcommittee. How many different selections include at least one member from each year level?

Total number of ways to choose 5 from all 11 students:

$$\binom{11}{5} = 462$$

We subtract cases that exclude at least one year level:
Exclude Year 10: choose 5 from $2 + 2 = 4$ students $\to$ not possible
Exclude Year 11: choose 5 from Year 10 + Year 12 = 9

$$\binom{9}{5} = 126$$

Exclude Year 12: same count: $\binom{9}{5} = 126$

Double-counted (only Year 10): $\binom{7}{5} = 21$

So total to subtract:

$$126 + 126 - 21 = 231$$

Answer: $462 - 231 = \boxed{231}$

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Question 18

Consider the digits of the number, 1345789.

a. How many different 7-digit numbers can be made using all seven digits, if all the odd digits are placed together and no digits are repeated?

Odd digits: 1, 3, 5, 7, 9 (5 digits) Even digits: 4, 8 (2 digits)
We treat the block of 5 odd digits as a group. So total "objects" $= 3$ (odd block + 2 even digits)
Ways to permute 3 objects: $3! = 6$
Ways to arrange the 5 odd digits internally: $5! = 120$
Total $= 6 \times 120 = 720$

b. How many 4-digit numbers can be made using these digits if the number is even, lies between 3000 and 5000, and no digits are repeated?

Digits available: 1, 3, 4, 5, 7, 8, 9 Even digits: 4, 8
First digit must be 3 or 4 (to ensure number is between 3000 and 5000). The last digit must be even: 4 or 8.
Case 1: First $= 3$, last $= 4$ → middle digits from 1,5,7,8,9 (5 left): $5 \times 4 = 20$
Case 2: First $= 3$, last $= 8$ → mid digits from 1,4,5,7,9: $5 \times 4 = 20$
Case 3: First $= 4$, last $= 8$ → mid from 1,3,5,7,9 $\rightarrow 5 \times 4 = 20$
Total $= 60$

c. How many numbers less than 1000 can be made using some or all of the digits, if the number is a multiple of 5 and digits can be repeated?

To be divisible by 5, number must end in 0 or 5. But 0 is not available, so only 5 is allowed as final digit.
We count 1-digit, 2-digit, and 3-digit numbers ending in 5.
1-digit: just 5 $\rightarrow 1$
2-digit: last digit 5, first digit any of the seven, 7 options.
3-digit: last digit $= 5$, first 2 digits: choose from 7 digits with repetition allowed: $7 \times 7 = 49$
Total $= 1 + 7 + 49 = 57$

Question 19

Nine cards are labelled: $1, 2, 2, 3, 3, 4, 6, 6, 6$

a. All nine cards are arranged in a line to make a 9-digit number.

i. How many such numbers can be formed if all the even digits appear together?

Even digits: $2, 2, 4, 6, 6, 6 \rightarrow 6$ cards
Odd digits: $1, 3, 3 \rightarrow 3$ cards
Treat even group as one block. There are $\frac{6!}{2!3!} = 60$ arrangements within.
We treat the even block as a single object and arrange it with the 3 odd digits $(1, 3, 3)$:
There are $\frac{4!}{2!} = 12$ ways to arrange those 4 objects.
Total $= 60 \times 12 = 720$

ii. How many such 9-digit numbers can be formed if the first and last digits are both odd?

Odd digits: $1, 3, 3 \rightarrow$ total of 3 odd cards. We must choose 2 odd digits to go at the first and last positions.
Since we have only one '1' and two '3's, the possible cases are:
Case 1: First and last digits are both 3
Used: $3, 3$. Remaining digits: $1, 2, 2, 4, 6, 6, 6$. Number of arrangements: $\frac{7!}{2! \times 3!} = 420$

Case 2: First $= 1$, Last $= 3$ (or vice versa)
Used: 1 and one 3. Remaining $= 2, 2, 3, 4, 6, 6, 6$. Repeated digits: two $2'$s, three 6's.
Arrangement: $\frac{7!}{2! \times 3!} = 420$
But this occurs in two ways: $(1, 3)$ or $(3, 1)$ $\rightarrow$ multiply by 2: $2 \times 420 = 840$

Total $= 420 + 840 = 1260$

b. Three of the nine cards are chosen and arranged to make a 3-digit number.

i. How many such numbers are possible if there are no repeated digits?

Digits available without repetition: 1, 2, 3, 4, 6 $\rightarrow$ 5 options Choose any 3 digits from these: $\binom{5}{3} = 10$ Each selection can be arranged: $3! = 6$
Total $= 10 \times 6 = \boxed{60}$

ii. How many such 3-digit numbers can be made if the number is between 200 and 300?

We require 2 **.
Remaining digits are 1,2,3,3,4,6,6,6. (5 unique)
Second digit is 1: = 4 ways to choose last digit.
Second digit is 2: = 4 ways to choose last digit.
Second digit is 3: = 5 ways to choose last digit.
Second digit is 4: = 4 ways to choose last digit.
Second digit is 6: = 5 ways to choose last digit.
So, a total of $4 \times 3 + 10 = 22$ ways.

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