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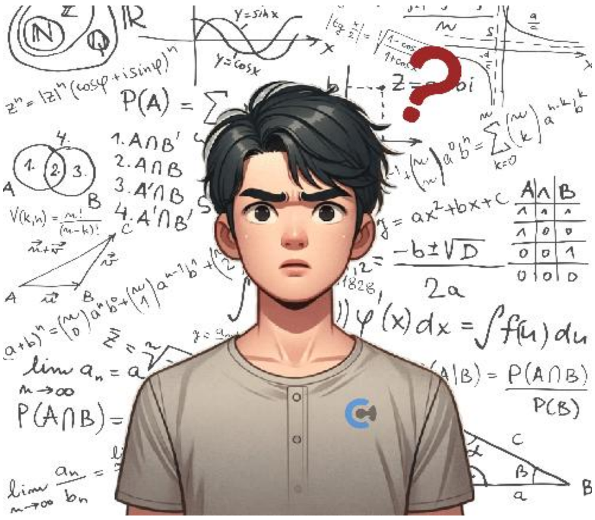
VCE Specialist Mathematics ½
Combinatorics I [5.1]
Workbook

Outline:

Section A: Introduction to Counting Methods

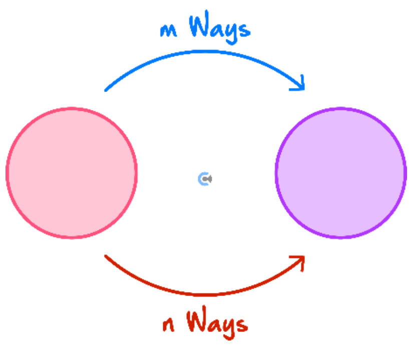
Sub-Section: Addition Principle

Context: Addition Principle



- Sam is choosing between his 4 pants and 3 shorts to go on his date with Emily.
- How many different options does he have? [7 / 12]
- We [added / multiplied] the options as Sam will choose one option [or / and] the other.

Addition Principle



- Associated with the use of the word "OR." alternatives

$$\text{Total Possibilities} = m + n$$

Question 1

The restaurant offers four vegan dishes and nine vegetarian dishes.

How many selections of one main meal does a customer have?

$4+9=13$ options

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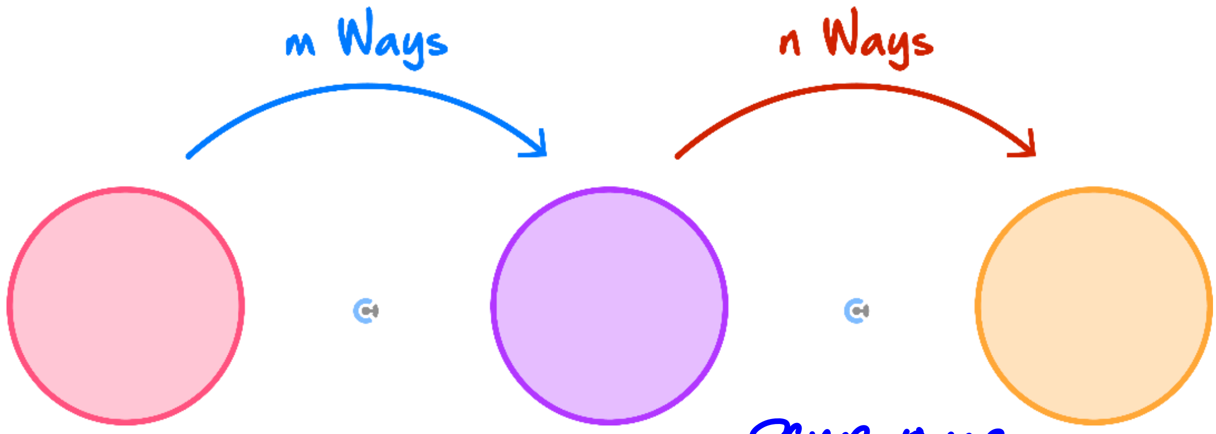
Sub-Section: Multiplication Principle

The Multiplication Principle



- Associated with the use of the word "AND." ← same time

$$\text{Total Possibilities} = m \times n$$

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Question 2

James has three different pants, four different tops and two different pairs of shoes.

How many different choices does she have for a complete outfit?

$3 \times 4 \times 2 = 24$ options

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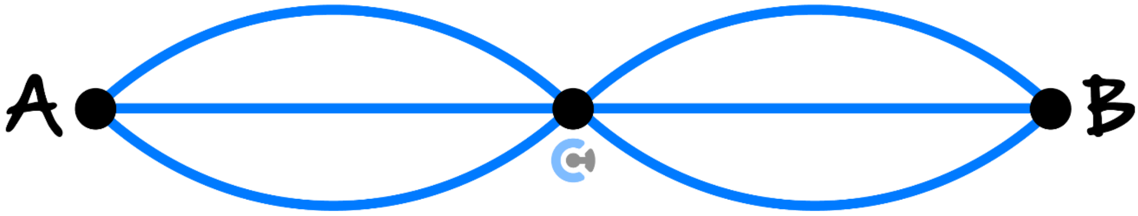
Question 3 Walkthrough.

Travelling from left to right, how many paths are there from point A to point B in the following diagram?



3×3

$= 9$ paths

NOTE: You go through the first, second and third bridges. Hence, we use the multiplication principle.

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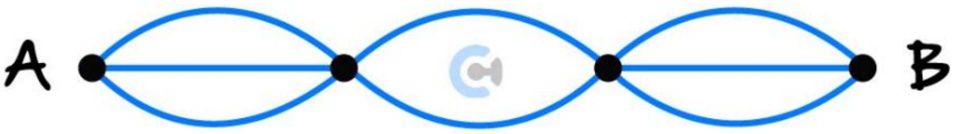
Question 4

Travelling from left to right, how many paths are there from point A to point B in the following diagram?



$3 \times 2 \times 3 = 18$ paths

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Section B: Arrangements (Permutations)

Sub-Section: Introduction to Arrangements

↳ permutations

What are arrangements?

Arrangements

- It is a study of the number of ways to Order things.

Discussion: How many ways can you arrange the letters a, b, c?

- abc
- acb
- bac
- bca
- cab
- cba

$3 \times 2 \times 1 = 6$ options

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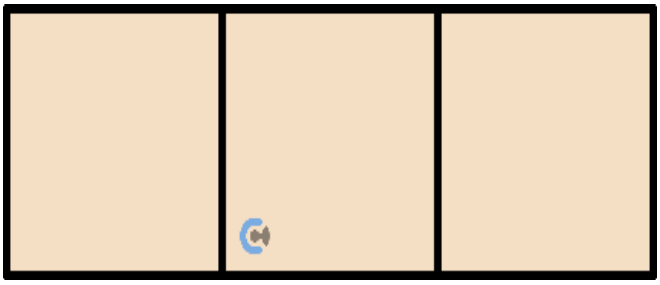
Is there a way to visualise the number of arrangements?

Definition

Box Diagram for Arrangements

- **Definition:** We can use it to write down a number of possiblities for each position represented by each box.



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Sub-Section: Arrangements for n Many Things in n Spots

Arrangements for n Many Things in n Spots

- When everything is ordered.

  Eg: 10 people sitting in 10 seats.

- It is given by factorial.

Ways to arrange/order n many things $=$ $\underline{n!}$

n $\times$	$n-1$ $\times$	$\ldots\ldots$ $\times$	2 $\times$	1

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Question 5 Walkthrough.

5 different letters are ordered for a 5 letter word.

How many different words can you get (even the incoherent ones)?

$\boxed{5 \times 4 \times 3 \times 2 \times 1} = 120$ ways

$5!$

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Question 6

A family of 3 sits next to each other and is interested in the number of ways they could be seated.

How many different ways can the family of 3 sit in their 3 seats?

$\boxed{3 \mid 2 \mid 1} = 6$ options

$3!$

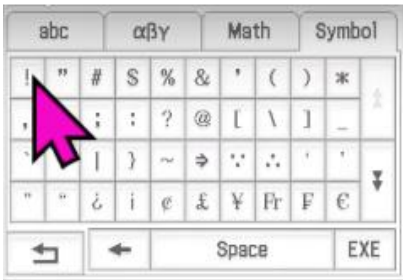
Calculator Commands: Factorial on Technology

- **Mathematica**
  - Exclamation Mark

    $x!$

- **TI-Nspire**
  - Menu 51

- **Casio Classpad**



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Sub-Section: General Arrangement

What would happen if we had n things to arrange in only r spots?

Exploration: General Arrangement

- Previously, we considered the case where **everything was ordered.**

How many ways can 5 people sit in 5 seats?

5	4	3	2	1

- What would happen if we don't have enough seats for everyone?

How many ways can 5 people sit in 2 seats?

5	4

- In summary:

don't arrange

$$\boxed{5}\,\boxed{4} = \frac{\boxed{5}\,\boxed{4}\,\boxed{3}\,\boxed{2}\,\boxed{1}}{\boxed{3}\,\boxed{2}\,\boxed{1}} = \frac{n!}{(n-r)!}$$

n = total

r = ordering

- Let's generalise this for n people with only r many seats! $(n > r)$.
  - The numerator represents a number of ways where we seat everyone.

    It is given by [$n!$, $r!$, $(n-r)!$].
  - The denominator represents a number of arrangements we missed out on due to a lack of a seat.

    How many people aren't sitting? [n, r, $(n-r)$]

    Hence, the denominator is given by [$n!$, $r!$, $(n-r)!$]

$$\text{Number of ways for } n \text{ people to sit in } r \text{ seats} = \frac{n!}{(n-r)!}$$

  - We call this ${}^nP_r$! Or Permutations!

Definition

General Arrangement

- Generally,

$$\text{Ways to arrange/order } n \text{ many things for } r \text{ spots} = \underline{\quad {}^nP_r \quad}$$

- We call this ${}^nP_r$:

$$^nP_r = \frac{n!}{(n-r)!}$$

Space for Personal Notes

Question 7 Walkthrough.

$r=4$ $n=6$

Sam is trying to make a four-digit number by using the numbers 1, 2, 3, 4, 5, and 6 without repeating them.

How many different numbers can Sam have?

^nP_r

$$^6P_4 = \frac{6!}{(6-4)!}$$

$$= \frac{6!}{2!}$$

$$= \frac{6\times 5!}{2} = 3\times 5!$$

$$= 3\times 120$$

$$= 360 \text{ ways}$$

box

6	5	4	3

$$= 6\cdot 5\cdot 4\cdot 3$$

$$= 30\cdot 12$$

$$= 360 \text{ ways}$$

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Question 8

The teacher decides to pick 3 students from 12 students in her class and appoints them as class captain and two different vice-captain roles.

How many different ways could the teacher do this?

$$^{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot \cancel{9!}}{\cancel{9!}}$$

$$= 1320 \text{ ways}$$

TIP: Only multiply the numbers after all the common factors are cancelled in the fraction.

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Calculator Commands: Arrangements on Technology

Mathematica

- FactorialPower

$^{n}P_{r}$

```
FactorialPower[n, r]
```

- OR make your own:

```
npr[n_, r_] := n! / (n - r)!
```

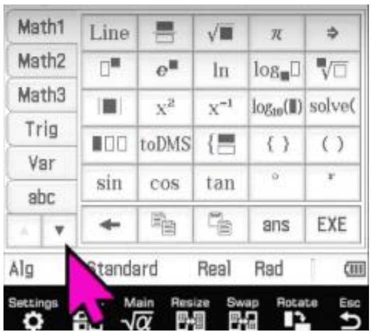
$^{12}P_3$

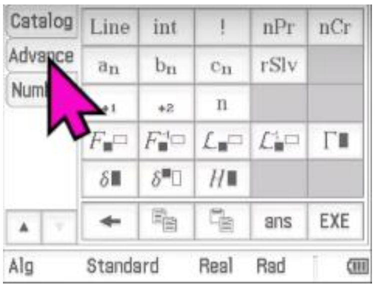
TI-Nspire

- Menu 52

$^{n}P_{r}(n, r)$

Casio Classpad





$^{n}P_{r}(n, r)$

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In[1]:= 5!

Out[1]= 120

In[2]:= FactorialPower[12, 3]
階乗ベキ

Out[2]= 1320

$^{n}C_{r}$ Binomial[ ]

In[4]:= Binomial[12, 3]
二項係数

Out[4]= 220

Question 9 Tech-Active.

A painter is to paint the five circles of the Olympic flag. He cannot remember the colours to use for any of the circles, but he knows they should all be different. He has ten colours of paint available.

In how many ways can he paint the circles on the flag?

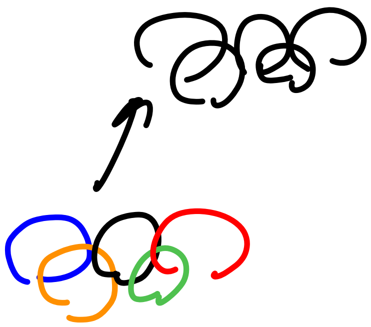
HINT: How many options are there for the first circle? How many for the second?

$$^{10}P_5 = 30240$$

```
In[5]:= FactorialPower[10, 5]
        階乗ベキ

Out[5]= 30 240
```

Space for Personal Notes



Sub-Section: Composite Arrangements

↳ something INSIDE another

Discussion: How can we arrange a, b, c, d if a and b need to be next to each other?

① $\boxed{a, b}$ ② $\boxed{c}$ ③ $\boxed{d}$

1. Order group $3!$

2. Order within group

① $2!$

② $1!$

③ $1!$

Composite Arrangements

- **Definition:** Occurs when an arrangement happens within another arrangement.
- Steps:
  - Consider each group as one object and find the arrangements.
  - Consider the arrangements within the groups and multiply.

Space for Personal Notes

Question 10 Walkthrough.

M F S D

Consider a family of 4 which consists of a mum, ~~dad~~ father, son and daughter.

If it is known that the mum must sit next to the ~~dad~~ father, how many different ways could they be sitting?

$\boxed{M\quad F}\ \boxed{S}\ \boxed{D}$

1 order groups $3!$

2 within $2!$

$3! \times 2! = 6 \cdot 2$

$= 12$ ways

Space for Personal Notes

Question 11

Pranav wants to install seating plans for his rowdy class of 8 students. Fearing the backlash of the students, he lets them choose one friend to sit next to each other.

How many different seating plans could Pranav come up with?

$\boxed{2!}\ \boxed{2!}\ \boxed{2!}\ \boxed{2!}$

① $4!$

② $(2!)^4$

$4! \times (2!)^4$

$= 24 \times 16$

$= 384$ ways

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Question 12 Tech-Active.

There are 12 animals in the animal farm; 5 dogs, 4 cats and 3 hamsters. Rei decides to label them with numbers ranging from 1-12. He decides to finish labelling all animals of the specific type of species before labelling animals from different species.

a. How many different ways could the animals be numbered?

dogs cats ham. 3!

5! 4! 3!

In[8]:= 3 ! * 5 ! * 4 ! * 3 !

Out[8]= 103 680

arrange = $3! \times 5! \times 4! \times 3!$

b. If Rei decided to label the animals without considering their species, are there more ways to number them compared to **part a.** or less?

→ 12!

more.

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Sub-Section: Arrangements with Restrictions

Discussion: What do we have to consider when making a 4-digit number with $0, 1, 2, 3$?

not 0

$\boxed{3}\boxed{3}\boxed{2}\boxed{1}$

Arrangements with Restrictions

- **Definition:** The general principle to deal with restrictions is to:
  - Use the boxes.
  - Fill in the number of options for the slot that has the restriction first.

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Question 13 Walkthrough.

How many different odd numbers between 10000-99999 exist, which only have the digits 2, 3, 4, 5, 6, 7, and 8 given each digit can only be used once?

7 option

3

6	5	4	3	$\boxed{3}$

= 1080 numbers

TIP: Consider the number of options for the last digit. And do this before considering the rest!

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Question 14

If no digit can be used more than once, find how many numbers can be formed from the digits 2, 3, 4, 5, 6, 7, 8, and 9 that are:

a. Four-digit numbers?

8 | 7 | 6 | 5 = 1680 ways

b. Odd three-digit numbers?

7 | 6 | 4 = 168 ways

c. Four-digit numbers greater than 7000?

3 | 7 | 6 | 5 = 630 ways

TIP: Target the digit with restrictions first.

Question 15 Tech-Active.

Three boys and four girls sit in a line on stools in front of a counter. Find the number of ways in which they can arrange themselves:

a. If one of the boys, Bob, insists on being on (either) one of the ends.

everyone else | B

6! | 1

2!

$6! \times 1 \times 2! = 1440$ ways

b. If two of the girls wish to sit at (either) end together.

2 G | everyone else

4×3 | $5!$

2!

$(4 \times 3) \times 5! \times 2!$

$= 2880$ ways

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Section C: Selections (Combinations)

Sub-Section: Introduction to Selections

↳ combinations

Discussion: When selecting two letters out of every letter in the alphabet, are AB and BA considered two different selections?

Same combination

Selection

- Definition:
  - Number of ways to select things.
  - For selections, the order does not matter.

Okay, so how do we solve for the number of selections (combinations)?

Exploration: Selections

- Consider the following example.

Number of selections when we select two alphabets from A, B, C?

- We can first solve for the number of arrangements.

Number of Arrangements $= {}^3P_2 = \frac{3!}{(3-2)!}$ $= 6$

How do we find the number of selections?

- Let's consider the number of arrangements with the same selection.

÷ overlaps

$$Eg: AB \text{ and } BA$$

- How many different arrangements with the same selection of letters? 2 = 1 selection

$$\text{Number of Selections} = \frac{\textit{Number of Arrangements}}{\text{number of double-ups}}$$

$$= \frac{\textit{Number of Arragenements}}{2}$$

- Let's generalise this for n people with only r many selected! $(n > r)$
  - The numerator represents the number of ways where we arrange n people in r spots.

    It is given by [^nP_r, $n!$, $r!$].

    3 ABC ACB

  - The denominator represents the number of arrangements in the same selections.

    It is given by [$n!$, $r!$, $(n-r)!$].

$$\text{Number of Selections} = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

no. of overlaps

  - We call this ^nC_r! Or combinations!

Space for Personal Notes

Sub-Section: General Selection

General Selection

- Generally,

$$\text{Ways to select } r \text{ things from } n \text{ many things} = \frac{^nP_r}{r!}$$

- We call this ^nC_r

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

- Where $r =$ number of selection spots.

Discussion: Why do we divide by $r!$?

Cancel out overlaps

r!= number of orders for the same selection. In the previous example, the number of arrangements for same selection were 2! As there were 2 slots.

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Question 16 Walkthrough.

A leadership team of four people is to be chosen from a group of ten students. How many different teams are possible?

$$^{10}C_4 = \frac{10!}{4!\,(10-4)!}$$

$$= \frac{10!}{4!\,6!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7\, 6!}{(\cancel{4} \cdot \cancel{3} \cdot \cancel{2})\, 6!}$$

$$= 10 \cdot 3 \cdot 7$$

$$= 210 \text{ ways}$$

Space for Personal Notes

Question 17

How many ways are there to choose exactly three pets from a store with 7 dogs and 13 cats?

$$ {}^{20}C_3 = \frac{20!}{3!(20-3)!} = \frac{20!}{3!\,17!} = \frac{20\cdot 19\cdot 18\cdot \cancel{17!}}{3!\,\cancel{17!}} $$

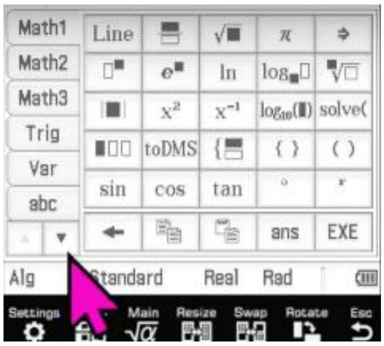
$$ = \frac{\cancel{20}^{\,10}\cdot 19\cdot \cancel{18}^{\,6}}{\cancel{3}\cdot \cancel{2}} $$

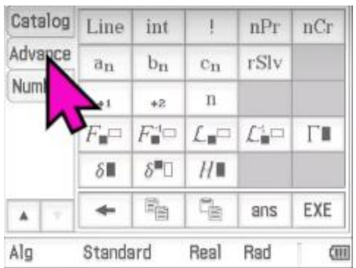
$$ = 1140 \text{ ways} $$

NOTE: We can treat dogs and cats the same although dogs are better.

Calculator Commands: Combinations of Technology

- **Mathematica**
  - Binomial $[n, r]$
- **TI-Nspire**
  - Menu 53
  - ${}^{n}C_{r}\,(n, r)$
- **Casio Classpad**





${}^{n}C_{r}\,(n, r)$

Question 18 Tech-Active.

"AND"

A team of four boys and four girls is to be chosen from a group of ten boys and eight girls. How many different teams are possible?

$^{10}C_4 \times {}^{8}C_4$

$=$

```
In[10]:= Binomial[10, 4] * Binomial[8, 4]
         二項係数            二項係数
Out[10]= 14700
```

Space for Personal Notes

How many words can be formed by "arrangements"

order matter

$n!$

$$\frac{12!}{2!\,2!\,2!\,2!} =$$

```
In[11]:= 12! / 16
Out[11]= 29937600
```

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