4.3

CONTOUREDUCATION

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VCE Specialist Mathematics ½
Transformations II [4.3]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [4.3.1]** – Transformations of Graphs
- ☐ **SM12 [4.3.2]** – Rotations Around Points
- ☐ **SM12 [4.3.3]** – Reflections in Lines

Section A: Recap of [4.2] Transformations

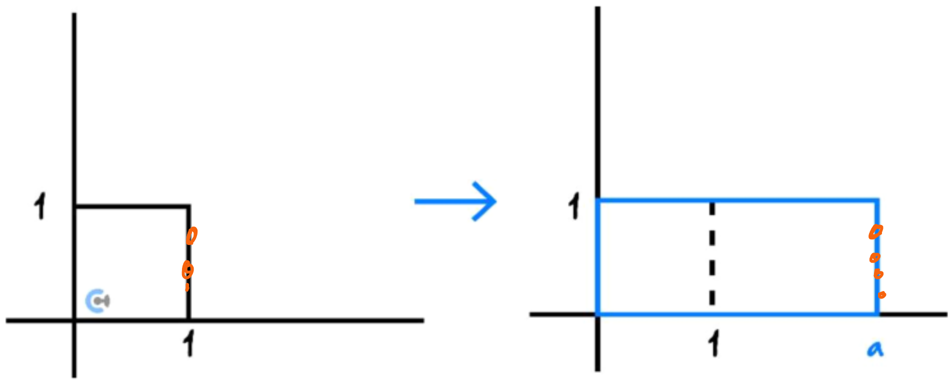
Let's do a quick recap of what we did last week!

Linear Transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- The (x', y) represents the new points and is called an image.
- Original point (x, y) is called the pre-image.
- A is the transformation matrix.

Dilation from the y-axis



Dilation by a factor a from the y-axis

- Dilation from the y-axis changes the x- value.

$$\textit{Transformation Matrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

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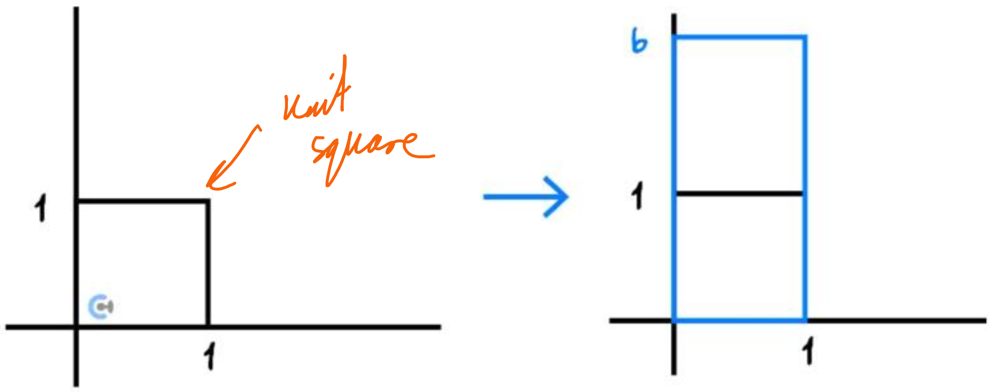
Definition

Dilation from the x-axis



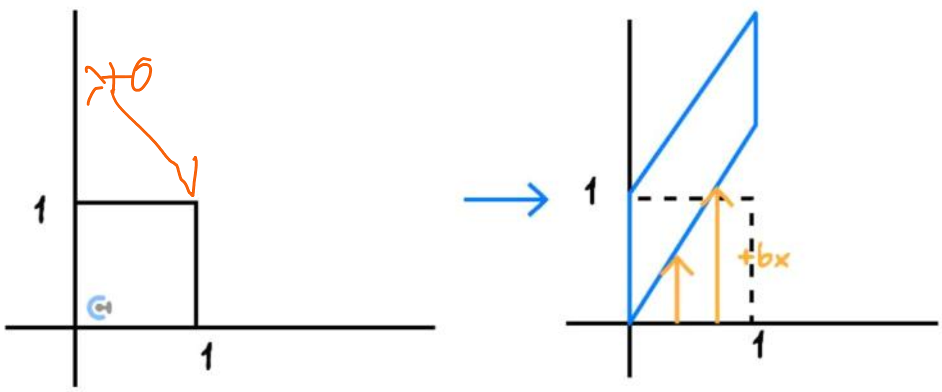
Dilation by a factor b from the x-axis

- Dilation from the x-axis changes the y-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

Definition

Shear Parallel to the y-axis



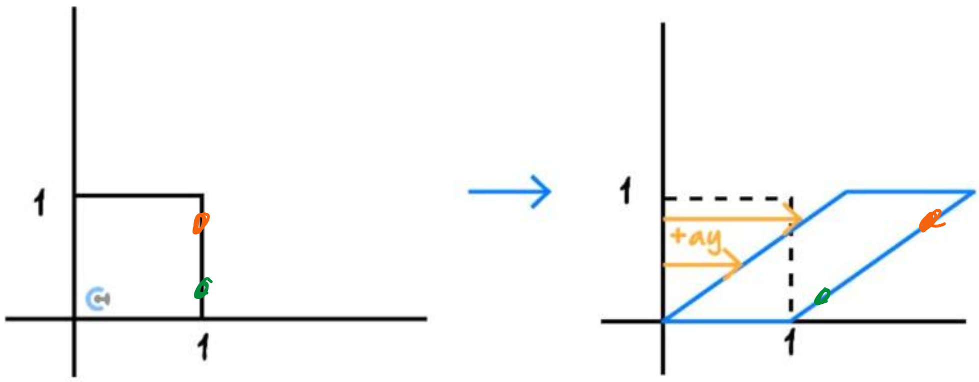
Shear of a factor b parallel to the y-axis

- Shear parallel to y-axis changes the y-value by a multiple of the x-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

Definition

Shear Parallel to the x-axis



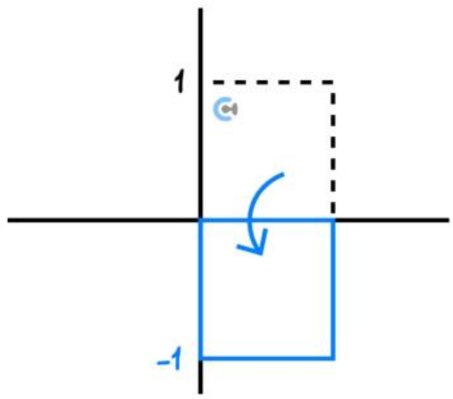
Shear of a factor a parallel to the x-axis

- Shear parallel to x-axis changes the x-value by a multiple of the y-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

Definition

Reflection Around x-axis



Reflection in the x-axis

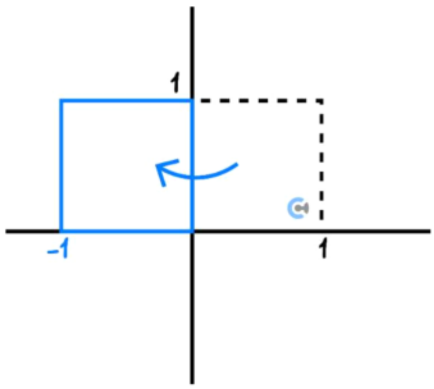
- Reflection in the x-axis changes the y-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

negative

Definition

<u>Reflection Around y-axis</u>



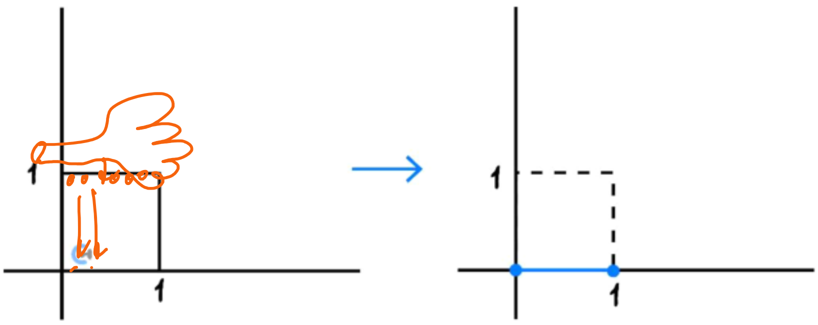
Reflection in the y-axis

- Reflection in the y-axis changes the x-value.

$$\textit{Transformation Matrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition

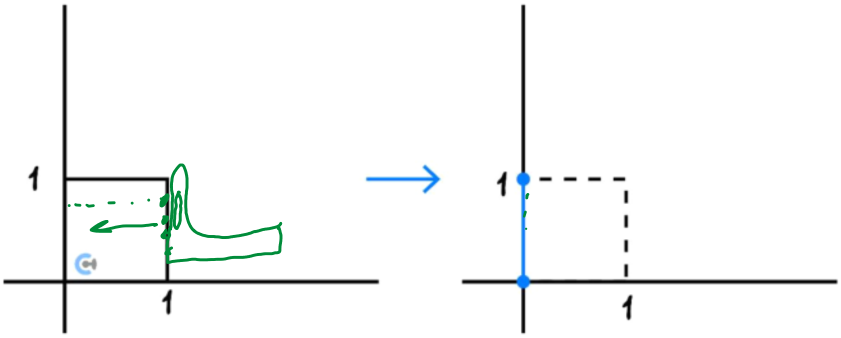
<u>Projections</u>



Projection onto x-axis

- The y-values becomes 0.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto y-axis

- The x-values becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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Question 1 Walkthrough.

a. State the transformation matrix for the shear of a factor 2 parallel to the y-axis and dilation by a factor 3 from the x-axis.

y-value

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 3 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate $(3, 1)$.

$$\begin{bmatrix} 1 & 0 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 21 \end{bmatrix}$$

$(3, 21)$

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Question 2

a. State the transformation matrix for dilation by a factor 2 from the y-axis and the shear of a factor 3 parallel to the x-axis.

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate $(2, 4)$.

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \end{bmatrix}$$

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Section B: Transformations of Graphs

Discussion: If we can transform points, how can we transform functions/graphs?

$\begin{bmatrix} & \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Transformation of Functions/Graphs

$$y = f(x) \to y' = f(x')$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

- **Steps:**

1. Find $x' = f(x)$ and $y' = g(y)$.
2. Rearrange and make x, y the subject.
3. Substitute into the original function.
4. Remove ′ on the variables.

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Question 3 Walkthrough.

change x - value

a. State the transformation matrix for dilation by a factor $\frac{1}{2}$ from the y-axis and reflection around the x-axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}$$

b. Find the image of (x, y) under the transformation described in **part a**.

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x \\ -y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x \\ -y \end{bmatrix}$$

Consider a function $f(x) = \sqrt{x+3} - 1$. It is known that all the points of $f(x)$ have been transformed by the transformation matrix found in **part a**.

c. Find the transformed graph.

$x' = \frac{1}{2}x$, $x = 2x'$

$y' = -y$, $y = -y'$

$y = \sqrt{x+3} - 1$

$-y' = \sqrt{2x' + 3} - 1$,

$y' = -\sqrt{2x'+3} + 1$

$y = -\sqrt{2x+3} + 1$

Your turn!

Question 4

a. State the transformation matrix for dilation by a factor of 5 from the y-axis and a reflection around the y-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Find the image of (x, y) under the transformation described in **part a**.

$$\begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5x \\ y \end{bmatrix}$$

Consider a function $f(x) = (x-4)^2 + 3$.

It is known that all the points of $f(x)$ have been transformed by the transformation matrix found in **part a**.

c. Find the transformed graph.

$x' = -5x \Rightarrow x = -\frac{x'}{5}$

$y' = y \Rightarrow y = y'$

$y = (x-4)^2 + 3$

$y' = \left(-\frac{x'}{5} - 4\right)^2 + 3$

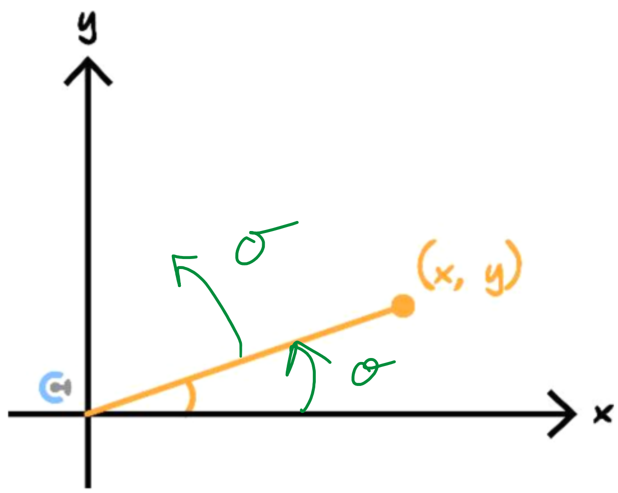
$y = \left(-\frac{x}{5} - 4\right)^2 + 3$

Section C: Rotations

Sub-Section: Rotations Around the Origin

How do we rotate a point around the origin?

Rotation Around the Origin



Rotation θ in the Anticlockwise Direction

$$Transformation\ Matrix = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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Question 5 Walkthrough.

a. State the transformation matrix for rotation around the origin 60° anticlockwise.

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b. Hence, find the image of (3, 1) after it has been rotated around the origin 60° anticlockwise.

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} + \frac{1}{2} \end{bmatrix}$$

$$\left(\frac{3}{2} - \frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2} + \frac{1}{2} \right)$$

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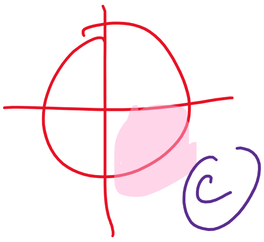
Question 6

a. State the transformation matrix for rotation around the origin 30° clockwise.



$$\begin{bmatrix} \cos -30^\circ & -\sin -30^\circ \\ \sin -30^\circ & \cos -30^\circ \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

b. Hence, find the image of $(1, 2)$ after it has been rotated around the origin 30° clockwise.

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\sqrt{3}}{2} \\ -\frac{1}{2} + \sqrt{3} \end{bmatrix}$$

$$\left(1 + \frac{\sqrt{3}}{2}, -\frac{1}{2} + \sqrt{3}\right)$$

NOTE: If the angle is clockwise, we measure it negatively.

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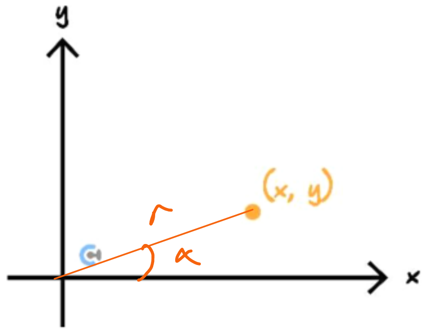
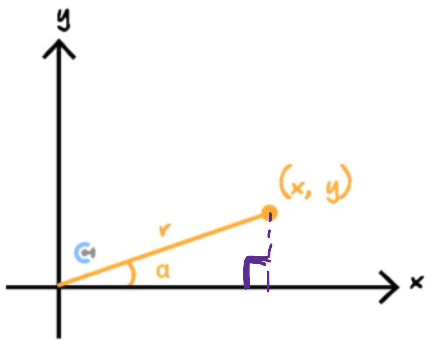
How does this work?

Exploration: Understanding Rotations Around the Origin

- Consider a pre-image (x, y).



- Let's say the point (x, y) ___r___ is away from the origin and has an angle of ___α___ anticlockwise from the x-axis.



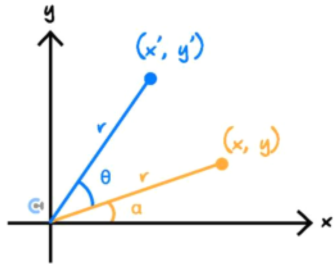
- Using SOHCAHTOA, how can we define x and y in terms of r and α?

$x =$ ___r cos(α)___

$y =$ ___r sin(α)___

- What happens when we rotate θ anticlockwise around the origin?

  Let's visualise the diagram together.



- Using SOHCAHTOA, how can we define x' and y' in terms of r and α?

$$x' = r\cos(\alpha+\theta)$$
$$y' = r\sin(\alpha+\theta)$$

- Using the compound angle formulas, expand the following and substitute in the original x and y!

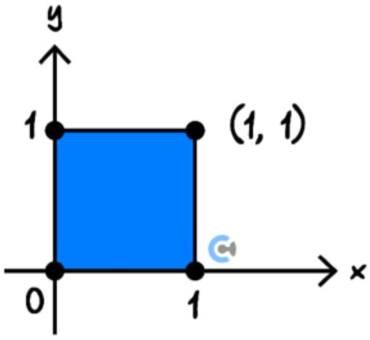
$$x' = r(\cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta))$$
$$= r\cos(\alpha)\cos\theta - r\sin(\alpha)\sin\theta$$
$$x' = x\cos\theta - y\sin\theta$$
$$y' = r(\sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta))$$
$$= r\sin(\alpha)\cos(\theta) + r\cos(\alpha)\sin(\theta)$$
$$y' = y\cos\theta + x\sin\theta$$

- Hence, in summary:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

REMINDER: Determinant of Transformation Matrix:

- Given that A = Transformation matrix.



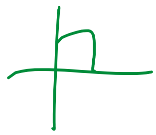
- The unit square was used to visualise how a transformation affects different points.

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

- Determinant of the transformation matrix tells us how the area of the unit circle changes.

Discussion: What would the determinant of the rotation matrix be?

1



Exploration: Determinant of the Rotation Matrix

- Consider the rotation transformation matrix:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

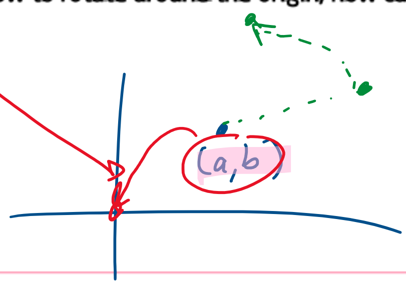
- What does the determinant equal to? Evaluate it using algebra!

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta)$$
$$= \cos^2\theta + \sin^2\theta$$
$$= 1$$

Sub-Section: Rotations Around Any Point

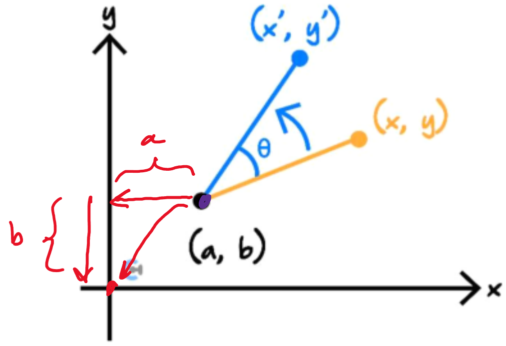
Discussion: Since we know how to rotate around the origin, how can we rotate around any point (a, b)?

Take (a,b) to origin

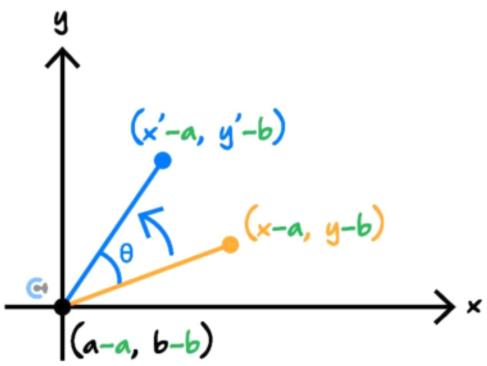


Rotations Around Any Point (a, b)

- Consider the rotation θ around the point (a, b) in the anticlockwise direction.



- Since we know how to rotate from the origin, let us translate ourselves to the origin!



- How do we go from $(x-a, y-b)$ to $(x'-a, y'-b)$?

$$\begin{bmatrix} x'-a \\ y'-b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

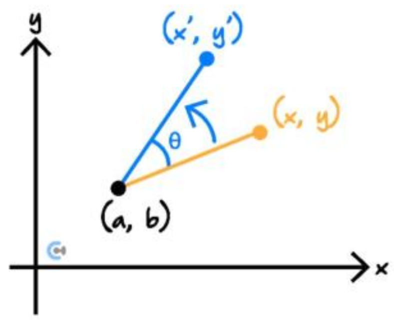
- Now, expand the matrices and make $\begin{bmatrix} x' \\ y' \end{bmatrix}$ the subject!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) + \begin{bmatrix} a \\ b \end{bmatrix}$$

Definition

Rotation Around Any Point (a, b)



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{2} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{1} \right) + \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{3}$$

- The idea is that we:

1. Translate the points by $(-a, -b)$ so that the centre becomes the origin.
2. Rotate the point around the origin using the transformation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
3. Translate the points by (a, b) so that we go back to (a, b) being the centre.

Question 7 Walkthrough.

State the image of $(1, 1)$ after the rotation around the point $(2, 1)$, $60°$ anticlockwise.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}}_{2} \Bigg(\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{1} \Bigg) + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{3}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 - \frac{\sqrt{3}}{2} \end{bmatrix}$$

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$$\left(\frac{3}{2}, 1 - \frac{\sqrt{3}}{2} \right)$$

Question 8

State the image of $(4, 2)$ after the rotation around the point $(-1, 1)$, $30°$ anticlockwise.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underset{(2)}{\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}} \left(\underset{(1)}{\begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right) + \underset{(3)}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5\sqrt{3}}{2} - \frac{1}{2} \\ \frac{5}{2} + \frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{3}}{2} - \frac{3}{2} \\ \frac{7}{2} + \frac{\sqrt{3}}{2} \end{bmatrix}$$

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$$\left(\frac{5\sqrt{3} - 3}{2}, \frac{7 + \sqrt{3}}{2} \right)$$

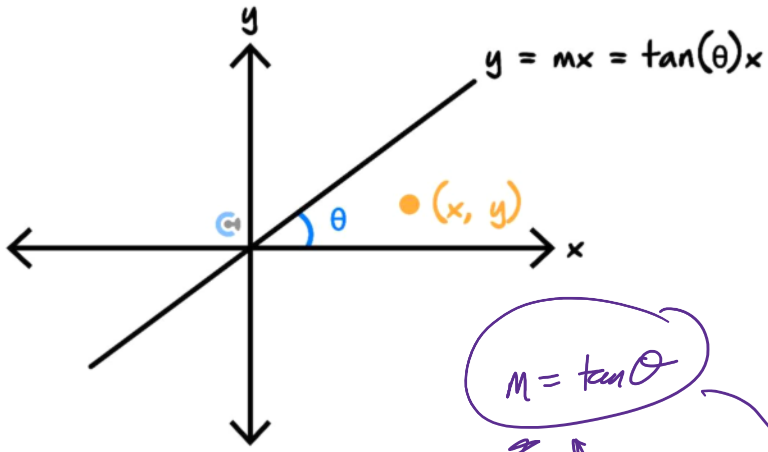
Section D: General Reflections

Sub-Section: Reflections Across a Line $y = mx$

How do we reflect a point around $y = mx$?

Definition

Reflections Across a Line $y = mx$



$m = \tan\theta$

Reflection around $y = mx = \tan(\theta)x$

- θ is the angle the reflection line meets with the x-axis.

$$\textit{Transformation Matrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

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Question 9 Walkthrough.

a. State the transformation matrix for the reflection around $y = \frac{1}{\sqrt{3}}x$.

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = 30^\circ$

$\times 2$

60°

$$\begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

b. Hence, find the image of $(3, 1)$ after it has been reflection around $y = \frac{1}{\sqrt{3}}x$.

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} - \frac{1}{2} \end{bmatrix}$$

$$\left(\frac{3}{2} + \frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2} - \frac{1}{2} \right)$$

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Question 10

$\theta = \tan^{-1}(-\sqrt{3})$

$= -60^\circ$

$\times 2$

-120°

a. State the transformation matrix for the reflection around $y = -\sqrt{3}x$.

$$\begin{bmatrix} \cos(-120^\circ) & \sin(-120^\circ) \\ \sin(-120^\circ) & -\cos(-120^\circ) \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b. Hence, find the image of $(-1, 1)$ after it has been reflection around $y = -\sqrt{3}x$.

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} + \frac{1}{2} \end{bmatrix}$$

NOTE: If the angle is clockwise, we measure it negatively.

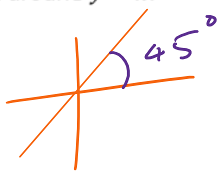
Space for Personal Notes

What about reflection around $y = x$?

inverse

Exploration: Understanding Reflection Around $y = x$

- Consider a transformation matrix for reflection around $y = x$.



- What angle does $y = x$ make with the x-axis?

  $45^\circ = \theta$

  Hence, construct the transformation matrix below.

$$\begin{array}{cc} 90^\circ & 90^\circ \\ \end{array}$$
$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$90^\circ$$

- Apply the transformation to the point (x, y). What do you see?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

- As you can see, x and y swaps!
- That makes sense as inverse relation is found by reflecting around $y = x$.

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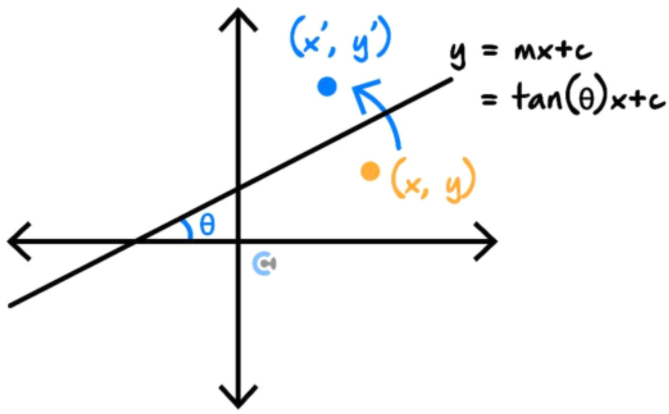
Sub-Section: Reflections Around a Line $y = mx + c$

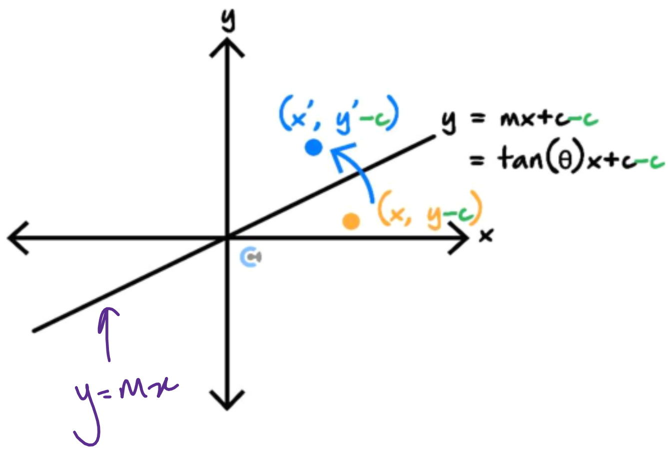
Discussion: Since we know how to reflect around $y = mx$, how can we reflect around $y = mx + c$?

Exploration: Reflection Around $y = mx + c$

- Consider the reflection around $y = mx + c$.



- Since we know how to reflect around $y = mx$, let's translate it down by c.



- How do we go from $(x, y - c)$ to $(x', y' - c)$?

$$\begin{bmatrix} x' \\ y' - c \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y - c \end{bmatrix}$$

- Now, expand the matrices and make $\begin{bmatrix} x' \\ y' \end{bmatrix}$ the subject!

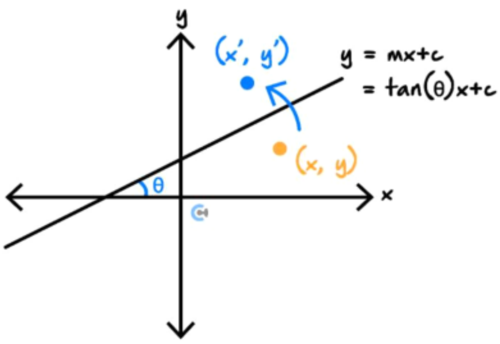
$+[\quad]$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

Definition

Reflection Across a Line $y = mx + c$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}}_{2} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ c \end{bmatrix}}_{1} \right) + \underbrace{\begin{bmatrix} 0 \\ c \end{bmatrix}}_{3}$$

- The idea is that we:

1. Translate the points by $(0, -c)$ so that the line $y = mx + c$ becomes $y = mx$.
2. Reflect the point around $y = mx$ using the transformation matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.
3. Translate the points by $(0, c)$ so that we go back to the line $y = mx + c$.

Question 11 Walkthrough.

$\theta = \tan^{-1}(\sqrt{3}) = 60^\circ$

State the image of $(1, 1)$ after the reflection around the line $y = \sqrt{3}x + 1$.

$d\ 1$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(120^\circ) & \sin(120^\circ) \\ \sin(120^\circ) & -\cos(120^\circ) \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}$$

Space for Personal Notes

Question 12

State the image of $(2, -1)$ after the reflection around the line $y = \frac{1}{\sqrt{3}}x - 1$.

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= 30^\circ$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

2 1 3

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 \\ \sqrt{3} - 1 \end{bmatrix}$$

Contour Check

Learning Objective: [4.3.1] – Transformations of graphs

Key Takeaways

Transformation of Functions/Graphs:

$$y = f(x) \rightarrow y' = f(x')$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

- **Steps:**

1. Find $x' = f(x)$ and $y' = g(y)$.
2. Rearrange and make x, y the subject.
3. Substitute into the original function.
4. Remove $'$ on the variables.

Learning Objective: [4.3.2] – Rotations around points

Key Takeaways

Rotation θ in the Anticlockwise Direction:

Rotation θ in the Anticlockwise Direction

$$\text{Transformation Matrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- **Rotation Around Any Point (a, b):**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) + \begin{bmatrix} a \\ b \end{bmatrix}$$

2 1 3

- The idea is that we:

1. Translate the points by $(-a, -b)$ so that the centre becomes the origin.
2. Rotate the point around the origin using the transformation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
3. Translate the points by (a, b) so that we go back to (a, b) being the centre.

Learning Objective: [4.3.3] – Reflections in lines

Key Takeaways

- **Reflections Across a Line $y = mx$:**

Reflection around $y = mx = \tan(\theta)x$

- θ is the angle the reflection line meets with the x-axis.

$$\text{Transformation Matrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

- **Reflection across a line $y = mx + c$**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

2 1

- The idea is that we:

1. Translate the points by $(0, -c)$ so that the line $y = mx + c$ becomes $y = mx$.
2. Reflect the point around $y = mx$ using the transformation matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.
3. Translate the points by c so that we go back to the line $y = mx + c$.

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