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VCE Specialist Mathematics ½
Transformations II [4.3]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [4.3.1]** – Transformations of Graphs
- ☐ **SM12 [4.3.2]** – Rotations Around Points
- ☐ **SM12 [4.3.3]** – Reflections in Lines

Section A: Recap of [4.2] Transformations

Let's do a quick recap of what we did last week!

Linear Transformations

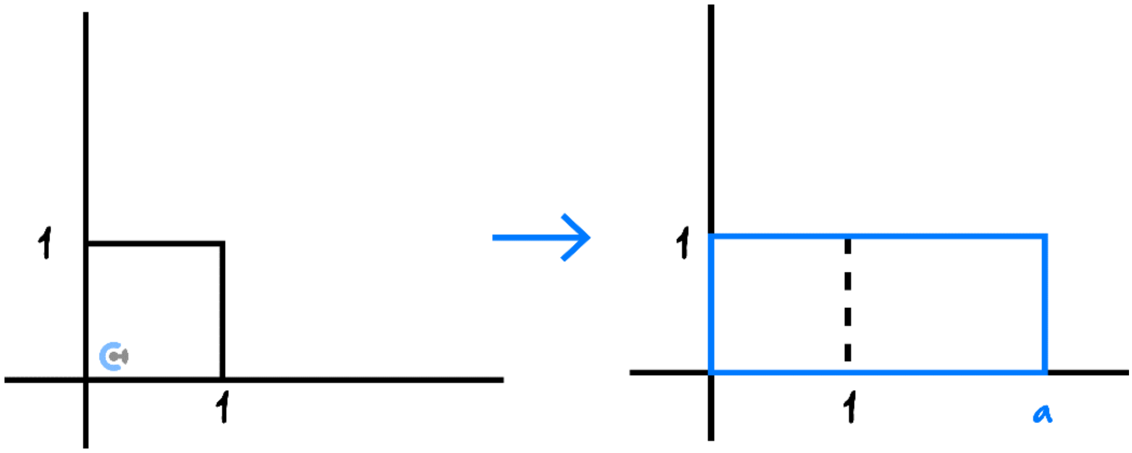
Definition

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- The (x', y) represents the new points and is called an __image__.
- Original point (x, y) is called the __pre-image__.
- A is the transformation matrix.

Dilation from the y-axis

Definition



Dilation by a factor a from the y-axis

- Dilation from the y-axis changes the __x-Value__.

$$\textit{Transformation Matrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

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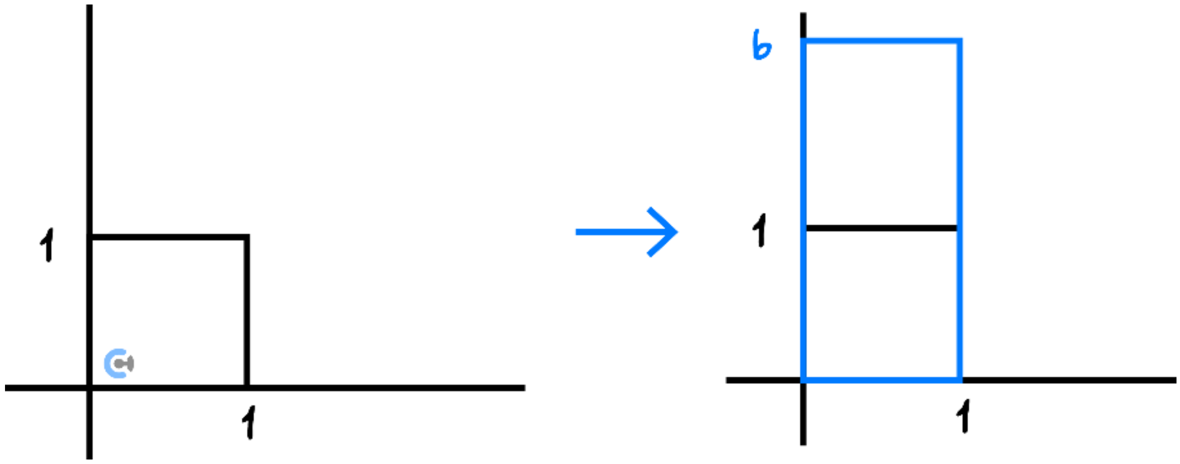
Definition

Dilation from the x-axis



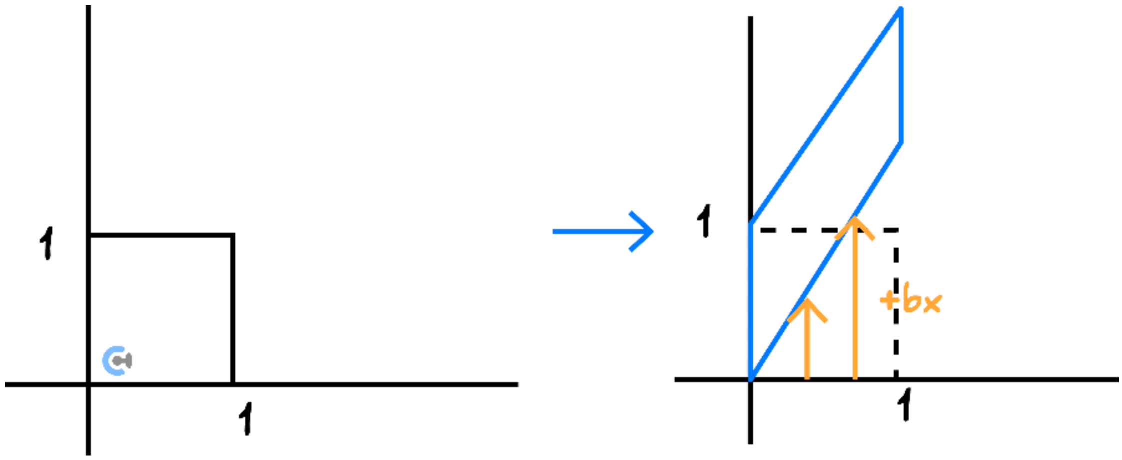
Dilation by a factor b from the x-axis

- Dilation from the x-axis changes the y-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

Definition

Shear Parallel to the y-axis



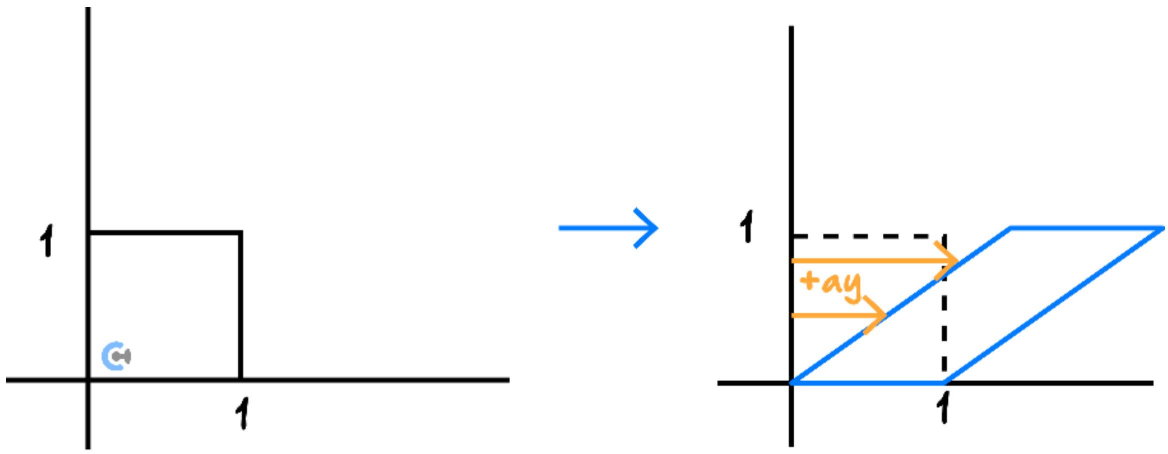
Shear of a factor b parallel to the y-axis

- Shear parallel to y-axis changes the y by a multiple of x

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

Definition

Shear Parallel to the x-axis



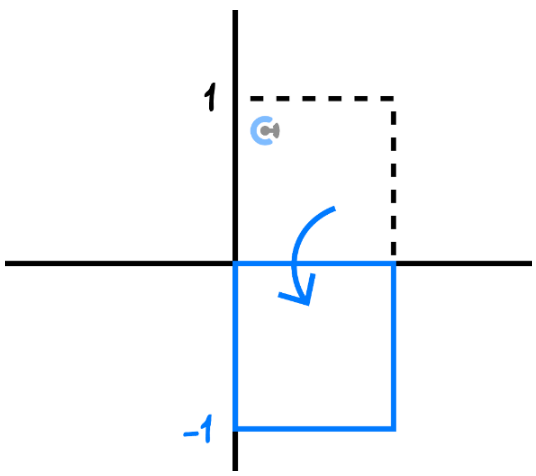
Shear of a factor a parallel to the x-axis

- Shear parallel to x-axis changes the x by a multiple of y

$$Transformation\ Matrix = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

Definition

Reflection Around x-axis



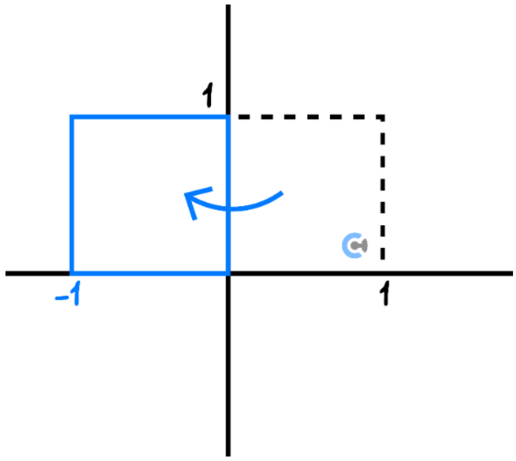
Reflection in the x-axis

- Reflection in the x-axis changes the y.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definition

Reflection Around y-axis



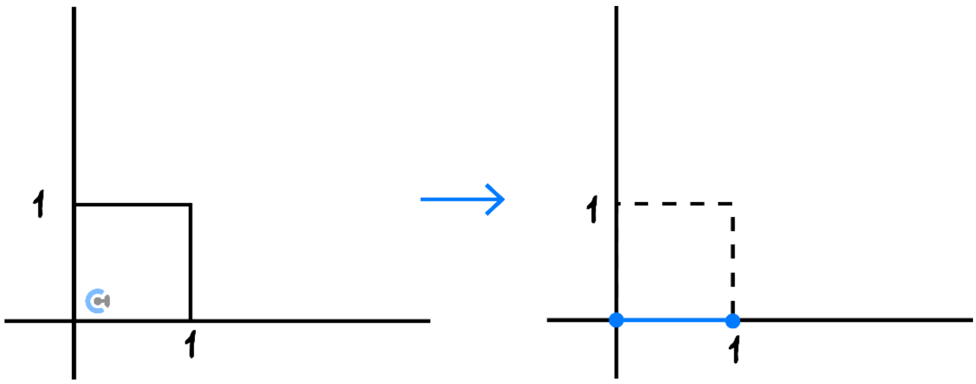
Reflection in the y-axis

- Reflection in the y-axis changes the ____x____.

$$\textit{Transformation Matrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition

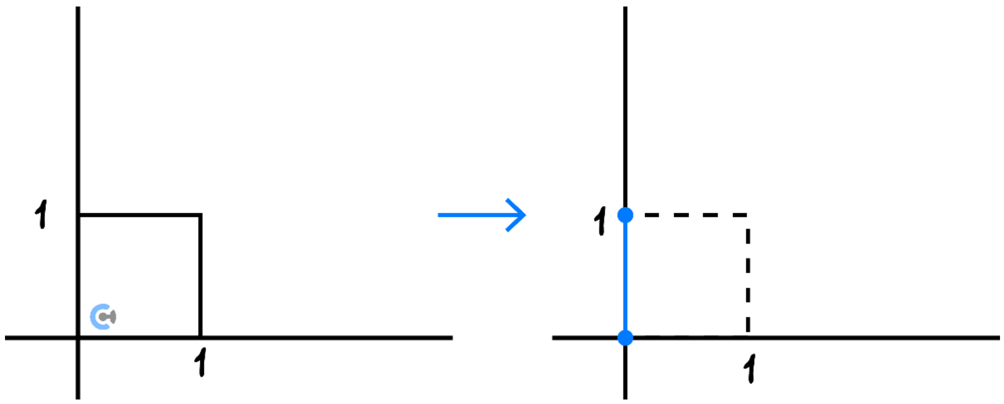
Projections



Projection onto x-axis

- The ____y____ becomes 0.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto y-axis

- The ____x____ becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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Question 1 Walkthrough.

a. State the transformation matrix for the shear of a factor 2 parallel to the y-axis and dilation by a factor 3 from the x-axis.

① ②

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+0(2) & 1(0)+0(1) \\ 0(1)+3(2) & 0(0)+3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 3 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate $(3, 1)$.

$$\begin{bmatrix} 1 & 0 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(3)+0(1) \\ 6(3)+3(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 21 \end{bmatrix}$$

$(3, 21)$

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Question 2

a. State the transformation matrix for dilation by a factor 2 from the y-axis and the shear of a factor 3 parallel to the x-axis.

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate $(2, 4)$.

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \end{bmatrix}$$

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Section B: Transformations of Graphs

Discussion: If we can transform points, how can we transform functions/graphs?

$x' = 3x + 2$

$\hookrightarrow x = \ldots$

$\hookrightarrow$ Sub it in

Transformation of Functions/Graphs

Definition

$$y = f(x) \to y' = f(x')$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

- Steps:

1. Find $x' = f(x)$ and $y' = g(y)$.
2. Rearrange and make x, y the subject.
3. Substitute into the original function.
4. Remove $'$ on the variables.

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Question 3 Walkthrough.

a. State the transformation matrix for dilation by a factor $\frac{1}{2}$ from the y-axis and reflection around the x-axis.

① ②

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}$$

b. Find the image of (x, y) under the transformation described in **part a**.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x \\ -y \end{bmatrix}$$

$$\begin{cases} x' = \frac{1}{2}x \\ y' = -y \end{cases}$$

Consider a function $f(x) = \sqrt{x+3} - 1$. It is known that all the points of $f(x)$ have been transformed by the transformation matrix found in **part a**.

c. Find the transformed graph.

$$\begin{cases} x = 2x' \\ y = -y' \end{cases}$$

$$y = \sqrt{x+3} - 1$$

$$-y' = \sqrt{2x'+3} - 1$$

$$y' = -\sqrt{2x+3} + 1$$

$$f(x) = -\sqrt{2x+3} + 1$$

Your turn!

Question 4

a. State the transformation matrix for dilation by a factor of 5 from the y-axis and a reflection around the y-axis.

① ②

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Find the image of (x, y) under the transformation described in **part a**.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5x \\ y \end{bmatrix}$$

Consider a function $f(x) = (x-4)^2 + 3$.

It is known that all the points of $f(x)$ have been transformed by the transformation matrix found in **part a**.

c. Find the transformed graph.

$$x' = -5x \qquad y' = y \Rightarrow x = -\frac{1}{5}x' \qquad y = y'$$

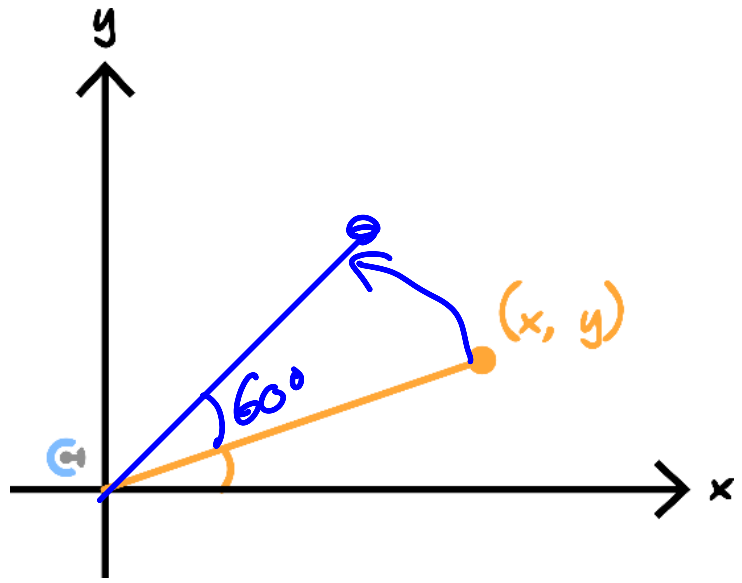
$$y' = \left(-\frac{1}{5}x' - 4\right)^2 + 3 \Rightarrow y = \left(\frac{1}{5}x + 4\right)^2 + 3$$

Section C: Rotations

Sub-Section: Rotations Around the Origin

How do we rotate a point around the origin?

Rotation Around the Origin



Rotation θ in the Anticlockwise Direction

$$Transformation\ Matrix = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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Question 5 Walkthrough.

a. State the transformation matrix for rotation around the origin 60° anticlockwise.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b. Hence, find the image of $(3, 1)$ after it has been rotated around the origin 60° anticlockwise.

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \frac{3\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1+3\sqrt{3}}{2} \end{bmatrix}$$

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Question 6

a. State the transformation matrix for rotation around the origin 30° clockwise.

$$\begin{bmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

b. Hence, find the image of $(1, 2)$ after it has been rotated around the origin 30° clockwise.

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}+1 \\ -\frac{1}{2}+\sqrt{3} \end{bmatrix}$$

NOTE: If the angle is clockwise, we measure it negatively.

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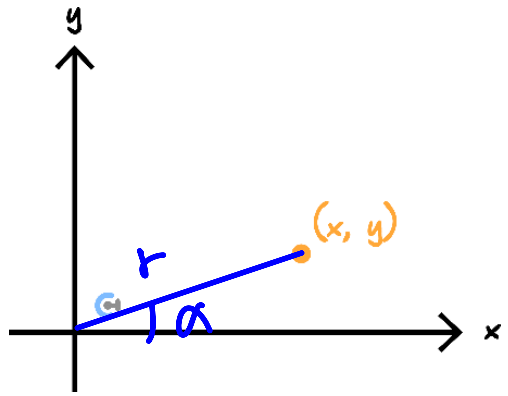
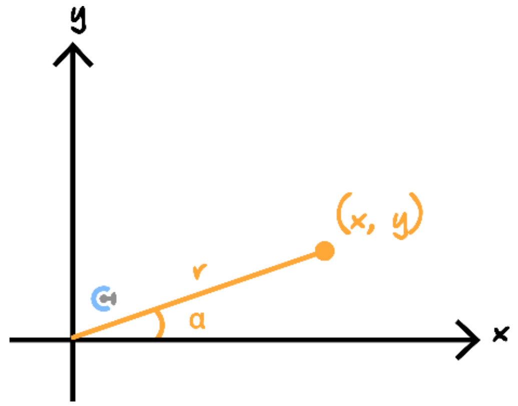
How does this work?

Exploration: Understanding Rotations Around the Origin

- Consider a pre-image (x, y).



- Let's say the point (x, y) __r__ is away from the origin and has an angle of __α__ anticlockwise from the x-axis.



$\cos\alpha = \frac{x}{r}$

$\sin\alpha = \frac{y}{r}$

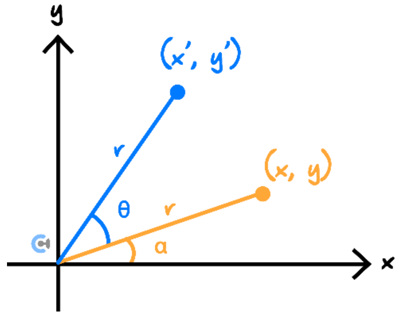
- Using SOHCAHTOA, how can we define x and y in terms of r and α?

$$x = r \cdot \cos\alpha$$

$$y = r \cdot \sin\alpha$$

➤ What happens when we rotate θ anticlockwise around the origin?

Let's visualise the diagram together.



➤ Using SOHCAHTOA, how can we define x' and y' in terms of r and α?

$$x' = r \cdot \cos(\alpha + \theta)$$

$$y' = r \cdot \sin(\alpha + \theta)$$

➤ Using the compound angle formulas, expand the following and substitute in the original x and y!

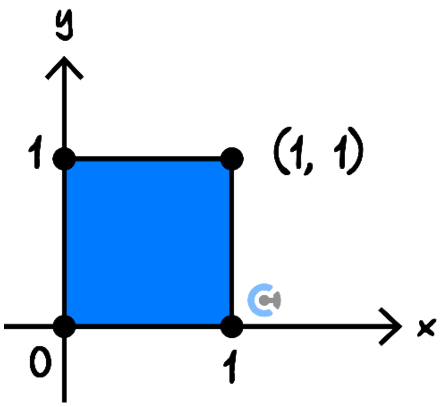
$$\begin{aligned} x' &= r \cdot \cos(\alpha + \theta) \\ &= r \cdot (\cos\alpha \cos\theta - \sin\alpha \sin\theta) \\ &= x \cos\theta - y \sin\theta \\ y' &= r \cdot \sin(\alpha + \theta) \\ &= r(\sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)) \\ &= y \cos\theta + x \sin\theta \end{aligned}$$

➤ Hence, in summary:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

REMINDER: Determinant of Transformation Matrix:

- Given that $A =$ Transformation matrix.



- The unit square was used to visualise how a transformation affects different points.

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

- Determinant of the transformation matrix tells us how the area of the unit circle changes.

Discussion: What would the determinant of the rotation matrix be?

won't change

Exploration: Determinant of the Rotation Matrix

- Consider the rotation transformation matrix:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

ad - bc

- What does the determinant equal to? Evaluate it using algebra!

$\cos^2\theta + \sin^2\theta = 1$

Sub-Section: Rotations Around Any Point

Discussion: Since we know how to rotate around the origin, how can we rotate around any point (a, b)?

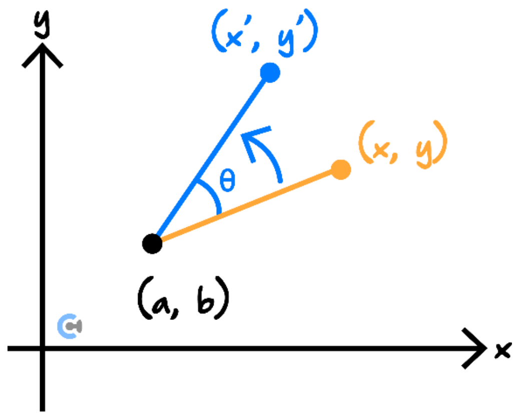
→ translate TO origin

→ Same as before (origin)

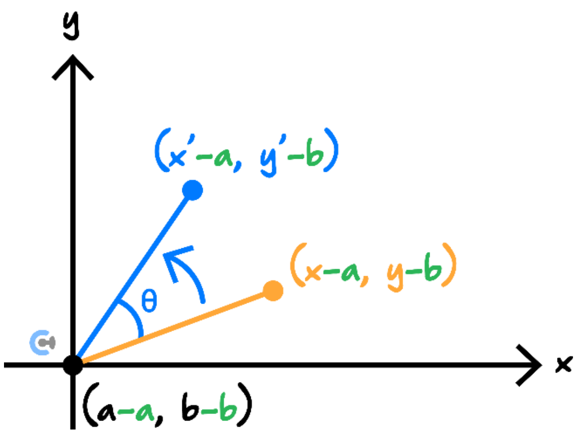
→ translate BACK

Rotations Around Any Point (a, b)

- Consider the rotation θ around the point (a, b) in the anticlockwise direction.



- Since we know how to rotate from the origin, let us translate ourselves to the origin!



- How do we go from $(x-a, y-b)$ to $(x'-a, y'-b)$?

$$\begin{bmatrix} x'-a \\ y'-b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

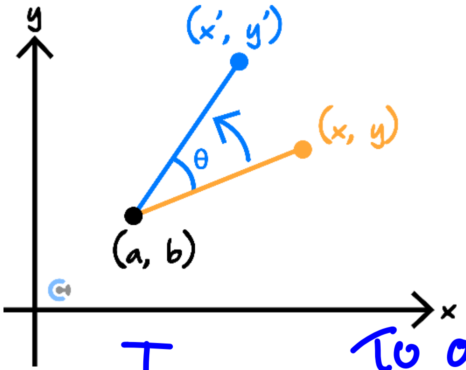
- Now, expand the matrices and make $\begin{bmatrix} x' \\ y' \end{bmatrix}$ the subject!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) + \begin{bmatrix} a \\ b \end{bmatrix}$$

Definition

<u>Rotation Around Any Point (a, b)</u>

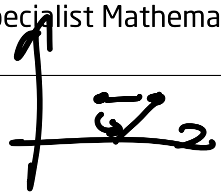


T To origin back

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) + \begin{bmatrix} a \\ b \end{bmatrix}$$

- The idea is that we:

1. Translate the points by $(-a, -b)$ so that the centre becomes the origin.
2. Rotate the point around the origin using the transformation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
3. Translate the points by (a, b) so that we go back to (a, b) being the centre.



Question 7 Walkthrough.

State the image of $(1, 1)$ after the rotation around the point $(2, 1)$, $60°$ anticlockwise.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} 1 - 2 \\ 1 - 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 - \frac{\sqrt{3}}{2} \end{bmatrix}$$

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$$\left(\frac{3}{2}, 1 - \frac{\sqrt{3}}{2} \right)$$

Question 8

State the image of $(4, 2)$ after the rotation around the point $(-1, 1)$, $30°$ anticlockwise.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix} \begin{bmatrix} 4+1 \\ 2-1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5\sqrt{3}-1}{2} \\ \frac{5+\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{3}-3}{2} \\ \frac{7+\sqrt{3}}{2} \end{bmatrix}$$

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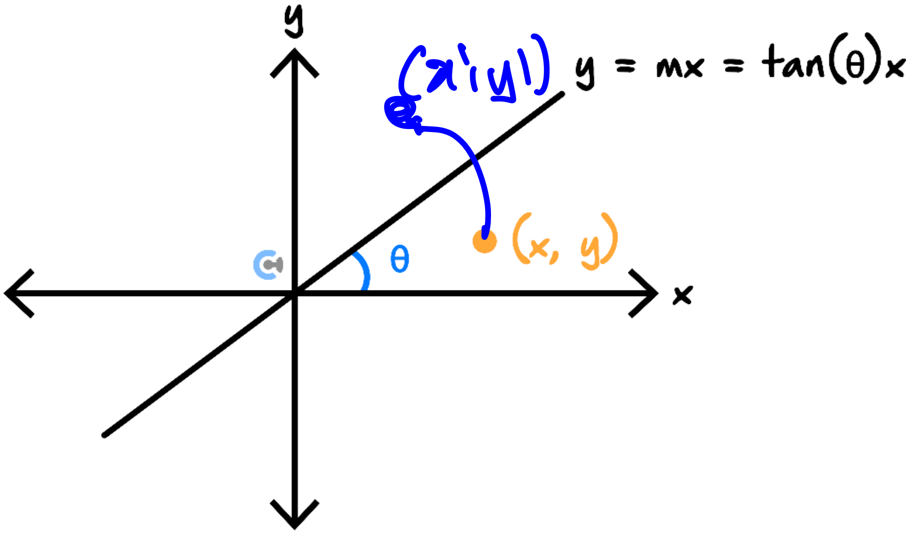
Section D: General Reflections

Sub-Section: Reflections Across a Line $y = mx$

How do we reflect a point around $y = mx$?

Definition

Reflections Across a Line $y = mx$



Reflection around $y = mx = \tan(\theta)x$

- θ is the angle the reflection line meets with the x-axis.

$$Transformation\ Matrix = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

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Question 9 Walkthrough.

a. State the transformation matrix for the reflection around $y = \frac{1}{\sqrt{3}}x$.

$\rightarrow \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = \frac{\pi}{6}$

$$A = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

b. Hence, find the image of $(3, 1)$ after it has been reflection around $y = \frac{1}{\sqrt{3}}x$.

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3+\sqrt{3}}{2} \\ \frac{3\sqrt{3}-1}{2} \end{bmatrix}$$

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Question 10

a. State the transformation matrix for the reflection around $y = -\sqrt{3}x$.

$$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

$$A = \begin{bmatrix} \cos\left(-\frac{2\pi}{3}\right) & \sin\left(-\frac{2\pi}{3}\right) \\ \sin\left(-\frac{2\pi}{3}\right) & -\cos\left(-\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b. Hence, find the image of $(-1, 1)$ after it has been reflection around $y = -\sqrt{3}x$.

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{\sqrt{3}+1}{2} \end{bmatrix}$$

NOTE: If the angle is clockwise, we measure it negatively.

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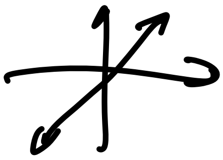
What about reflection around $y = x$?

Exploration: Understanding Reflection Around $y = x$

- Consider a transformation matrix for reflection around $y = x$.
- What angle does $y = x$ make with the x-axis?



  - 45°
  - Hence, construct the transformation matrix below.

$$\begin{bmatrix} \cos(\cancel{2\theta}\,90^\circ) & \sin(\cancel{2\theta}\,90^\circ) \\ \sin(\cancel{2\theta}\,90^\circ) & -\cos(\cancel{2\theta}\,90^\circ) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Apply the transformation to the point (x, y). What do you see?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

- As you can see, x & y swaps!
- That makes sense as inverse relation is found by reflecting around $y = x$.

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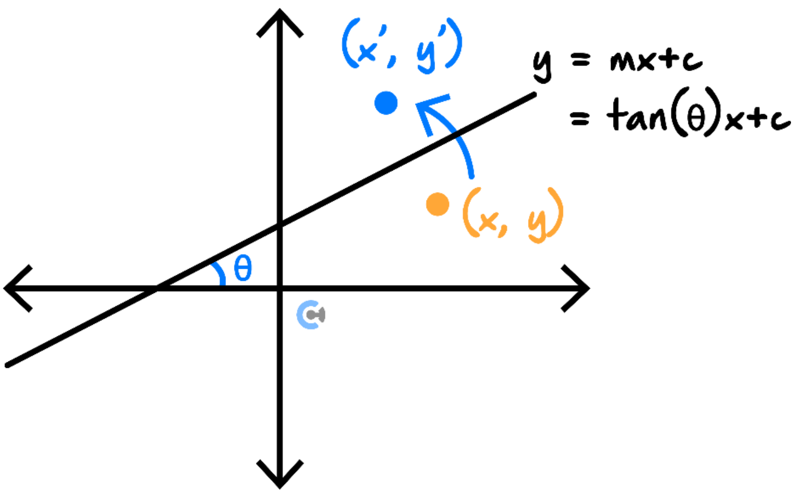
Sub-Section: Reflections Around a Line $y = mx + c$

Discussion: Since we know how to reflect around $y = mx$, how can we reflect around $y = mx + c$?

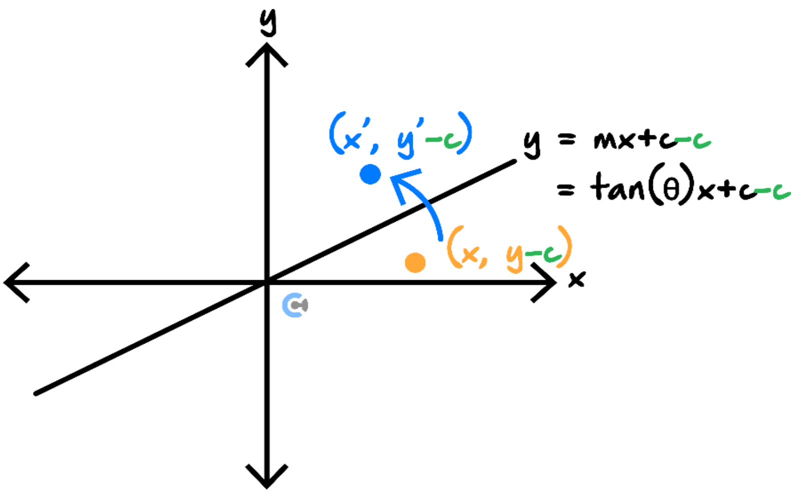
→ origin
→ rotate
→ back

Exploration: Reflection Around $y = mx + c$

- Consider the reflection around $y = mx + c$.



- Since we know how to reflect around $y = mx$, let's translate it down by c.



- How do we go from $(x, y - c)$ to $(x', y' - c)$?

$$\begin{bmatrix} x' \\ y' - c \end{bmatrix} = \begin{bmatrix} \qquad\qquad \\ \qquad\qquad \end{bmatrix} \begin{bmatrix} x \\ y - c \end{bmatrix}$$

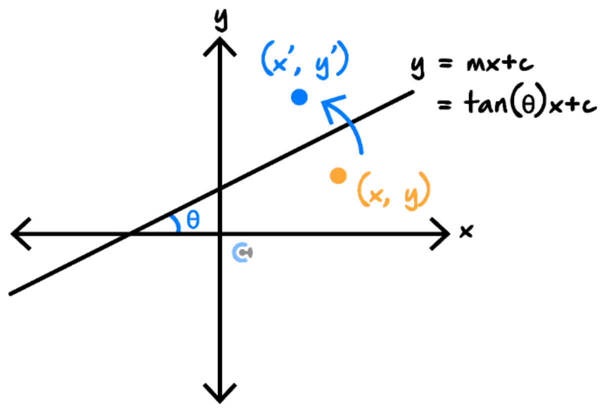
- Now, expand the matrices and make $\begin{bmatrix} x' \\ y' \end{bmatrix}$ the subject!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}}_{\text{transform}} \left(\underbrace{\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix}}_{\text{bring to origin}} \right) + \underbrace{\begin{bmatrix} 0 \\ c \end{bmatrix}}_{\text{bring back}}$$

Definition

Reflection Across a Line $y = mx + c$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

- The idea is that we:

1. Translate the points by $(0, -c)$ so that the line $y = mx + c$ becomes $y = mx$.
2. Reflect the point around $y = mx$ using the transformation matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.
3. Translate the points by $(0, c)$ so that we go back to the line $y = mx + c$.

Question 11 Walkthrough.

State the image of $(1, 1)$ after the reflection around the line $y = \sqrt{3}x + 1$.

$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y - c \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & -\cos(\frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} + 1 \end{bmatrix}$$

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Question 12

State the image of $(2,-1)$ after the reflection around the line $y = \frac{1}{\sqrt{3}}x - 1$.

$\hookrightarrow \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) \end{bmatrix} \begin{bmatrix} x \\ y-c \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \sqrt{3}-1 \end{bmatrix}$$

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Contour Check

Learning Objective: [4.3.1] – Transformations of graphs

Key Takeaways

- Transformation of Functions/Graphs:

$$y = f(x) \rightarrow y' = f(x')$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

- Steps:

1. Find $x' = f(x)$ and $y' = g(y)$.
2. Rearrange and make x, y the subject.
3. Substitute into the original function.
4. Remove ′ on the variables.

Learning Objective: [4.3.2] – Rotations around points

Key Takeaways

- Rotation θ in the Anticlockwise Direction:

Rotation θ in the Anticlockwise Direction

$$\text{Transformation Matrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- **Rotation Around Any Point (a, b):**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) + \begin{bmatrix} a \\ b \end{bmatrix}$$

  - The idea is that we:

    1. Translate the points by $(-a, -b)$ so that the centre becomes the origin.
    2. Rotate the point around the origin using the transformation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
    3. Translate the points by (a, b) so that we go back to (a, b) being the centre.

Learning Objective: [4.3.3] - Reflections in lines

Key Takeaways

- **Reflections Across a Line $y = mx$:**

$$\text{Reflection around } y = mx = \tan(\theta)x$$

- θ is the angle the reflection line meets with the x-axis.

$$Transformation\ Matrix = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Reflection across a line $y = mx + c$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

- The idea is that we:

1. Translate the points by $(0, -c)$ so that the line $y = mx + c$ becomes $y=mx$.
2. Reflect the point around $y = mx$ using the transformation matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.
3. Translate the points by $(0,c)$ so that we go back to the line $y = mx + c$.

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Transformations II [4.3]

Test

21 Marks. 1 Minute Reading. 20 Minutes Writing.

Results:

Test Questions	____________ / 21

Section A: Test Questions (21 Marks)

Question 1 (3 marks)

State whether the statement is **true** or **false**.

Statement	True	False
a. To transform a function, we simply substitute in the x and y in terms of x' and y'.		
b. Rotations and reflections preserve the length of shapes, but dilations and shears do not.		
c. The rotation of θ clockwise around the origin is given by the following matrix: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$		
d. A rotation by $60°$ anti-clockwise about the point $(1, 2)$ is the same as translating one unit down, two units to the left, rotating $60°$ about the origin, and then translating one unit upward, and finally translating two units to the right.		
e. To reflect around a point (a, b), we first translate a units right and b units up.		
f. To reflect a point about the line $y = -2x + 1$, first you need to translate the point one unit down, and then reflect it about the line $y = -2x$.		

Space for Personal Notes

Question 2 (3 marks)

Find the equation of the line $y = \frac{1}{2}x - 1$ after it undergoes a shear of factor -2 parallel to the y-axis.

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Question 3 (4 marks)

Find the matrix corresponding to each of the following linear transformations, and hence find the image of the point $(1,2)$ after undergoing each of the transformations.

a. Rotation by $60°$ anticlockwise. (2 marks)

b. Reflection in the line $y = \frac{1}{\sqrt{3}}x$. (2 marks)

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Question 4 (5 marks)

a. Find the matrix that will reflect the point (x, y) in the line through the origin at an angle of $30°$ to the positive direction of the x-axis. (2 marks)

b. Find the matrix that will reflect the point (x, y) in the line $y = 2x$. (3 marks)

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Question 5 (6 marks)

Find the equation of the graph of $y = 3x + 1$ under a reflection in the line $y = \frac{1}{\sqrt{3}}x + 0$

$\rightarrow \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$$A = \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

transform graph:

(1.) multiply $\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

(2) Rearrange

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$AX = X'$

$X = A^{-1} \cdot X'$

$$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \\ \frac{\sqrt{3}}{2}x' - \frac{y'}{2} \end{bmatrix}$$

Space for Personal Notes

sub

$$y = \frac{(3\sqrt{3}-1)\left((\sqrt{3}-3)x-2\right)}{26}$$

$$= \left(\frac{6}{13} - \frac{5\sqrt{3}}{13}\right)x + \left(1 - \frac{3\sqrt{3}}{13}\right)$$

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