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VCE Specialist Mathematics ½
Transformations II [4.3]
Workbook

Outline:

Recap of [4.2] Transformations	Pg 2-7	Rotations ➤ Rotations Around the Origin ➤ Rotations Around Any Point	Pg 11-20
Transformations of Graphs	Pg 8-10	General Reflections ➤ Reflections Across a Line $y = mx$ ➤ Reflections Around a Line $y = mx + c$	Pg 21-28

Learning Objectives:

- ☐ **SM12 [4.3.1]** – Transformations of Graphs
- ☐ **SM12 [4.3.2]** – Rotations Around Points
- ☐ **SM12 [4.3.3]** – Reflections in Lines

Section A: Recap of [4.2] Transformations

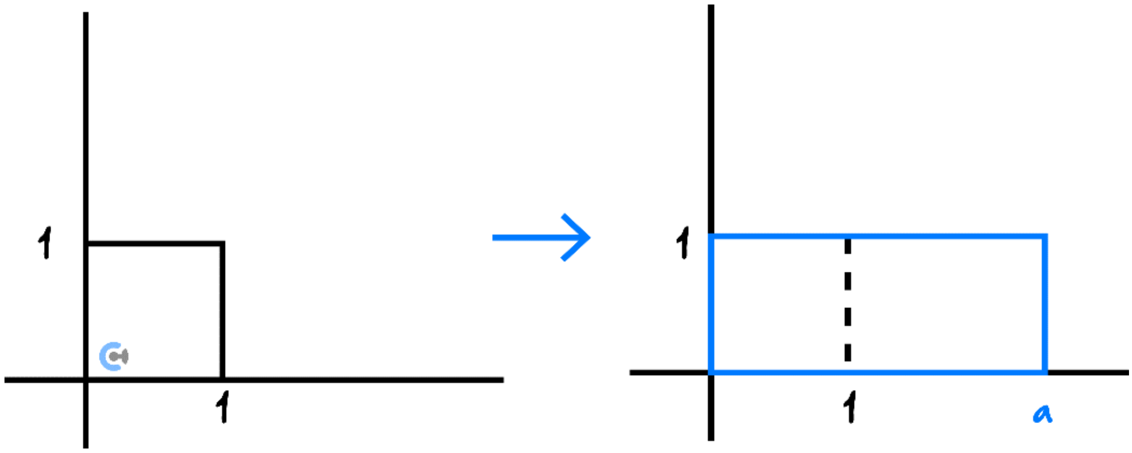
Let's do a quick recap of what we did last week!

Linear Transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- The (x', y) represents the new points and is called an ________________.
- Original point (x, y) is called the ______________________.
- A is the transformation matrix.

Dilation from the y-axis



Dilation by a factor a from the y-axis

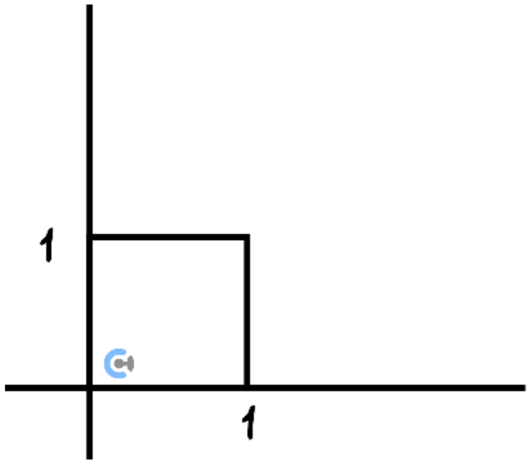
- Dilation from the y-axis changes the _____________________.

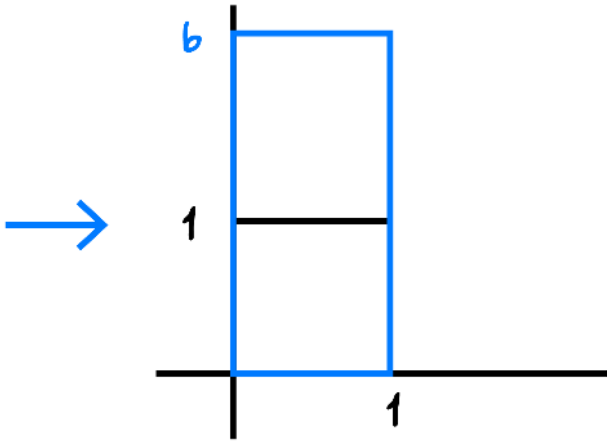
$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

Space for Personal Notes

Definition

Dilation from the x-axis





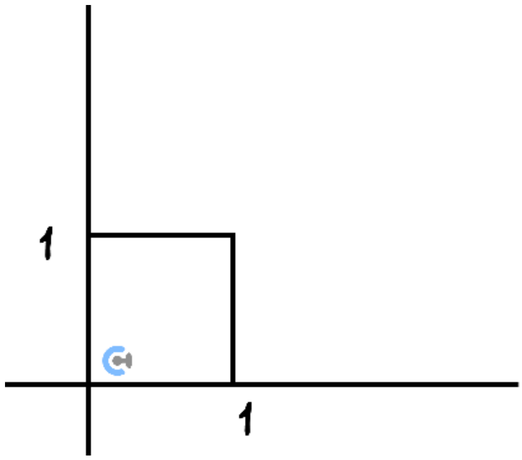
Dilation by a factor b from the x-axis

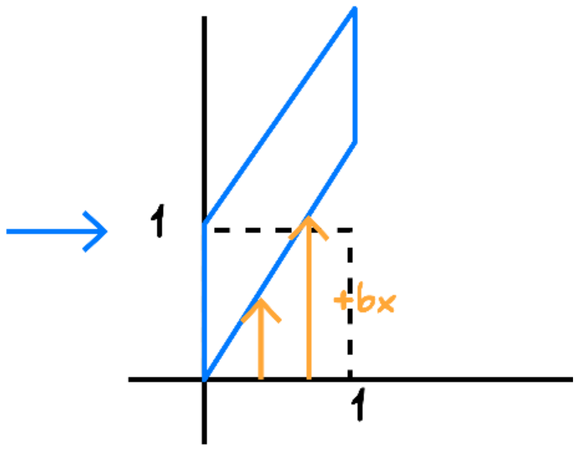
- Dilation from the x-axis changes the ____________________________.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

Definition

Shear Parallel to the y-axis





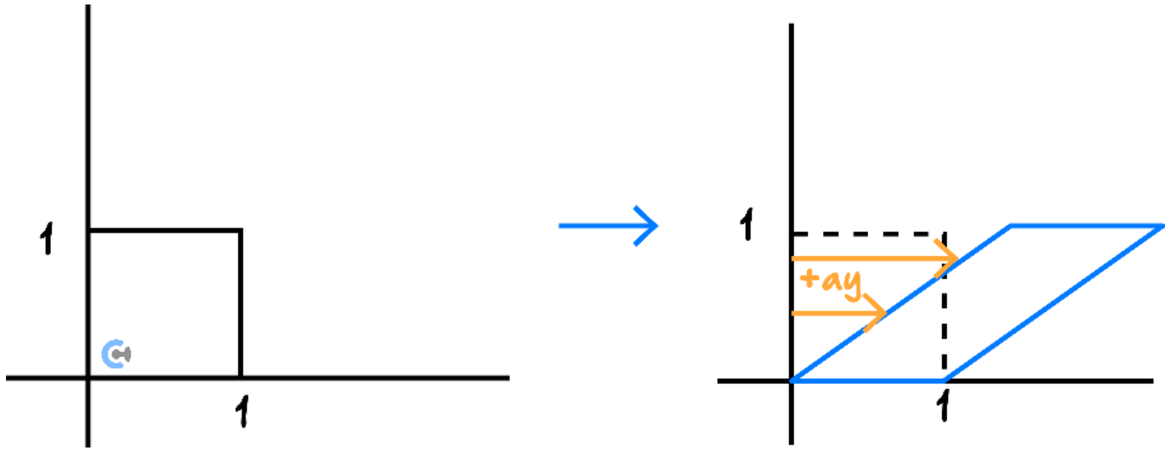
Shear of a factor b parallel to the y-axis

- Shear parallel to y-axis changes the ____.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

Definition

<u>Shear Parallel to the x-axis</u>



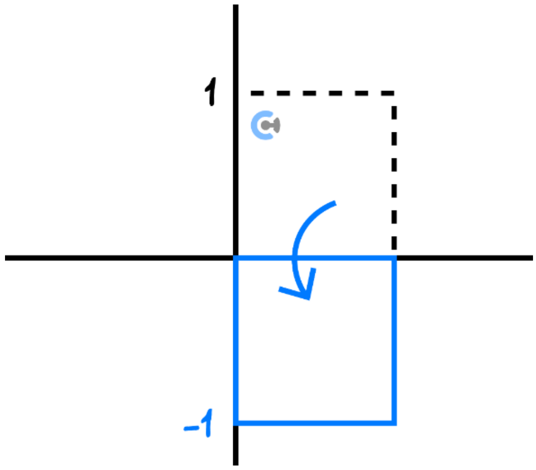
Shear of a factor a parallel to the x-axis

- Shear parallel to x-axis changes the ____________________________________.

$$\boldsymbol{Transformation\ Matrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

Definition

<u>Reflection Around x-axis</u>



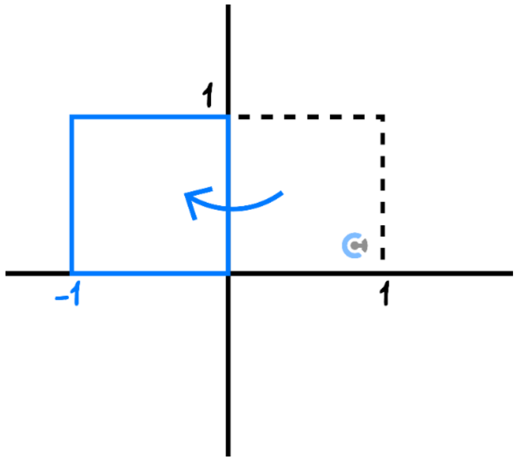
Reflection in the x-axis

- Reflection in the x-axis changes the ____________________.

$$\boldsymbol{Transformation\ Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definition

Reflection Around y-axis



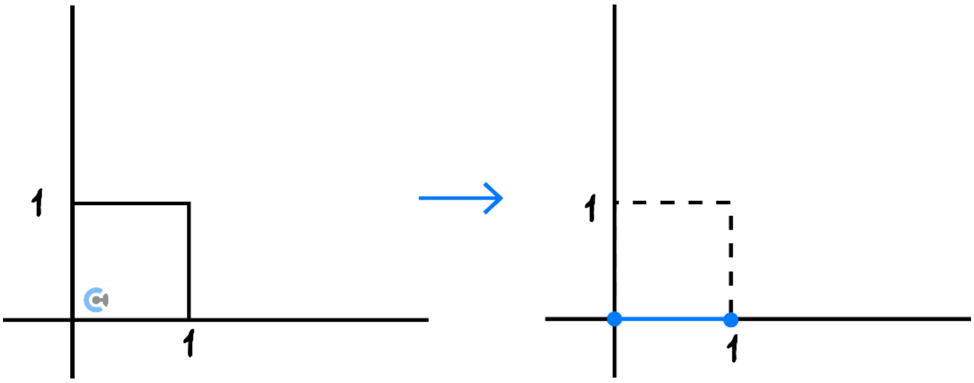
Reflection in the y-axis

- Reflection in the y-axis changes the ____________________.

$$Transformation\ Matrix = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition

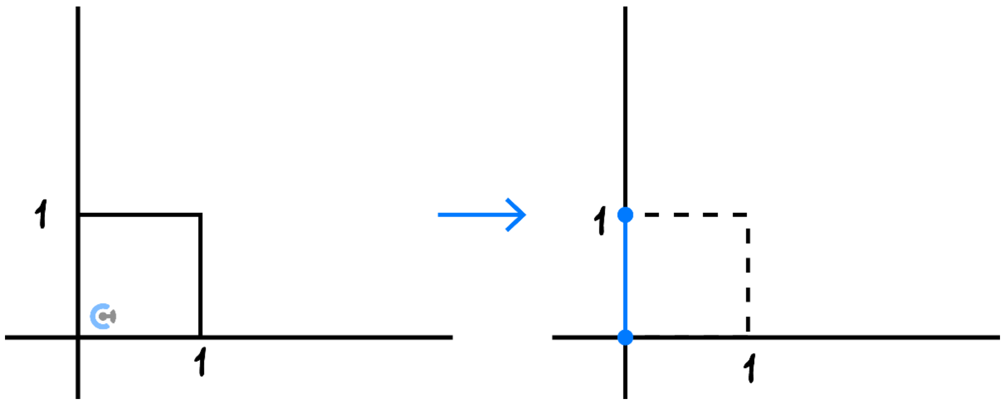
Projections



Projection onto x-axis

- The ____________________ becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto y-axis

- The ______________________ becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Space for Personal Notes

Question 1 Walkthrough.

a. State the transformation matrix for the shear of a factor 2 parallel to the y-axis and dilation by a factor 3 from the x-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate $(3, 1)$.

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Question 2

a. State the transformation matrix for dilation by a factor 2 from the y-axis and the shear of a factor 3 parallel to the x-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate $(2, 4)$.

Space for Personal Notes

Section B: Transformations of Graphs

Discussion: If we can transform points, how can we transform functions/graphs?

Transformation of Functions/Graphs

$$y = f(x) \rightarrow y' = f(x')$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

- Steps:
  1. Find $x' = f(x)$ and $y' = g(y)$.
  2. Rearrange and make x, y the subject.
  3. Substitute into the original function.
  4. Remove $'$ on the variables.

Space for Personal Notes

Question 3 Walkthrough.

a. State the transformation matrix for dilation by a factor $\frac{1}{2}$ from the y-axis and reflection around the x-axis.

b. Find the image of (x, y) under the transformation described in **part a**.

Consider a function $f(x) = \sqrt{x+3} - 1$. It is known that all the points of $f(x)$ have been transformed by the transformation matrix found in **part a**.

c. Find the transformed graph.

Your turn!

Question 4

a. State the transformation matrix for dilation by a factor of 5 from the y-axis and a reflection around the y-axis.

b. Find the image of (x, y) under the transformation described in **part a**.

Consider a function $f(x) = (x - 4)^2 + 3$.

It is known that all the points of $f(x)$ have been transformed by the transformation matrix found in **part a**.

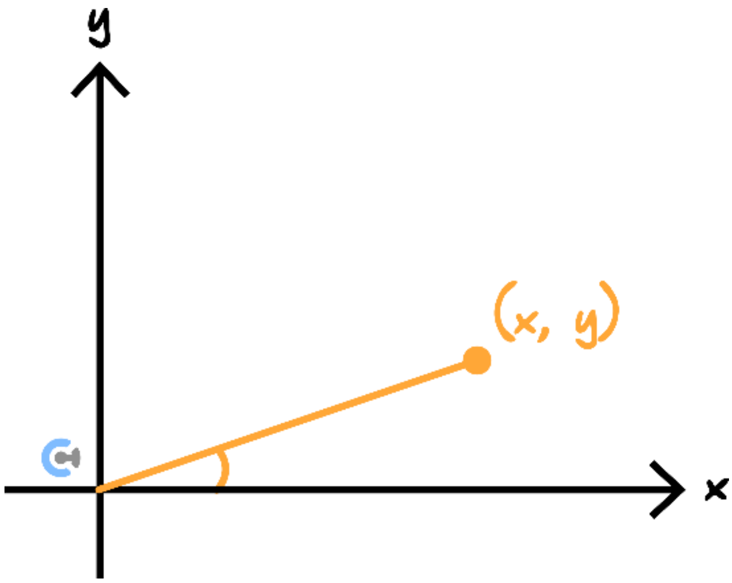
c. Find the transformed graph.

Section C: Rotations

Sub-Section: Rotations Around the Origin

How do we rotate a point around the origin?

Rotation Around the Origin



Rotation θ in the Anticlockwise Direction

$$Transformation\ Matrix = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Space for Personal Notes

Question 5 Walkthrough.

a. State the transformation matrix for rotation around the origin $60°$ anticlockwise.

b. Hence, find the image of $(3, 1)$ after it has been rotated around the origin $60°$ anticlockwise.

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Question 6

a. State the transformation matrix for rotation around the origin $30°$ clockwise.

b. Hence, find the image of $(1, 2)$ after it has been rotated around the origin $30°$ clockwise.

NOTE: If the angle is clockwise, we measure it negatively.

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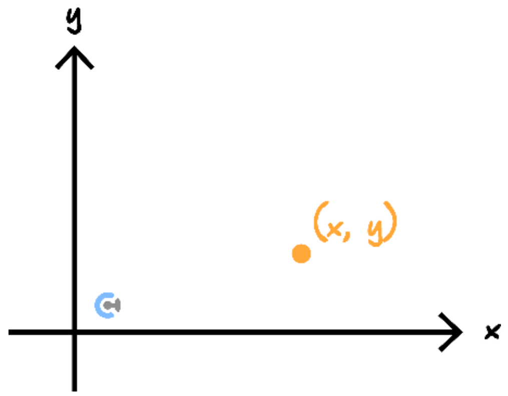
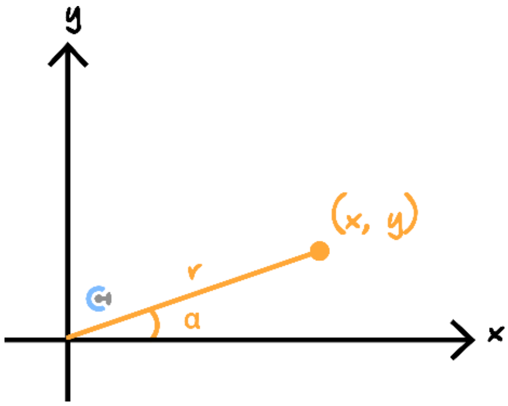
How does this work?

<u>Exploration</u>: Understanding Rotations Around the Origin

- Consider a pre-image (x, y).



- Let's say the point (x, y) ________ is away from the origin and has an angle of ________ anticlockwise from the x-axis.



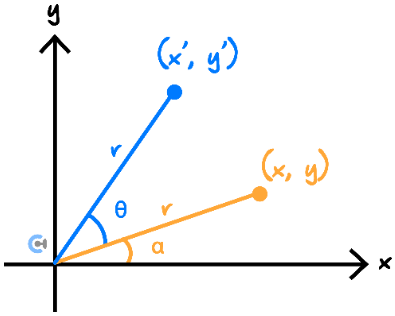
- Using SOHCAHTOA, how can we define x and y in terms of r and α?

$$x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$y = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- What happens when we rotate θ anticlockwise around the origin?

  Let's visualise the diagram together.



- Using SOHCAHTOA, how can we define x' and y' in terms of r and α?

$$x' = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$y' = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Using the compound angle formulas, expand the following and substitute in the original x and y!

$$x' = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$y' = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

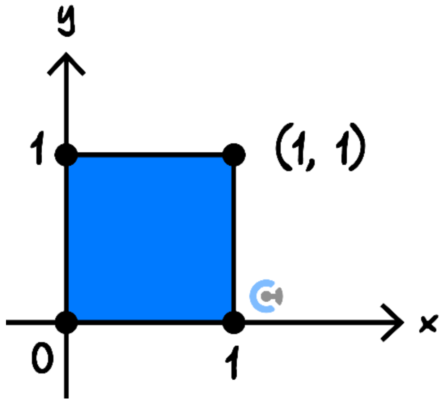
$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Hence, in summary:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix} = \begin{bmatrix} \\ \_\_\_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

REMINDER: Determinant of Transformation Matrix:

- Given that $A =$ Transformation matrix.



- The unit square was used to visualise how a transformation affects different points.

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

- Determinant of the transformation matrix tells us how the area of the unit circle changes.

Discussion: What would the determinant of the rotation matrix be?

Exploration: Determinant of the Rotation Matrix

- Consider the rotation transformation matrix:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

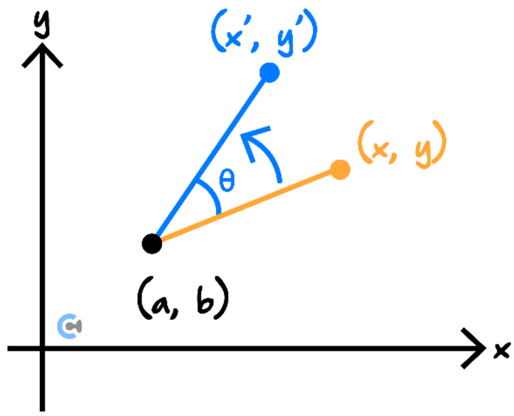
- What does the determinant equal to? Evaluate it using algebra!

Sub-Section: Rotations Around Any Point

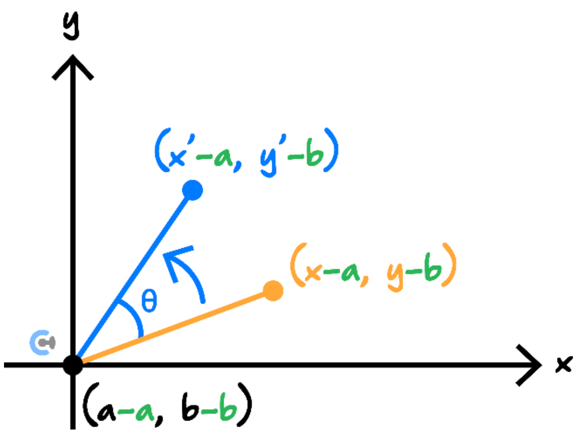
Discussion: Since we know how to rotate around the origin, how can we rotate around any point (a, b)?

Rotations Around Any Point (a, b)

- Consider the rotation θ around the point (a, b) in the anticlockwise direction.



- Since we know how to rotate from the origin, let us translate ourselves to the origin!



- How do we go from $(x - a, y - b)$ to $(x' - a, y' - b)$?

$$\begin{bmatrix} x' - a \\ y' - b \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

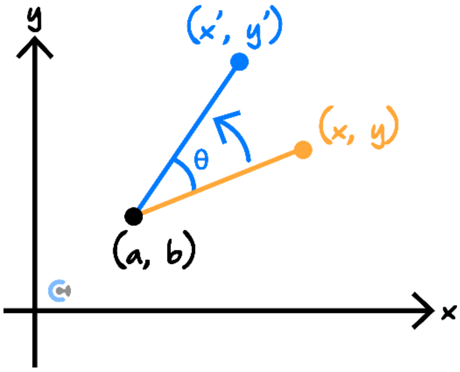
- Now, expand the matrices and make $\begin{bmatrix} x' \\ y' \end{bmatrix}$ the subject!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \_\_\_\_ \end{bmatrix} \right)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \quad \end{bmatrix} \right) + \begin{bmatrix} \_\_\_\_ \end{bmatrix}$$

Definition

<u>Rotation Around Any Point (a, b)</u>



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) + \begin{bmatrix} a \\ b \end{bmatrix}$$

- The idea is that we:

1. Translate the points by $(-a, -b)$ so that the centre becomes the origin.
2. Rotate the point around the origin using the transformation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
3. Translate the points by (a, b) so that we go back to (a, b) being the centre.

Question 7 Walkthrough.

State the image of $(1, 1)$ after the rotation around the point $(2, 1)$, $60°$ anticlockwise.

Space for Personal Notes

Question 8

State the image of $(4, 2)$ after the rotation around the point $(-1, 1)$, $30°$ anticlockwise.

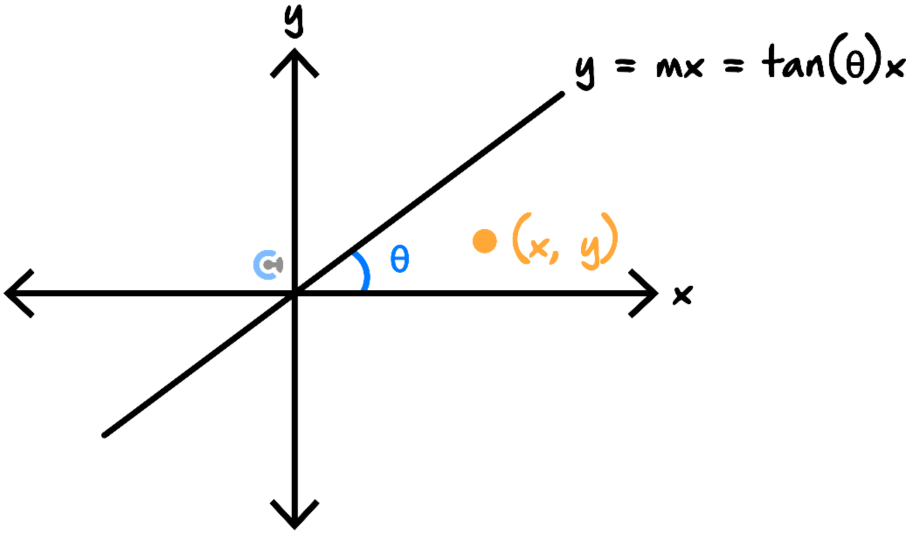
Space for Personal Notes

Section D: General Reflections

Sub-Section: Reflections Across a Line $y = mx$

How do we reflect a point around $y = mx$?

Reflections Across a Line $y = mx$



Reflection around $y = mx = \tan(\theta)x$

- θ is the angle the reflection line meets with the x-axis.

$$Transformation\ Matrix = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Space for Personal Notes

Question 9 Walkthrough.

a. State the transformation matrix for the reflection around $y = \frac{1}{\sqrt{3}}x$.

b. Hence, find the image of $(3, 1)$ after it has been reflection around $y = \frac{1}{\sqrt{3}}x$.

Space for Personal Notes

Question 10

a. State the transformation matrix for the reflection around $y = -\sqrt{3}x$.

b. Hence, find the image of $(-1, 1)$ after it has been reflection around $y = -\sqrt{3}x$.

NOTE: If the angle is clockwise, we measure it negatively.

Space for Personal Notes

What about reflection around $y = x$?

Exploration: Understanding Reflection Around $y = x$

- Consider a transformation matrix for reflection around $y = x$.
- What angle does $y = x$ make with the x-axis?
  - ________________
  - Hence, construct the transformation matrix below.

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

- Apply the transformation to the point (x, y). What do you see?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

- As you can see, ________________ swaps!
- That makes sense as ________________ relation is found by reflecting around $y = x$.

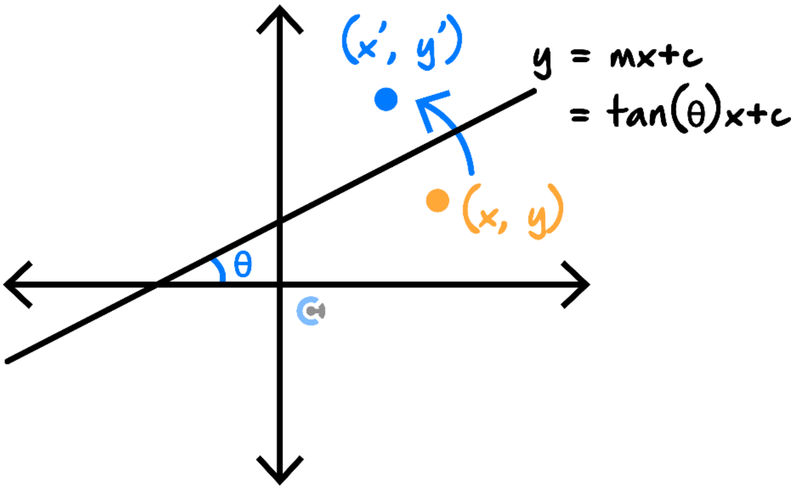
Space for Personal Notes

Sub-Section: Reflections Around a Line $y = mx + c$

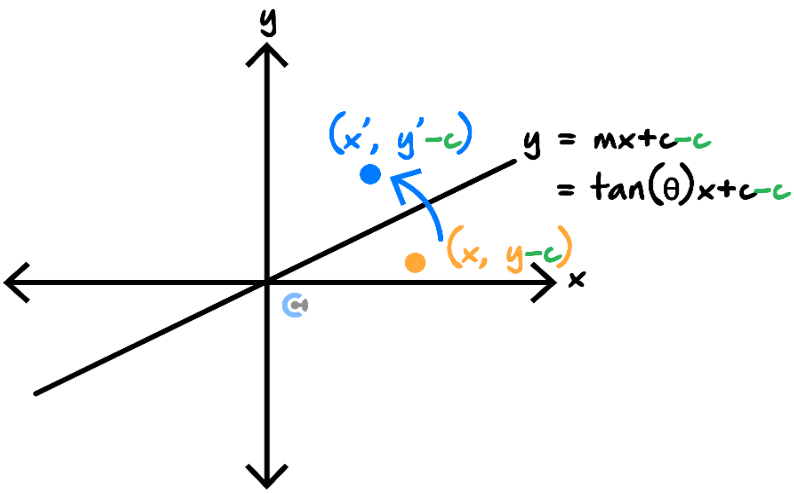
Discussion: Since we know how to reflect around $y = mx$, how can we reflect around $y = mx + c$?

Exploration: Reflection Around $y = mx + c$

- Consider the reflection around $y = mx + c$.



- Since we know how to reflect around $y = mx$, let's translate it down by c.



➤ How do we go from $(x, y - c)$ to $(x', y' - c)$?

$$\begin{bmatrix} x' \\ y' - c \end{bmatrix} = \begin{bmatrix} \underline{\hspace{6cm}} \end{bmatrix} \begin{bmatrix} x \\ y - c \end{bmatrix}$$

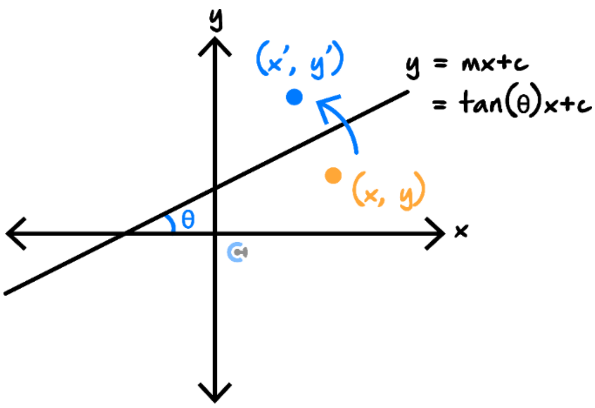
➤ Now, expand the matrices and make $\begin{bmatrix} x' \\ y' \end{bmatrix}$ the subject!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \underline{\hspace{2cm}} \end{bmatrix} \right)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \quad \\ \quad \end{bmatrix} \right) + \begin{bmatrix} \underline{\hspace{2cm}} \end{bmatrix}$$

Definition

<u>Reflection Across a Line $y = mx + c$</u>



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

➤ The idea is that we:

1. Translate the points by $(0, -c)$ so that the line $y = mx + c$ becomes $y = mx$.

2. Reflect the point around $y = mx$ using the transformation matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

3. Translate the points by $(0, c)$ so that we go back to the line $y = mx + c$.

Question 11 Walkthrough.

State the image of $(1, 1)$ after the reflection around the line $y = \sqrt{3}x + 1$.

Space for Personal Notes

Question 12

State the image of $(2, -1)$ after the reflection around the line $y = \frac{1}{\sqrt{3}}x - 1$.

Space for Personal Notes

Contour Check

Learning Objective: [4.3.1] – Transformations of graphs

Key Takeaways

- **Transformation of Functions/Graphs:**

$$y = f(x) \to y' = f(x')$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

- Steps:

1. Find $x' = f(x)$ and $y' = g(y)$.
2. Rearrange and make __________ the subject.
3. Substitute into the original ____________.
4. Remove $'$ on the variables.

Learning Objective: [4.3.2] – Rotations around points

Key Takeaways

- **Rotation θ in the Anticlockwise Direction:**

Rotation θ in the Anticlockwise Direction

$$\textit{Transformation Matrix} = \begin{bmatrix} \quad \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{bmatrix}$$

- Rotation Around Any Point (a, b):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) + \begin{bmatrix} a \\ b \end{bmatrix}$$

  - The idea is that we:
    1. Translate the points by $(-a, -b)$ so that the centre becomes the ______________.
    2. Rotate the point around the origin using the transformation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
    3. ____________________ the points by (a, b) so that we go back to (a, b) being the centre.

Learning Objective: [4.3.3] - Reflections in lines

Key Takeaways

- Reflections Across a Line $y = mx$:

Reflection around $y = mx = \tan(\theta)x$

- θ is the angle the reflection line meets with the x-axis.

$$Transformation\ Matrix = \begin{bmatrix} \quad \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{bmatrix}$$

Reflection across a line $y = mx + c$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

- The idea is that we:

  1. Translate the points by $(0, -c)$ so that the line $y = mx + c$ becomes __________.
  2. Reflect the point around $y = mx$ using the transformation matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.
  3. Translate the points by _________ so that we go back to the line $y = mx + c$.

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