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VCE Specialist Mathematics ½
Transformations II [4.3]
Homework Solutions

Admin Info & Homework Outline:

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Section A: Compulsory Questions

Sub-Section [4.3.1]: Transformations of Graphs

Question 1

a. State the transformation matrix for dilation by factor 2 from the x-axis, followed by a reflection in the y-axis.

$$D_x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T = R_y D_x = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

b. Find the image of the point (x, y) under this transformation.

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ 2y \end{bmatrix}$$

c. Consider the graph $f(x) = (x-3)^2 - 1$. All points of the graph have been transformed by the matrix in **part a.** Write the equation of the transformed graph.

We have $x' = -x \implies x = -x'$ and $y' = 2y \implies y = \frac{y}{2}$. Sub these into the equation without the dashes.

$$\frac{y}{2} = (-x-3)^2 - 1 \Rightarrow y = 2(x+3)^2 - 2$$

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Question 2

a. State the transformation matrix for dilation by factor 3 from the y-axis, followed by a reflection in the x-axis, and then a shear in the x-direction with shear factor 1.

$$D_y = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$T = S_x R_x D_y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix}$$

b. Find the image of the point (x, y) under this transformation.

$$\begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - y \\ -y \end{bmatrix}$$

c. Consider the graph $f(x) = 6x + 7$. All points of the graph have been transformed by the matrix in **part a.** Write an equation for the transformed graph.

We have $x' = 3x - y$ and $y' = -y \implies y = -y'$. Thus $x' = 3x + y' \implies x = \frac{x'}{3} - \frac{y'}{3}$.
Removing the dashes and subbing in to $y = f(x)$ we have:

$$-y = 2x - 2y + 7 \implies y = 2x + 7$$

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Question 3

a. State the transformation matrix for a dilation by factor $\frac{1}{2}$ from the x-axis, followed by a rotation of $\frac{\pi}{2}$ clockwise.

$$D_x = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T = RD_x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{bmatrix}$$

b. Find the image of (x, y) under this transformation.

$$\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{y}{2} \\ -x \end{bmatrix}$$

c. Consider the graph $f(x) = x^2 + 2x$. All points have been transformed by the matrix in **part a.** Write the equation of the transformed graph.

Use $x' = \frac{1}{2}y \implies y = 2x'$ and $y' = -x \implies x = -y'$:

$$2x = y^2 - 2y \implies x = \frac{y^2}{2} - y$$

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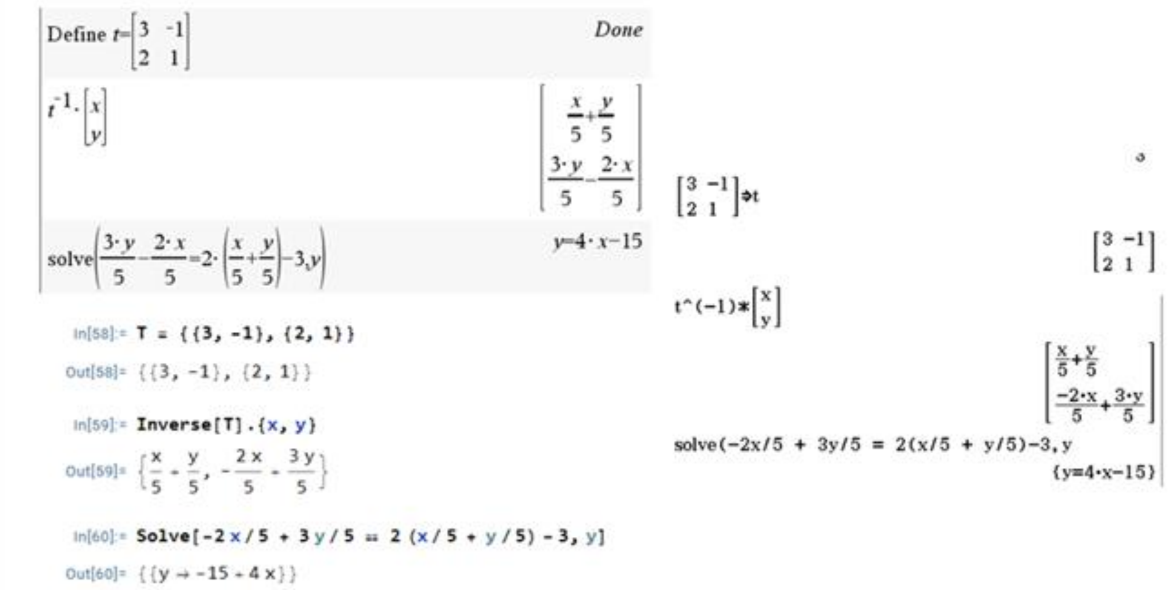
Question 4 Tech-Active.

Find the equation of the line $y = 2x - 3$ under the transformation matrix $T = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$.

$y = 4x - 15$



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Sub-Section [4.3.2]: Rotations Around Points

Question 5

a. A point $P = (3, 1)$ is rotated 90° anticlockwise about the origin. Find the coordinates of the image point.

A 90° anticlockwise rotation about the origin maps $(x, y) \to (-y, x)$. So, $(3, 1) \to (-1, 3)$.

b. The graph of $f(x) = x^2$ is rotated 180° about the origin. Write the equation of the transformed graph.

A 180° rotation about the origin maps $(x, y) \to (-x, -y)$. So the new graph is $-y = (-x)^2 \implies y = -x^2$.

Question 6

a. A triangle has vertices $A(2, 0)$, $B(2, 3)$, $C(4, 0)$. The triangle is rotated 90° clockwise about the origin. Find the coordinates of the new vertices.

A 90° clockwise rotation maps $(x, y) \to (y, -x)$.

$$A(2, 0) \to (0, -2)$$
$$B(2, 3) \to (3, -2)$$
$$C(4, 0) \to (0, -4)$$

b. The graph of $f(x) = \sqrt{x}$ is rotated 270° anticlockwise about the origin. Write the equation of the transformed graph.

A 270° anticlockwise rotation is the same as a 90° clockwise rotation: $(x, y) \to (y, -x)$.
So $x' = y$ and $y' = -x \implies x = -y$.
So the new graph is given by $x = \sqrt{-y}$

Question 7

a. A point $A\ (6, 2)$ is rotated 90° anticlockwise about the point $(3, 1)$. Find the image of point A.

Translate so rotation is about origin: $A \to (3, 1)$. Relative position: $(6, 2) - (3, 1) = (3, 1)$. Rotate: $(3, 1) \to (-1, 3)$. Translate back: $(-1, 3) + (3, 1) = (2, 4)$.

b. The graph of $f(x) = \frac{1}{x-1}$ is rotated 90° clockwise about the point $(1, 0)$. Write the equation of the transformed graph.

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y+1 \\ 1-x \end{bmatrix}$$

So $x' = y + 1 \implies y = x' - 1$ and $y' = 1 - x \implies x = 1 - y'$
Thus $x - 1 = \frac{1}{-y} \implies y = \frac{1}{1-x}$.

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Question 8 Tech-Active.

A point $(3, -2)$ is rotated $135°$ anticlockwise about the origin. Find the image of the point.

$$\left(-\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$$

Define $r(x)=\begin{bmatrix}\cos(x) & -\sin(x)\\ \sin(x) & \cos(x)\end{bmatrix}$ *Done*

$r\left(\frac{3\cdot\pi}{4}\right)\cdot\begin{bmatrix}3\\-2\end{bmatrix}$ $\begin{bmatrix}\frac{-\sqrt{2}}{2}\\ \frac{5\cdot\sqrt{2}}{2}\end{bmatrix}$

define $r(x) = \begin{bmatrix}\cos(x) & -\sin(x)\\ \sin(x) & \cos(x)\end{bmatrix}$ done

$r(3\pi/4)*\begin{bmatrix}3\\-2\end{bmatrix}$ $\begin{bmatrix}\frac{-\sqrt{2}}{2}\\ \frac{5\cdot\sqrt{2}}{2}\end{bmatrix}$

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In[69]:= r[x_] := {{Cos[x], -Sin[x]},
            {Sin[x], Cos[x]}}

In[68]:= r[3 Pi / 4].{3, -2} // FullSimplify
```

Out[68]= $\left\{-\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right\}$

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Sub-Section [4.3.3]: Reflections in Lines

Question 9

a. Reflect the point $(3, 5)$ in the line $y = x$.

Reflecting in $y = x$ swaps the coordinates:

$$(3, 5) \mapsto (5, 3)$$

b. Reflect the point $(-2, 4)$ in the line $y = -x$.

Reflecting in $y = -x$ maps (x, y) to $(-y, -x)$:

$$(-2, 4) \mapsto (-4, 2)$$

Question 10

a. Reflect the point $(4, 0)$ in the line $y = 2x + 1$.

Translate the line so it passes through the origin and apply a reflection matrix:
The reflection matrix is obtained by letting $\theta = \arctan(2)$. Then we use double angle formulas to find $\cos(2\theta)$ and $\sin(2\theta)$. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$ and $\sin(2\theta) = 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{4}{5}$.

$$R = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

Then apply this matrix to the translated point and shift back:

$$(4, 0) \mapsto \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(-\frac{16}{5}, \frac{18}{5}\right)$$

b. Reflect the point $(-3,-2)$ in the line $y=-\frac{1}{2}x+4$.

Use reflection matrix for $y=-\frac{1}{2}x$:

$$R=\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

Apply to translated point then shift back:

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} -3 \\ -6 \end{bmatrix}+\begin{bmatrix} 0 \\ 4 \end{bmatrix}=\begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

So $(-3,-2)\mapsto(3,10)$

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Question 11

a. Reflect the graph of the line $y = 3x - 1$ in the line $y = 2x$.

Choose points $A = (0, -1)$, $B = (1, 2)$. Apply reflection matrix for $y = 2x$:

$$R = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Multiply both sides on the left by the inverse matrix to get:

$$x = -\frac{3x'}{5} + \frac{4y'}{5} \text{ and } y = \frac{4x'}{5} + \frac{3y'}{5}$$

Thus, $\frac{4x}{5} + \frac{3y}{5} = 3\left(-\frac{3x'}{5} + \frac{4y'}{5}\right) - 1.$

$\implies y = \frac{1}{9}(13x + 5)$

b. Reflect the line $y = -x + 5$ in the line $y = \frac{1}{2}x + 1$.

Pick points $A = (0, 5), B = (5, 0)$ that are both on the first line. Use reflection matrix:

$$R = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

Then

$$A' = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{16}{5} \\ -\frac{7}{5} \end{bmatrix}$$

Similarly, we find $B' = \left(\frac{11}{5}, \frac{28}{5}\right)$. Then the reflected line is the line through these points. Thus

$$y = 21 - 7x$$

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Question 12 Tech-Active.

Reflect the point $(-5, 3)$ in the line $y = 2x + 6$.

$\left(\frac{3}{5}, \frac{1}{5}\right)$

Define $t(x)=\begin{bmatrix}\cos(2\cdot x) & \sin(2\cdot x)\\ \sin(2\cdot x) & -\cos(2\cdot x)\end{bmatrix}$ *Done*

$\text{tExpand}\left(t(\tan^{-1}(2))\cdot\begin{bmatrix}-5\\3-6\end{bmatrix}+\begin{bmatrix}0\\6\end{bmatrix}\right)$ $\begin{bmatrix}\frac{3}{5}\\\frac{1}{5}\end{bmatrix}$

define t(x) = $\begin{bmatrix}\cos(2*x) & \sin(2*x)\\ \sin(2*x) & -\cos(2*x)\end{bmatrix}$ done

simplify(t(tan⁻¹(2))*$\begin{bmatrix}-5\\3-6\end{bmatrix}+\begin{bmatrix}0\\6\end{bmatrix}$ $\begin{bmatrix}\frac{3}{5}\\\frac{1}{5}\end{bmatrix}$

```
In[79]:= t[x_] := {{Cos[2 x], Sin[2 x]},
           {Sin[2 x], -Cos[2 x]}}

In[81]:= t[ArcTan[2]].{-5, 3 - 6} + {0, 6} // FullSimplify

Out[81]= {3/5, 1/5}
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Sub-Section: The 'Final Boss'

Question 13

A graph of the function $f(x) = \sqrt{x}$ is transformed in a sequence of steps.

a. The function is first reflected in the y-axis, then translated 2 units right and 3 units up. Write the equation of the resulting function.

Reflection in the y-axis gives $f(-x) = \sqrt{-x}$. Translation right 2 and up 3 gives:

$$y = \sqrt{-(x-2)} + 3$$

b. The point $P(4, f(4))$ is rotated 90° anti-clockwise about the origin. Find the coordinates of the image point.

$f(4) = \sqrt{4} = 2$, so $P = (4, 2)$.

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Thus, a 90° anti-clockwise rotation about the origin maps $(x, y) \to (-y, x)$. Thus,

$$(4, 2) \mapsto (-2, 4)$$

c. Reflect the line $y = 2x - 3$ in the line $y = -2x$. Find the equation of the reflected line.

To reflect a point in the line $y = -2x$, we use the matrix:

$$T = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Apply to (x, y) we get: $(x', y') = \left(-\frac{3x}{5} - \frac{4y}{5}, \frac{3y}{5} - \frac{4x}{5}\right)$.

So $x = \frac{1}{5}(-3x' - 4y')$ and $y = \frac{1}{5}(-4x' + 3y')$.

Put these values into $y = 2x - 3$ with the dashes removed to get the reflection:

$$y = \frac{1}{11}(-15 - 2x)$$

Section B: Supplementary Questions

Sub-Section [4.3.1]: Transformations of Graphs

Question 14

a. State the transformation matrix for dilation by a factor of 6 from the y-axis and a reflection around the x-axis.

The combined transformation matrix is:

$$T = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$$

b. Find the image of (x, y) under the transformation described in **part a.**

The image of the point (x, y) is:

$$\begin{bmatrix} 6x \\ -y \end{bmatrix}$$

c. Consider an exponential function $j(x) = e^x$. The transformed graph is:

$$j_t(x) = -e^{\frac{x}{6}}$$

Question 15

a. State the transformation matrix for dilation by a factor of 3 parallel to the x-axis and a reflection around the y-axis.

The combined transformation matrix is:

$$T = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Find the image of (x, y) under the transformation described in **part a.**

The image of the point (x, y) is:

$$\begin{bmatrix} -3x \\ y \end{bmatrix}$$

c. Consider a trigonometric function $n(x) = \cos(x)$. The transformed graph is:

The graph of $n'(x) = \cos\left(-\frac{1}{3}x\right)$.

Question 16

a. State the transformation matrix for dilation by a factor of 2 from the x-axis and a reflection around the y-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

b. Find the image of (x, y) under the transformation described in **part a.**

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} -x \\ 2 \cdot y \end{bmatrix}$$

Consider a function $f(x) = 2\sin^{-1}(x-1) + \frac{\pi}{6}$.

It is known that all the points of $f(x)$ have been transformed by the transformation matrix found in **part a.**

c. Find the transformed graph.

$4\sin^{-1}(-x-1) + \frac{\pi}{3}$ or $-4\sin^{-1}(x+1) + \frac{\pi}{3}$

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Sub-Section [4.3.2]: Rotations Around Points

Question 17

a. State the transformation matrix for a rotation around the origin $45°$ counterclockwise.

The transformation matrix is:

$$R = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

b. Hence, find the image of $(3, 3)$ after it has been rotated around the origin $45°$ counterclockwise.

The image of the point (3,3) is:

$$R \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix}$$

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Question 18

a. State the transformation matrix for a rotation around the origin $60°$ clockwise.

The transformation matrix is:

$$R = \begin{bmatrix} \cos(-60^\circ) & \sin(-60^\circ) \\ -\sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b. Hence, find the image of $(0, 2)$ after it has been rotated around the origin $60°$ clockwise.

The image of the point (0,2) is:

$$R \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

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Question 19

State the image of $(0, 0)$ after the rotation around the point $(3, -2)$ $120°$ anticlockwise.

$$R(120)\cdot\left(\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}\right)+\begin{bmatrix}3\\-2\end{bmatrix} \qquad \begin{bmatrix}\frac{9}{2}-\sqrt{3}\\ \frac{-3\cdot\sqrt{3}}{2}-3\end{bmatrix}$$

Question 20

State the image of $(1, -5)$ after the rotation around the point $(-2, 1)$ $90°$ clockwise.

$$R(-90)\cdot\left(\begin{bmatrix}1\\-5\end{bmatrix}-\begin{bmatrix}-2\\1\end{bmatrix}\right)+\begin{bmatrix}-2\\1\end{bmatrix} \qquad \begin{bmatrix}-8\\-2\end{bmatrix}$$

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Sub-Section [4.3.3]: Reflections in Lines

Question 21

Reflect the point $(-1, 3)$ across the line $y = \sqrt{3}x$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = t\left(\sqrt{3}\right) \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} x' = \frac{3 \cdot \sqrt{3}}{2} + \frac{1}{2} \\ y' = \frac{3}{2} - \frac{\sqrt{3}}{2} \end{bmatrix}$$

Question 22

Reflect the point $(0, 1)$ across the line $y = 5x$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = t(5) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x' = \frac{5}{13} \\ y' = \frac{12}{13} \end{bmatrix}$$

Space for Personal Notes

Question 23

State the image of $(-1, 1)$ after the reflection around the line $y = x - 4$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = t(1) \cdot \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -4 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ -4 \end{bmatrix} \qquad \begin{bmatrix} x'=5 \\ y'=-5 \end{bmatrix}$$

Question 24

State the image of $(3, 2)$ after the reflection around the line $y = -2\left(x + \frac{1}{2}\right)$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = t(-2) \cdot \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} x'=\frac{-21}{5} \\ y'=\frac{-8}{5} \end{bmatrix}$$

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Question 25

a. Find the equation of the line $y = 3x - 1$ after it undergoes a reflection in the line $y = -2x$.

$$y = -\frac{1}{3}(x + 1)$$

b. Find the equation of the line $y = x + 4$ after it undergoes a reflection in the line $y = 3x + 2$.

$$y = 12 - 7x$$

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