4.2

# CONTOUREDUCATION

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# VCE Specialist Mathematics ½
## Transformations I [4.2]
## Workbook

### Outline:



### Learning Objectives:

- ☐ **SM12 [4.2.1]** - Using Matrices for Linear Transformations
- ☐ **SM12 [4.2.2]** - Dilations, Reflections, Translations, Shears and Projections
- ☐ **SM12 [4.2.3]** - Inverse Transformations
- ☐ **SM12 [4.2.4]** - Composite Transformations

## Section A: Linear Transformations

### Sub-Section: Introduction to Linear Transformations

**Context: Linear Transformations**

- Consider a point $(1, 4)$.
- What would the new $x$-value be if it's triple the current $x$-values plus double the current $y$-value?

$x_{new} = 3x + 2y = 11$

- What would the new $y$-value be if it's double the current $x$-values minus half the current $y$-value?

$y_{new} = 2x - \frac{1}{2}y$
$= 2 - 2 = 0$

- Hence, what would the new point be?

$(11, 0)$

**Linear Transformations**

$$(x, y) \to (ax + by, cx + dy) = (x', y')$$

- The $(x', y')$ represents the new points and is called an image.
- Original point $(x, y)$ is called the pre-image

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**Question 1**

Find the image of the point $(2, 1)$ under the transformation with rule $(x, y) \to (3x - 5y, 2x - 4y)$.

$$(x, y) = (6 - 5, 4 - 4)$$

$$= (1, 0)$$

REMINDER: Matrix Multiplication

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + (-1) \times 1 \\ 1 \times 3 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\text{Number of Columns of } 1^{st} \text{ Matrix} = \text{Number of Rows of } 2^{nd}$$

- The answer will always be a matrix.

*How can we represent the transformation using matrices?*

Exploration: Matrices for Linear Transformations

- Consider the following matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- Evaluate the answer for the above multiplication!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \text{Transformation Matrix}$$

**Question 2 Walkthrough.**

Consider a point $(x, y)$ which is represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

Find the image given by $\begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

$$= \begin{bmatrix} -x + 3y \\ 5x - 3y \end{bmatrix}$$

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**Question 3**

Consider a point $(x, y)$ which is represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

Find the transformed point given by $\begin{bmatrix} 3 & 2 \\ 2 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

$$= \begin{bmatrix} 3x + 2y \\ 2x - \frac{1}{2}y \end{bmatrix}$$

NOTE: $\begin{bmatrix} 3 & 2 \\ 2 & -\frac{1}{2} \end{bmatrix}$ is called a transformation matrix.

**Discussion:** Considering the answer from above, why is it called linear transformation?

$ax+by$

Linear equation

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**Question 4**

**a.** Find the matrix of the linear transformation with the rule $(x, y) \to (x - 2y, 3x + y)$.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

**b.** Use the matrix to find the image of the point $(2, 3)$ under the transformation.

$$\boxed{\begin{matrix} x = -4 \\ y = 9 \end{matrix}}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-6 \\ 6+3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$$

**c.** The image of a point $(c, d)$ under the linear transformation is $(2, 3)$. Find $c$ and $d$.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} c-2d \\ 3c+d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left.\begin{matrix} c-2d=2 \\ 3c+d=3 \end{matrix}\right\}$$

$$c = \frac{8}{7}, \ d = \frac{-3}{7}$$

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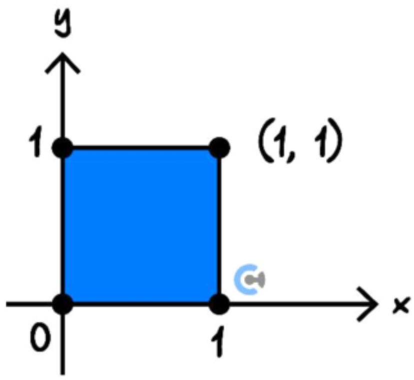
## Sub-Section: Unit Square

*What would be the best way to visualise the linear transformations?*

**Transforming the Unit Square**

### Unit Square has a side length of $1$.




- Unit square has a coordinate $(0,0)$ $(1,0)$ $(0,1)$ $(1,1)$.
- Apply the transformation to $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ to see the effect of the transformations.

**NOTE:** We use unit squares to visualise how the transformation affects different points.

**Discussion:** Does it have to be a square then?

**Question 5 Walkthrough.**

A linear transformation is represented by the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$.

**a.** Find the image of the points of the unit square $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ under this transformation and write the image points as column vectors.

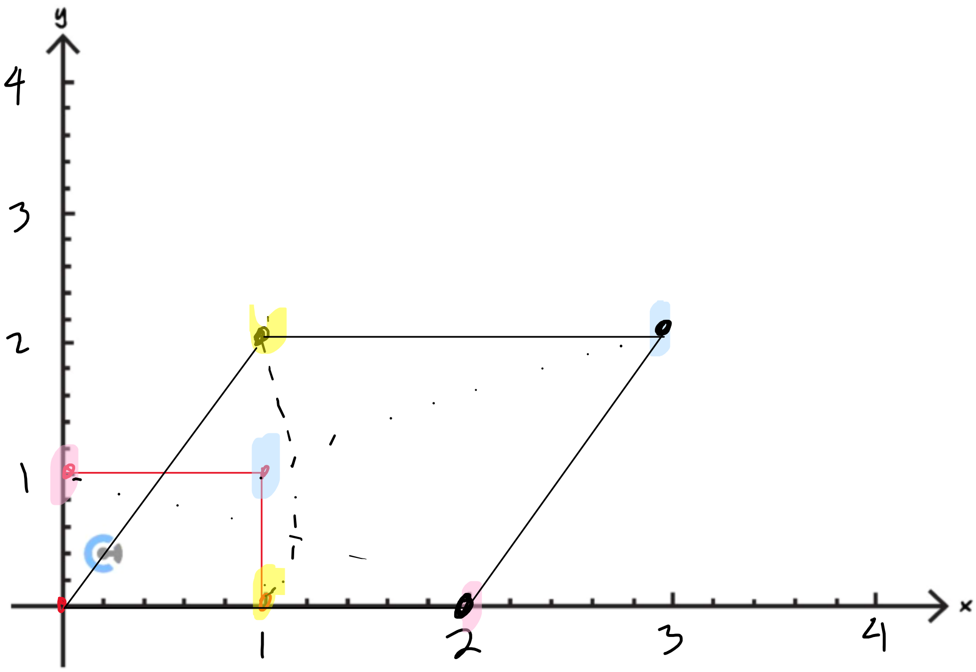
$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

**b.** Sketch the unit square and its image on the axes below.



**NOTE:** Unit square simply helps us to understand how the transformation affects the points.

**Discussion:** How could we have done the linear transformations for $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ using one matrix multiplication?

Put all the points in one matrix!

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

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## Sub-Section: Determinant and Area of Unit Square

REMINDER: Determinant of a $2 \times 2$ Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

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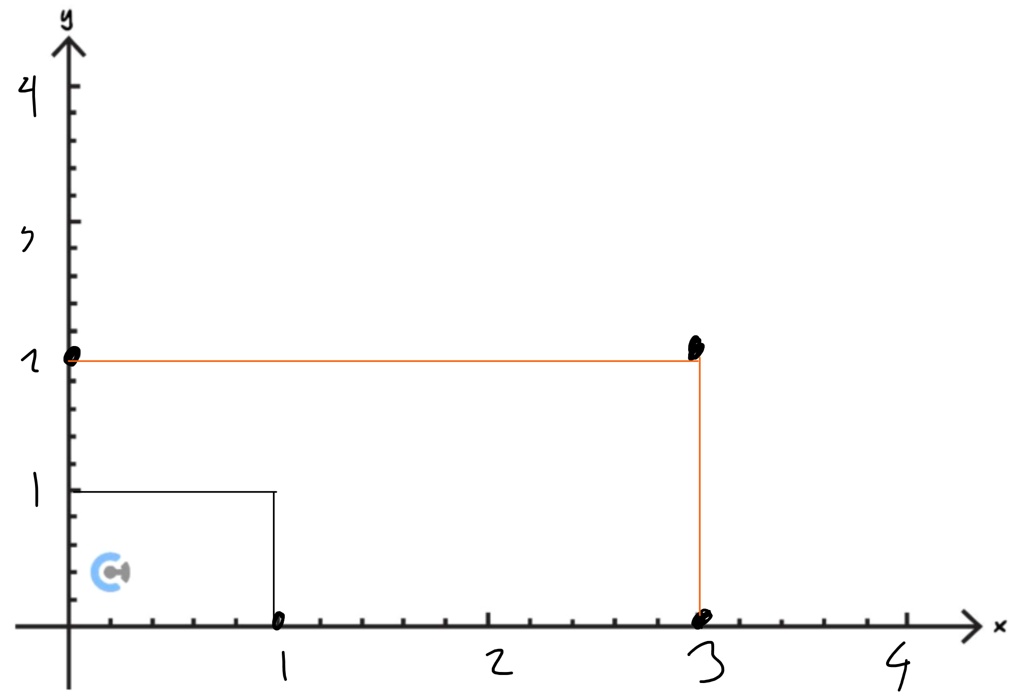
**Question 6 Walkthrough.**

A linear transformation is represented by the transformation matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

**a.** Find the image of the points of the unit square $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ under this transformation and write the image points as column vectors.

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

**b.** Sketch the unit square and its image on the axes below.



**c.** State the area of the unit square and its image.

Area = 1

Area = 6

**d.** Find the determinant of the transformation matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

$$\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6 - 0$$
$$= 6$$

**Discussion:** What do you notice? What does the determinant of the transformation matrix tell us?

Area of transformed unit square.

**Determinant of Transformation Matrix**

- Given that $A$ = Transformation matrix.

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

- Determinant could be negative hence we put the modulus.

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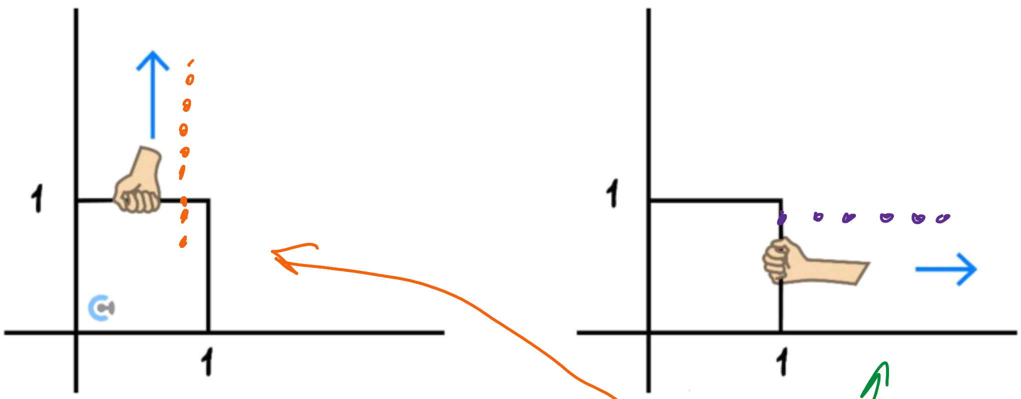
## Section B: Types of Transformations

### Sub-Section: Dilations

*What do dilations do?*

**Exploration: Understanding Dilations**

- Let's say Krish is bored that the unit square has a length of 1, and decides to stretch the unit square from the $x$ and the $y$-axis.




- From the diagram above, state which one is dilation from the $x$-axis and $y$-axis.

- Which variable ($x$ or $y$) does the dilation from the $x$-axis change?

y- value

- Which variable ($x$ or $y$) does the dilation from the $y$-axis change?

x-value

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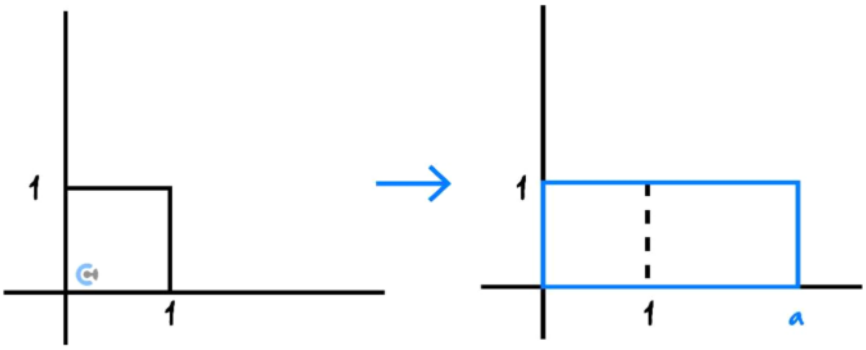
Definition

Dilation from the $y$-axis




Dilation by a factor $a$ from the $y$-axis.

- Dilation from the $y$-axis changes the x- value.

$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

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**Question 7**

**a.** State the transformation matrix for dilation by a factor of 3 from the $y$-axis.

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

**b.** Apply the transformation matrix found in **part a.** to the coordinate $(x, y)$.

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 3x \\ y \end{bmatrix}$$

**NOTE:** The $x$-value is tripled for dilation by a factor 3 from the $y$-axis.

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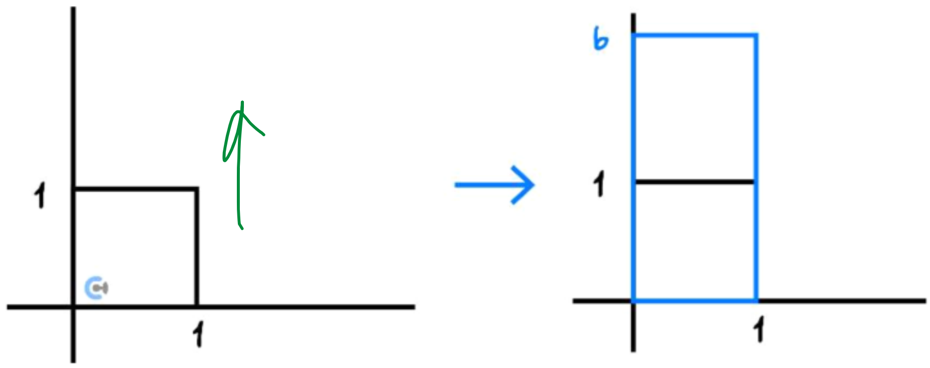
Definition

## Dilation from the $x$-axis




**Dilation by a factor $b$ from the $x$-axis.**

- Dilation from the $x$-axis changes the y-value

$$\textbf{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

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**Question 8**

**a.** State the transformation matrix for dilation by factor 2 from the $x$-axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

**b.** Apply the transformation matrix found in **part a.** to the coordinate $(x, y)$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$

**NOTE:** The $y$-value is doubled for dilation by a factor 2 from the $x$-axis.

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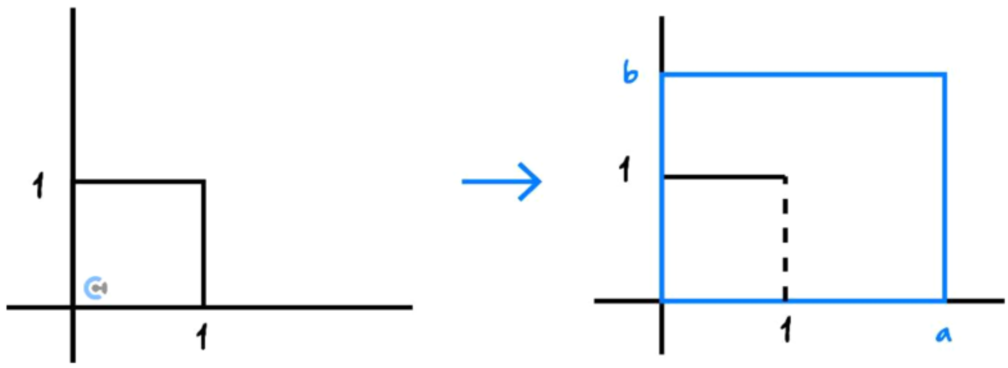
**Let's combine.**

Dilation and its Transformation Matrix




Dilation by a factor $a$ from the $y$-axis.

Dilation by a factor $b$ from the $x$-axis.

$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Discussion: Find the determinant of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Does it make sense?

$$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab - 0$$
$$= ab$$

Yes

Area of new image

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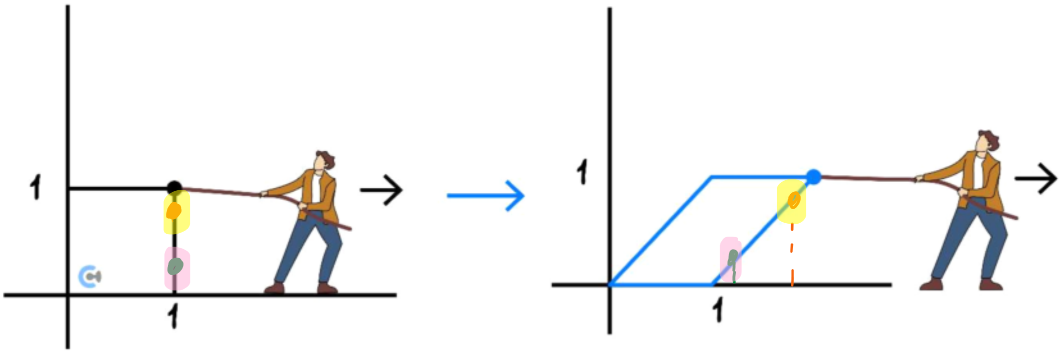
## Sub-Section: Shear

### *What about "shear"?*

**Exploration: Understanding Shear Parallel to the $x$-axis**

- Let's bring Krish back again.
- He ties a rope on the point $(1,1)$ of the "malleable" unit square and pulls it parallel to $x$-axis.




- Which variable ($x$ or $y$) would change?

$x$ - value

- Would all the points move the same distance parallel to the $x$-axis?

No

- Does the point move more if they are further from the $x$-axis or closer?

Further

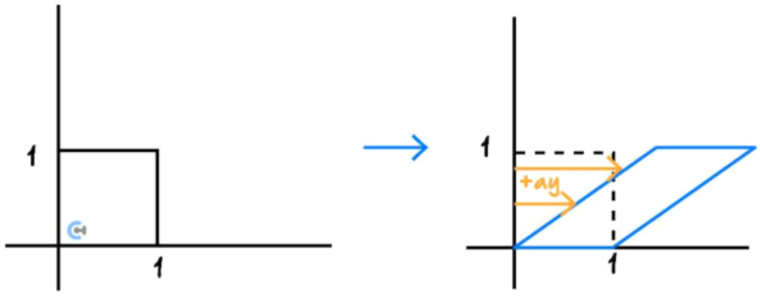
- Therefore, what does the change in $x$-value correspond to?

$y$-value

**NOTE:** The $x$-value changes with respect to how big their $y$-value is.

Definition

<u>Shear Parallel to the $x$-axis</u>




Shear of a factor $a$ parallel to the $x$-axis.

- Shear parallel to $x$-axis changes the x-value by a multiple of y-value

$$Transformation\ Matrix = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

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**Question 9**

**a.** State the transformation matrix for the shear of a factor 3 parallel to the $x$-axis.

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

**b.** Apply the transformation matrix found in **part a.** to the coordinate $(x, y)$.

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ y \end{bmatrix}$$

**NOTE:** The $x$-value is added by tripling the $y$-value.

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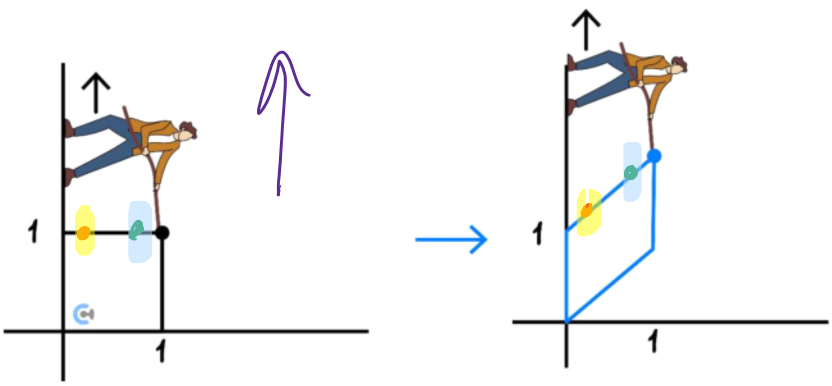
## *What about in the direction of the $y$-axis?*

**Exploration: Understanding Shear Parallel to the $y$-axis**

- Let's bring Krish back again $\times 2$.
- He ties a rope on the point $(1, 1)$ of the "malleable" unit square and pulls it parallel to $y$-axis.




- Which variable ($x$ or $y$) would change?

y - value

- Would all the points move the same distance parallel to the $y$-axis?

No

- Does the point move more if they are further from the $y$-axis or closer?

Further

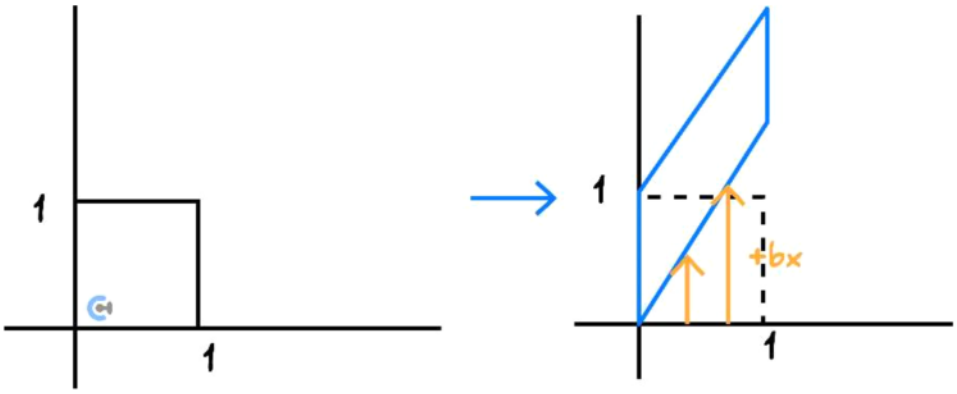
- Therefore, what does the change in $y$-value correspond to?

x - value

**NOTE:** The $y$-value changes with respect to how big their $x$-value is.

Definition

Shear Parallel to the $y$-axis




## Shear of a factor $b$ parallel to the $y$-axis.

- Shear parallel to $y$-axis changes the y-value by a multiple of x-value

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

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**Question 10**

**a.** State the transformation matrix for the shear of a factor 2 parallel to the $y$-axis.

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

**b.** Apply the transformation matrix found in **part a.** to the coordinate $(x, y)$.

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x+y \end{bmatrix}$$

**NOTE:** The $y$-value is added by doubling the $x$-value.

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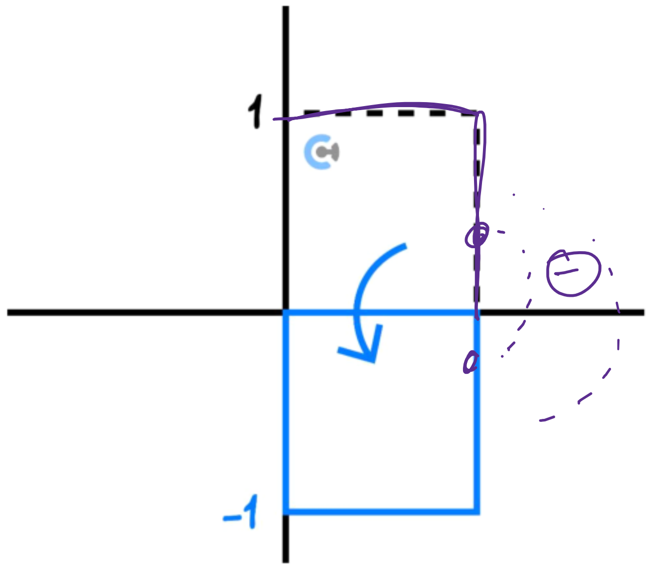
## Sub-Section: Reflections around $x$ and $y$-axis

**Discussion:** If you reflect something around the $x$-axis, what would happen? What about the $y$-axis?

negative

### Reflection around $x$-axis




**Reflection in the $x$-axis.**

- Reflection in the $x$-axis changes the y-value.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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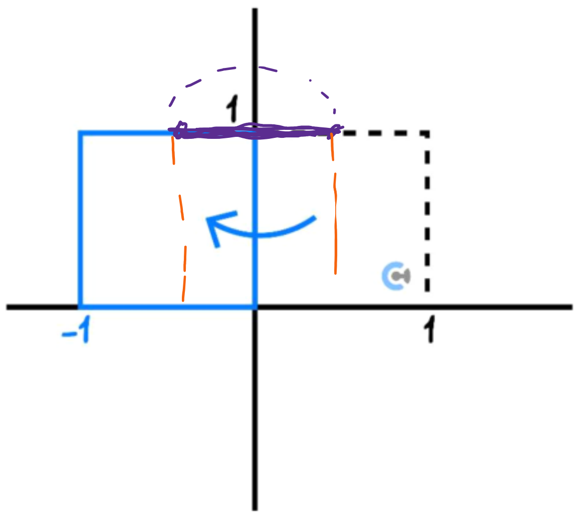
*Now around $y$-axis.*

Reflection around $y$-axis




## Reflection in the $y$-axis

- Reflection in the $y$-axis changes the $x$-value.

$$Transformation\ Matrix = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

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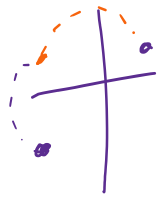
**Question 11**

**a.** State the transformation matrix for reflection in both $x$ and $y$-axis.



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

**b.** Apply the transformation matrix found in **part a.** to the coordinate $(x, y)$.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

__Discussion:__ Consider the size of the determinant of the reflection transformation matrix. Does it make sense?

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 - 0$$

$$= 1$$

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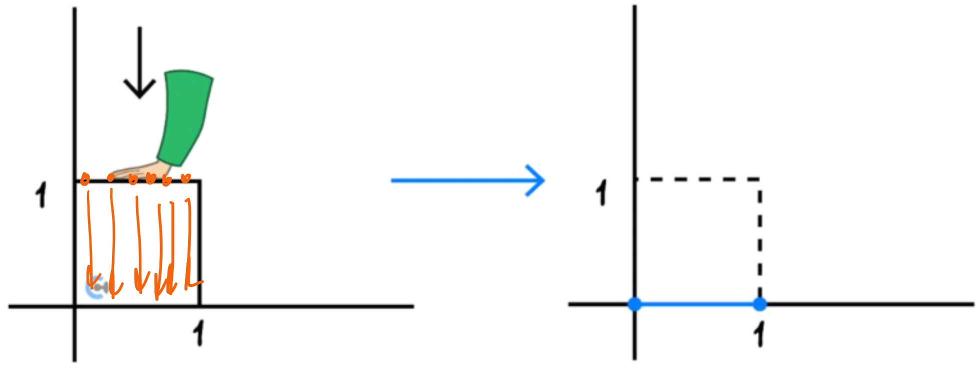
Reflections don't change area!

## Sub-Section: Projections

### *What about "projections"?*

**Exploration: Understanding Projection**

- Bringing Krish back again $\times$ 3.
- He wants to squish the unit square onto the $x$-axis.

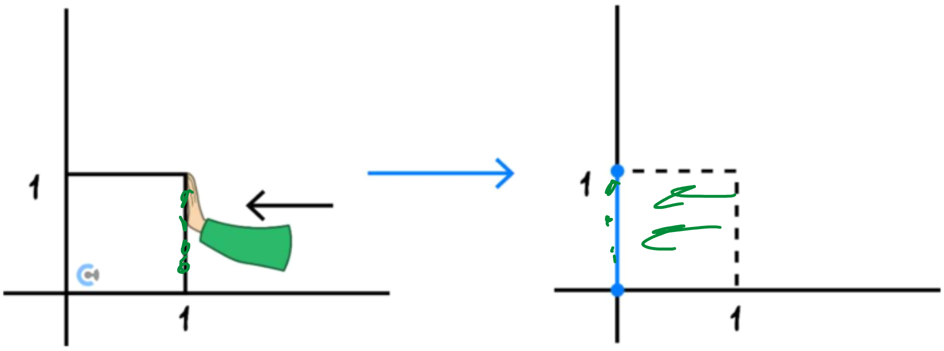



- What would happen?

y - values become 0.

- Would this be a "projection" onto the $x$-axis or $y$-axis?

x-axis.



- How about now?
- What would happen?

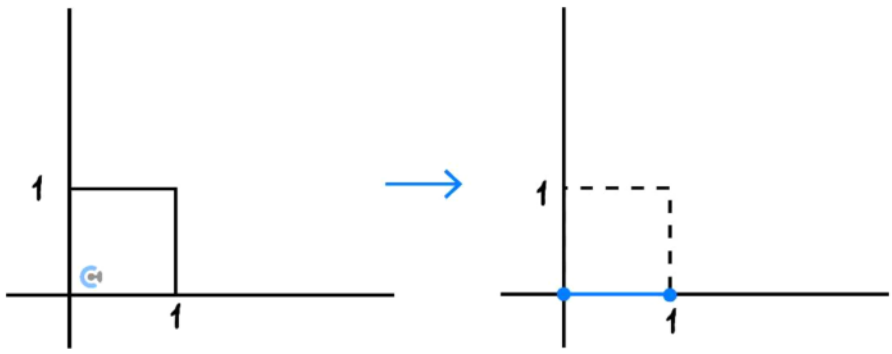
$x$-values become 0

- Would this be a "projection" onto the $x$-axis or $y$-axis?

y-axis.

Definition

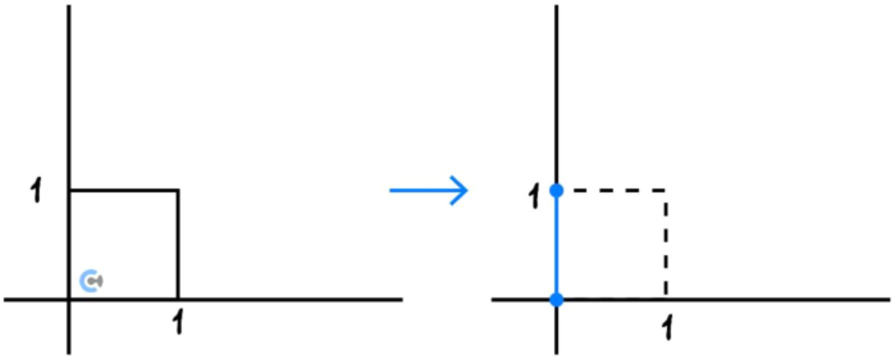
<u>Projections</u>



Projection onto the $x$-axis:

- The y-values becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$




## Projection onto the $y$-axis:

- The x-values becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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**Question 12**

**a.** State the transformation matrix for projection onto $y$-axis.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

**b.** Find the image of $(2, 1)$ after the transformation projection onto $y$-axis.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**NOTE:** Projection onto $y$-axis only keeps the $y$-value.

**Discussion:** Consider the determinant of the projection transformation matrix. Does it make sense?

**YES, because the area becomes 0. Hence, the determinant should always be 0.**

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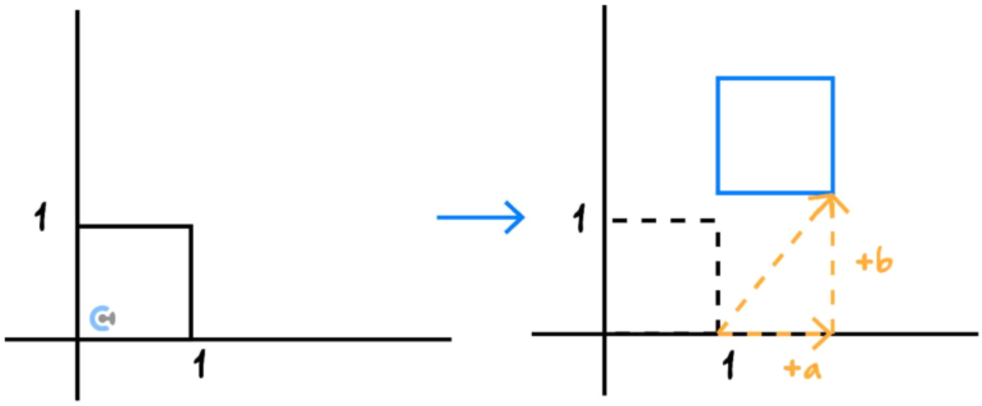
## Sub-Section: Translations

*Now translations!*

**Translation**




- Translation simply moves the point.

**Translation $a$ units in the positive direction of the $x$-axis.**

**Translation $b$ units in the positive direction of the $y$-axis.**

- We simply add/subtract the translation value to $x$ and $y$.

$$\textit{Transformation}: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

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**Question 13**

Consider the point $(4, 1)$.

The point has been translated 2 units in the positive direction of the $x$-axis and translated 3 units in the negative direction of the $y$-axis.

Find the image using matrices.

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

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## Section C: Inverse Transformations

### Sub-Section: Reversing Transformations

**REMINDER: Inverse of a $2 \times 2$ Matrix**

$AA^{-1} = I$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Inverse only exists for a square matrix.
- Matrix that has an inverse is called (invertible).

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**Question 14**

Consider a transformation matrix given by $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$.

**a.** Find the image of $(2, 3)$ after applying transformation $A$.

$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

**b.** Find the inverse matrix of $A$.

$$A^{-1} = \frac{1}{-2-3}\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$$

**c.** Find the image of $(7, 3)$ after applying transformation $A^{-1}$.

$$-\frac{1}{5}\underbrace{\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix}} = -\frac{1}{5}\begin{bmatrix} -10 \\ -15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

**TIP:** Take the factor out and multiply it afterwards.

**Discussion:** From the previous question, what do we do to reverse a transformation?

Use $A^{-1}$ matrix

*Let's also prove this using matrix algebra!*

**Exploration: Algebraic Proof of Inverse Transformation**

- Consider:

image pre-image

$$X' = AX$$

- Multiply $A^{-1}$ on both sides.

NOTE: We always multiply the matrices on the LHS.

$$A^{-1}X' = A^{-1}AX$$

- What does $AA^{-1}$ equal to?

$$A^{-1}X' = IX$$

$$A^{-1}X' = X$$

- What does $IA$ always equal to?

- We can multiply the inverse transformation matrix by the image to go back to the pre-image.

**Inverse Transformation**

$$\text{If } X' = AX \text{ then } A^{-1}X' = X.$$

- Multiply the inverse transformation matrix to the image to go back to the pre-image.

**Question 15**

A point $(x, y)$ has been transformed by $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and the image was given by $(2, 1)$.

**a.** Find $A^{-1}$.

$$A^{-1} = \frac{1}{2-12}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

$$= -\frac{1}{10}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

**b.** Hence, find the point $(x, y)$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{10}\begin{bmatrix} 1 \\ -7 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ \frac{7}{10} \end{bmatrix}$$

**Space for Personal Notes**

## Sub-Section: Validity of Inverse Transformations

**Discussion:** Do all matrices have an inverse?

No.

**REMINDER:** Determinant of a $2 \times 2$ Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

- If the determinant equals zero, then $A$ does not have an inverse.
- $A$ is not invertible.

**Discussion:** If a transformation matrix $A$ does not have an inverse $A^{-1}$, how can we reverse the transformation under $A$?

We can't reverse

Space for Personal Notes

*Why can't some transformations be reversed?*

**Question 16**

Consider a transformation matrix given by $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

**a.** Find the det $(A)$.

$$\det(A) = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 2 - 2 = \underline{0}$$

**b.** Find the image of $(3, 4)$ under the transformation given by $A$.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

**c.** Find the image of $(2, 5)$ under the transformation given by $A$.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

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**Discussion:** Looking at the question above, how can we reverse the transformation from the image: $(7, 14)$?

We can't

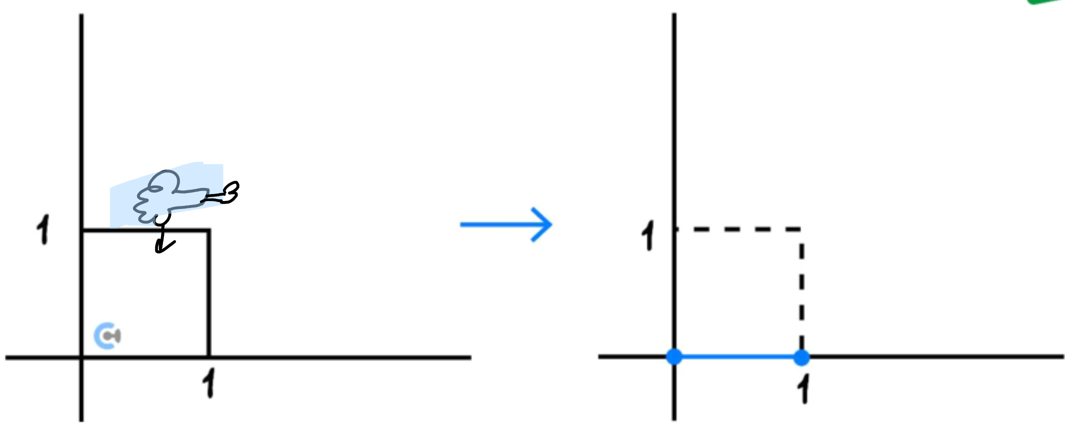
**Non-Invertible Matrix and Inverse Transformations**

$$X' = AX$$

$$If\ \det(A) = 0, then\ X\ cannot\ be\ solved\ as\ A^{-1}\ is\ undefined.$$

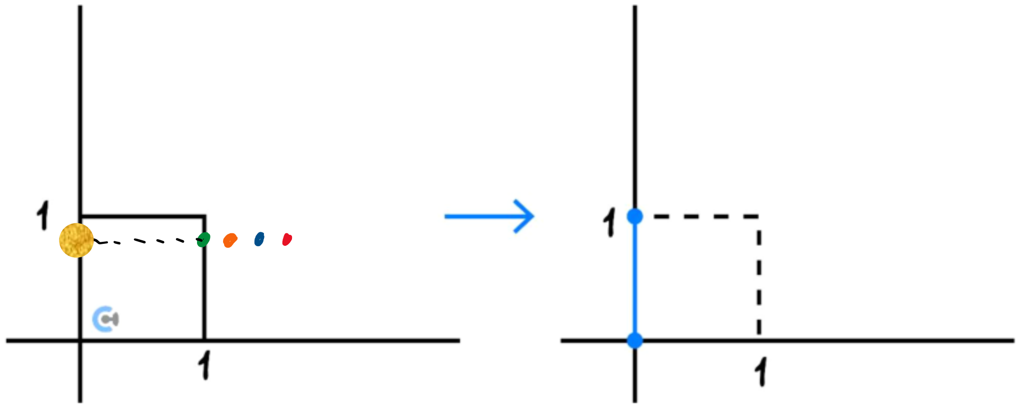
- The original point cannot be solved if the inverse matrix does not exist.
- The transformation cannot be reversed when $\det = 0$
- It happens as the image can be achieved from multiple pre-images.

**Active Recall:** Projection




Projection onto $x$-axis:

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$




Projection onto $y$-axis:

$\boldsymbol{Transformation\ Matrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Discussion: Consider the determinant of the projection transformation matrix. Can any projection transformation be reversed? Does that make sense?

det = 0 → Can't reverse transformations

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## Section D: Composite Transformations

### Sub-Section: Composite Transformations

**Discussion:** How do we do multiple transformations?

$\overset{③}{[\;\_\;]}\overset{②}{[\;\_\;]}\overset{①}{[\;\_\;]}\begin{bmatrix} x \\ y \end{bmatrix}$

**Composite Transformations**

- For transformation under $A$ and $B$ respectively,

$$X' = BAX$$

- Always multiply the next transformation matrix on the LHS.

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**Question 17 Walkthrough.**

**a.** State the transformation matrix for dilation by factor 2 from the $x$-axis and reflection in the $x$-axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

**b.** Apply the transformation matrix found in **part a.** to the coordinate $(3, 1)$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Space for Personal Notes

*Your turn!*

**Question 18**

**a.** State the transformation matrix for dilation by factor 3 from the $x$-axis, shear of factor 3 parallel to the $y$-axis and reflection in the $y$-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & 3 \end{bmatrix}$$

**b.** Hence, apply the transformation "dilation by factor 3 from the $x$-axis, shear of factor 3 parallel to the $y$-axis and reflection in the $y$-axis" to the coordinate $(-2, 5)$.

$$\begin{bmatrix} -1 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

**Space for Personal Notes**

## Contour Check

### Learning Objective: [4.2.1] – Using matrices for linear transformations

**Key Takeaways**

- **Linear Transformations:**

$$(x, y) \to (ax + by, cx + dy) = (x', y')$$

  - The $(x', y')$ represents the new points and is called an image.
  - Original point $(x, y)$ is called the pre-image.

- **Matrices for Linear Transformations:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \text{Transformation matrix}$$

- **Determinant of Transformation Matrix:**

  - Given that $A =$ Transformation matrix.

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

  - Determinant could be negative hence we put the modulus.

## Learning Objective: [4.2.2] – Dilations, reflections, translations, shears and projections

### Key Takeaways

- Dilation from the $y$-axis:

  **Dilation by a factor $a$ from the $y$-axis.**

  - Dilation from the $y$-axis changes the x-value

  $$\textit{Transformation Matrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

- Dilation from the $x$-axis:

  **Dilation by a factor $b$ from the $x$-axis.**

  - Dilation from the $x$-axis changes the y-value

  $$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

- Dilation and its Transformation Matrix:

  **Dilation by a factor $a$ from the $y$-axis.**

  **Dilation by a factor $b$ from the $x$-axis.**

  $$\textit{Transformation Matrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

- Shear Parallel to the $x$-axis:

  **Shear of a factor $a$ parallel to the $x$-axis.**

  - Shear parallel to $x$-axis changes the x-value by a multiple of y-value

  $$\textit{Transformation Matrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

- **Shear Parallel to the $y$-axis:**

## Shear of a factor $b$ parallel to the $y$-axis.

- Shear parallel to $y$-axis changes the y-value by a multiple of x-value

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

- **Reflection around $x$-axis:**

## Reflection in the $x$-axis:

- Reflection in the $x$-axis changes the y-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Reflection around $y$-axis:**

## Reflection in the $y$-axis:

- Reflection in the $y$-axis changes the x-value.

$$Transformation\ Matrix = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Projections:**

## Projection onto the $x$-axis:

- The y-value becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

## Projection onto the $y$-axis:

- The x-value becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Translation:**
  - Translation simply moves the point.

Translation $a$ units in the positive direction of the $x$-axis.

Translation $b$ units in the positive direction of the $y$-axis.

  - We simply add/subtract the translation value to $x$ and $y$.

$$Transformation: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

## Learning Objective: [4.2.3] – Inverse transformations

**Key Takeaways**

- **Inverse Transformation:**

If $X' = AX$ then $X =$ $A^{-1}X'$.

  - Multiply the inverse transformation matrix to the image to go back to the pre-image.

- **Non-Invertible Matrix and Inverse Transformations:**

$$X' = AX$$

$$if\ \det(A) = 0, then\ X\ cannot\ be\ solved\ as\ A^{-1}\ is\ undefined.$$

  - The original point cannot be solved if the inverse matrix does not exist.
  - The transformation cannot be reversed when det = 0.
  - It happens as the image can be achieved from multiple pre-images.

## Learning Objective: [4.2.4] – Composite transformations

**Key Takeaways**

- For transformation under $A$ and $B$ respectively:

$$X' = BAX$$

- Always multiply the next transformation matrix on the LHS.