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VCE Specialist Mathematics ½
Transformations I [4.2]
Workbook

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Outline:

Learning Objectives:

- ☐ SM12 [4.2.1] – Using Matrices for Linear Transformations
- ☐ SM12 [4.2.2] – Dilations, Reflections, Translations, Shears and Projections
- ☐ SM12 [4.2.3] – Inverse Transformations
- ☐ SM12 [4.2.4] – Composite Transformations

Section A: Linear Transformations

Sub-Section: Introduction to Linear Transformations

Context: Linear Transformations

- Consider a point $(1, 4)$.
- What would the new x-value be if it's triple the current x-values plus double the current y-value?

$x' = 3x+2y$

- What would the new y-value be if it's double the current x-values minus half the current y-value?

$y' = 2x - \frac{1}{2}y$

- Hence, what would the new point be?

Linear Transformations

$$(x, y) \rightarrow (ax + by, cx + dy) = (x', y')$$

- The (x', y') represents the new points and is called an image.
- Original point (x, y) is called the pre-image

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Question 1

Find the image of the point $(2, 1)$ under the transformation with rule $(x, y) \to (\overset{x'}{3x - 5y}, \overset{y'}{2x - 4y})$.

$x' = 3(2) - 5(1) = 6 - 5 = 1$

$y' = 2(2) - 4(1) = 4 - 4 = 0$

$\therefore$ image is $(1, 0)$

REMINDER: Matrix Multiplication

$$A \times B = \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2 \times 3 + (-1) \times 1 \\ 1 \times 3 + 2 \times 1 \end{bmatrix} = \underset{2\times 1}{\begin{bmatrix} 5 \\ 5 \end{bmatrix}}$$

$$Number\ of\ Columns\ of\ 1^{st}\ Matrix = Number\ of\ Rows\ of\ 2^{nd}$$

- The answer will always be a matrix.

How can we represent the transformation using matrices?

Exploration: Matrices for Linear Transformations

- Consider the following matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Evaluate the answer for the above multiplication!

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{transformation matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{pre-image}} = \underbrace{\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}}_{\text{image}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = Transformation\ Matrix$$

Question 2 Walkthrough.

Consider a point (x, y) which is represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

Find the image given by $\underbrace{\begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}}_{2\times 2 \text{ transformation matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{2\times 1 \text{ pre-image}}$. $= \begin{bmatrix} -x+3y \\ 5x-3y \end{bmatrix}$

image $= \begin{bmatrix} -x+3y \\ 5x-3y \end{bmatrix}$

$x' = -x+3y$

$y' = 5x-3y$

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Question 3

Consider a point (x, y) which is represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

Find the transformed point given by $\begin{bmatrix} 3 & 2 \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. $= \begin{bmatrix} 3x + 2y \\ 2x - \frac{1}{2}y \end{bmatrix}$

$\therefore$ transformed point is

$(x', y') = (3x + 2y, 2x - \frac{1}{2}y)$

NOTE: $\begin{bmatrix} 3 & 2 \\ 2 & -\frac{1}{2} \end{bmatrix}$ is called a transformation matrix.

<u>Discussion:</u> Considering the answer from above, why is it called linear transformation?

↳ max. power 1

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transformation x pre-image = image

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Question 4

a. Find the matrix of the linear transformation with the rule $(x, y) \to (x - 2y, 3x + y)$.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

b. Use the matrix to find the image of the point $(2, 3)$ under the transformation.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(2) - 2(3) \\ 3(2) + 1(3) \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 9 \end{bmatrix} \quad \therefore (-4, 9)$$

c. The image of a point (c, d) under the linear transformation is $(2, 3)$. Find c and d.

$$A \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} c - 2d \\ 3c + d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} c - 2d = 2 \Rightarrow c = 2d + 2 \\ 3c + d = 3 \end{cases}$$

sub in

$3(2d+2) + d = 3$

$6d + 6 + d = 3$

$7d = -3$

$d = -\frac{3}{7}$

$c = 2d + 2$

$c = 2(-\frac{3}{7}) + 2$

$c = -\frac{6}{7} + 2$

$c = \frac{8}{7}$

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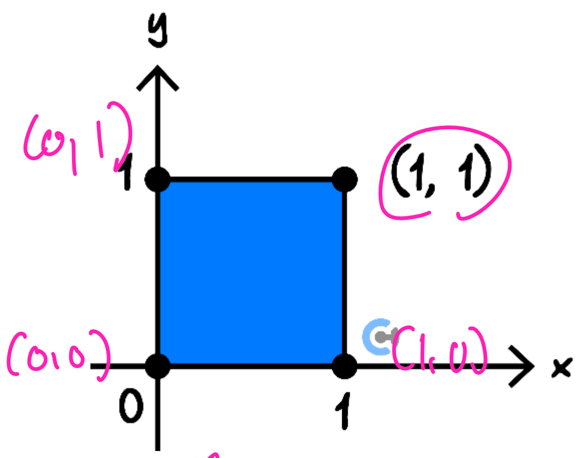
$$A^{-1}A\begin{bmatrix} c \\ d \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Sub-Section: Unit Square

What would be the best way to visualise the linear transformations?

Transforming the Unit Square

Unit Square has a side length of 1.

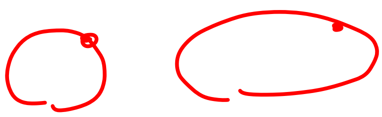


- Unit square has a coordinate ______________________________.
- Apply the transformation to $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ to see the effect of the transformations.

NOTE: We use unit squares to visualise how the transformation affects different points.

Discussion: Does it have to be a square then?

no.



Question 5 Walkthrough.

A linear transformation is represented by the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$.

a. Find the image of the points of the unit square $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ under this transformation and write the image points as column vectors.

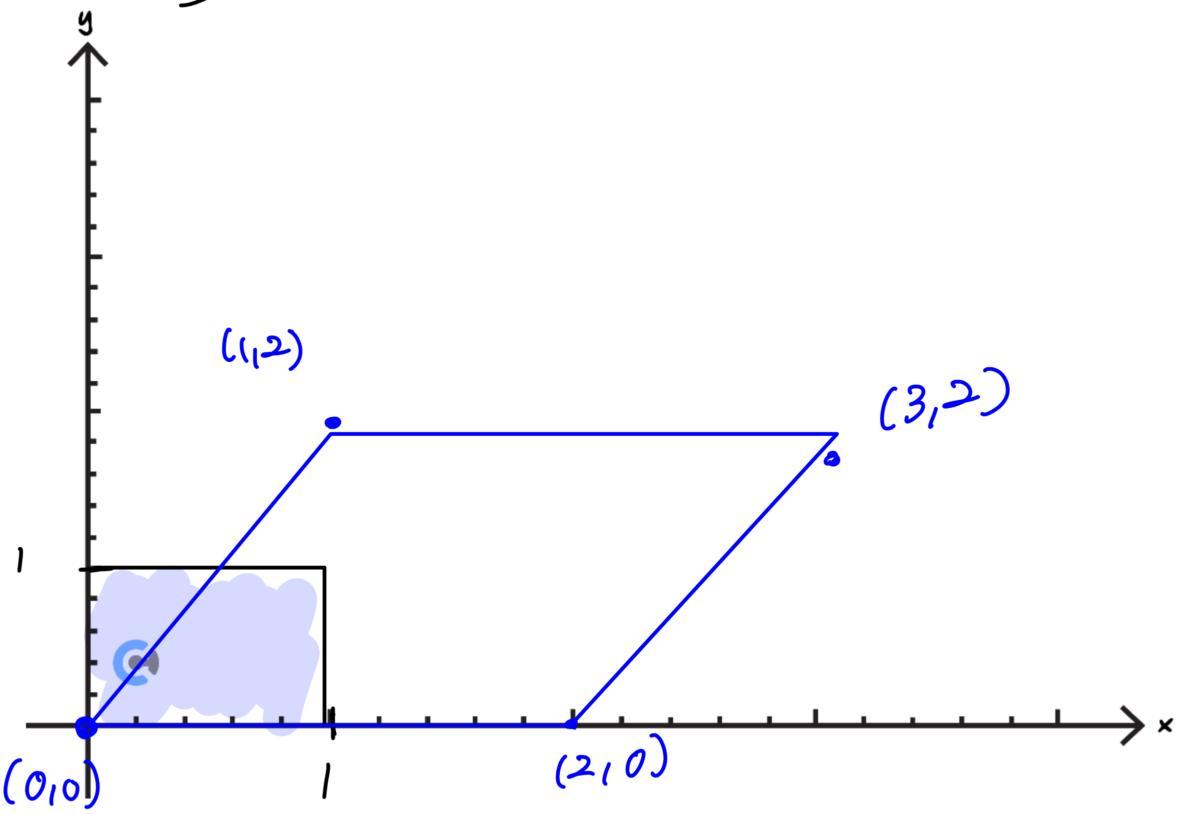
$(0,0) \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (0,0)$

$(1,0) \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow (1,2)$

$(0,1) \rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow (2,0)$

$(1,1) \rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \rightarrow (3,2)$

b. Sketch the unit square and its image on the axes below.



NOTE: Unit square simply helps us to understand how the transformation affects the points.

Discussion: How could we have done the linear transformations for $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ using one matrix multiplication?

$$[\text{transf.}] \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} (\ ) & (\ ) & (\ ) & (\ ) \end{bmatrix}$$

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Sub-Section: Determinant and Area of Unit Square

REMINDER: Determinant of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

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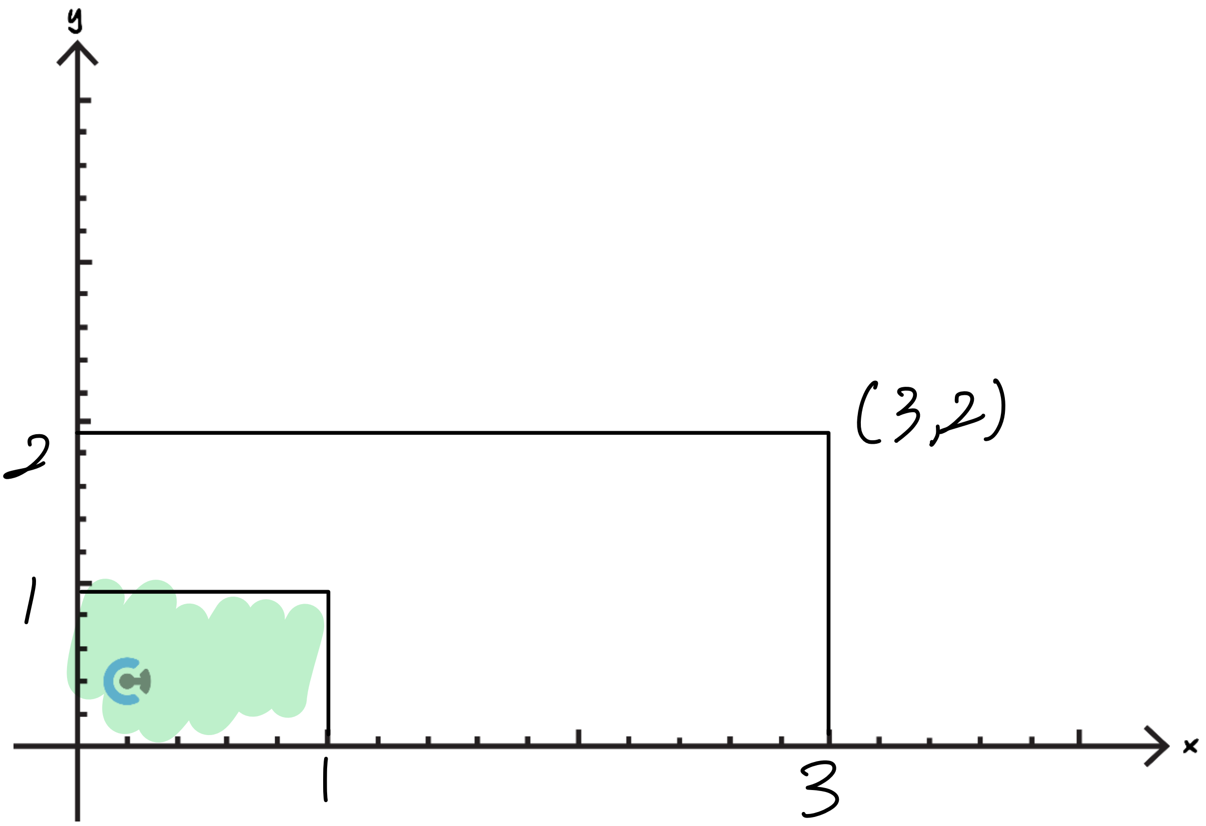
Question 6 Walkthrough.

A linear transformation is represented by the transformation matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

a. Find the image of the points of the unit square $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ under this transformation and write the image points as column vectors.

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

b. Sketch the unit square and its image on the axes below.



c. State the area of the unit square and its image.

$1 \times 1 = 1$ $3 \times 2 = 6$

d. Find the determinant of the transformation matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(B) = 1$

$\det(A) = 3 \cdot 2 - 0 \cdot 0$

$= 6$

Discussion: What do you notice? What does the determinant of the transformation matrix tell us?

$|\det| = \text{Area}$

Determinant of Transformation Matrix

- Given that $A =$ Transformation matrix.

not a unit square

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

- Determinant could be ___________ hence we put the modulus.

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Section B: Types of Transformations

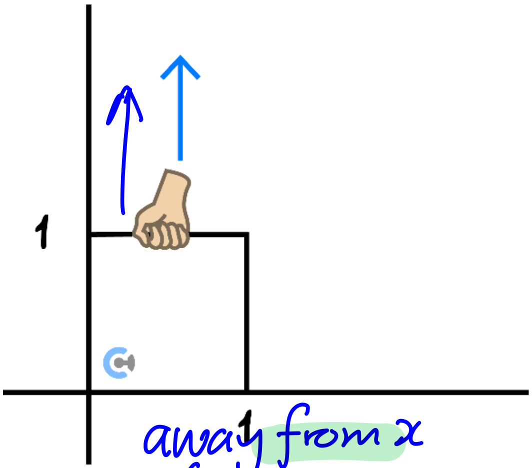
Dilation
Reflection
Translation
Shear

Sub-Section: Dilations

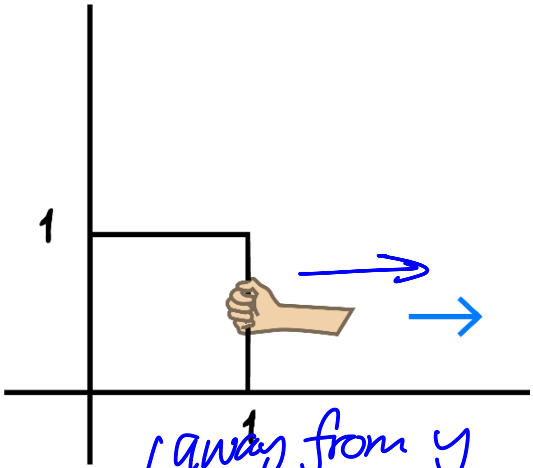
What do dilations do?

Exploration: Understanding Dilations

- Let's say Krish is bored that the unit square has a length of 1, and decides to stretch the unit square from the x and the y-axis.



away from x



away from y

- From the diagram above, state which one is dilation from the x-axis and y-axis.
- Which variable (x or y) does the dilation from the x-axis change?

Δy Δx

- Which variable (x or y) does the dilation from the y-axis change?

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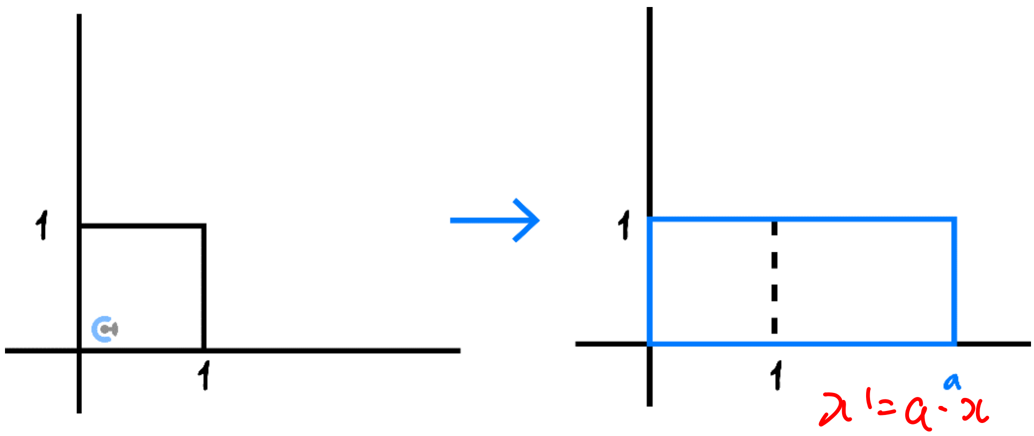
Definition

Dilation from the y-axis



Dilation by a factor a from the y-axis.

- Dilation from the y-axis changes the ______________.

$$\textit{Transformation Matrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Question 7 walkthrough

a. State the transformation matrix for dilation by a factor of 3 from the y-axis.

$$\begin{array}{c} x \rightarrow \\ y \rightarrow \end{array} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$x' = 3x$

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ y \end{bmatrix}$$

NOTE: The x-value is tripled for dilation by a factor 3 from the y-axis.

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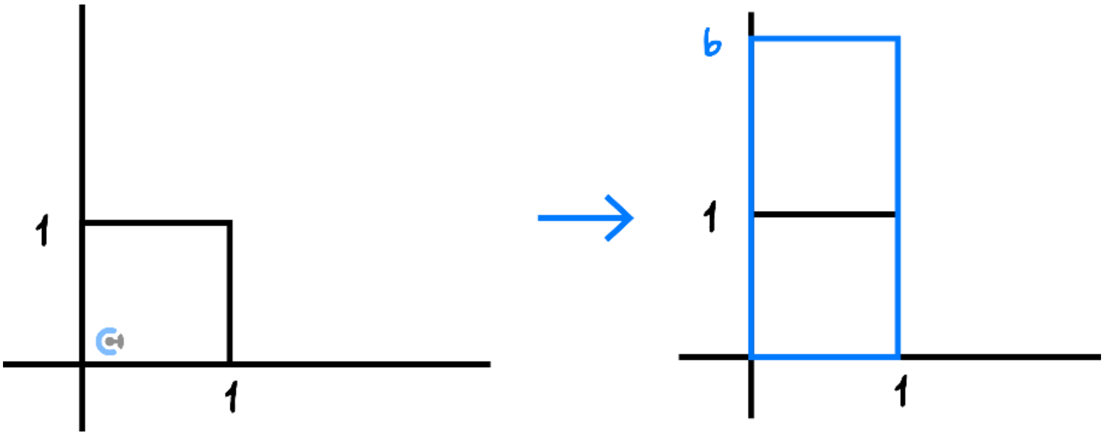
Definition

Dilation from the x-axis



Dilation by a factor b from the x-axis.

- Dilation from the x-axis changes the y.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

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Question 8

a. State the transformation matrix for dilation by factor 2 from the x-axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x+0y \\ 0x+2y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$

NOTE: The y-value is doubled for dilation by a factor 2 from the x-axis.

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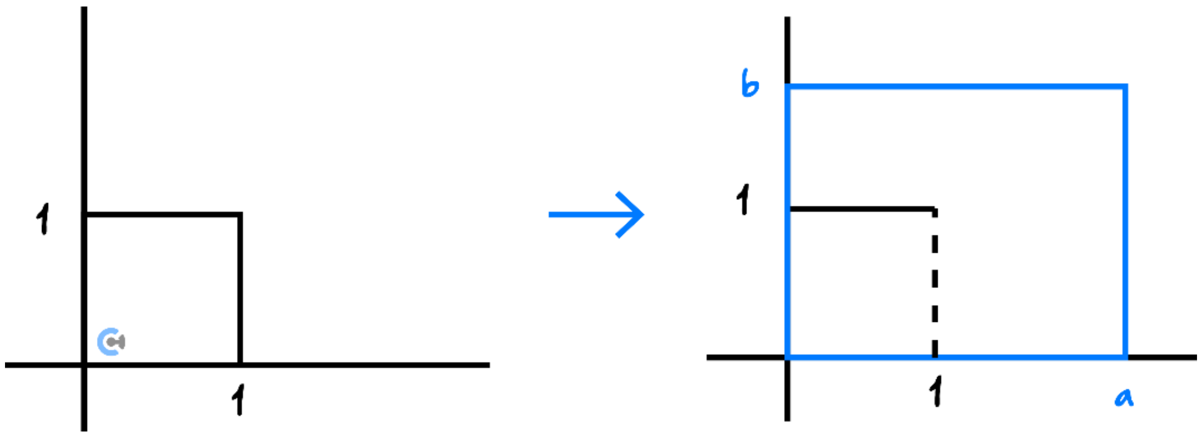
Let's combine.

Definition

Dilation and its Transformation Matrix



Dilation by a factor a from the y-axis.

Dilation by a factor b from the x-axis.

$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Discussion: Find the determinant of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Does it make sense?

x (from y)

y (from x)

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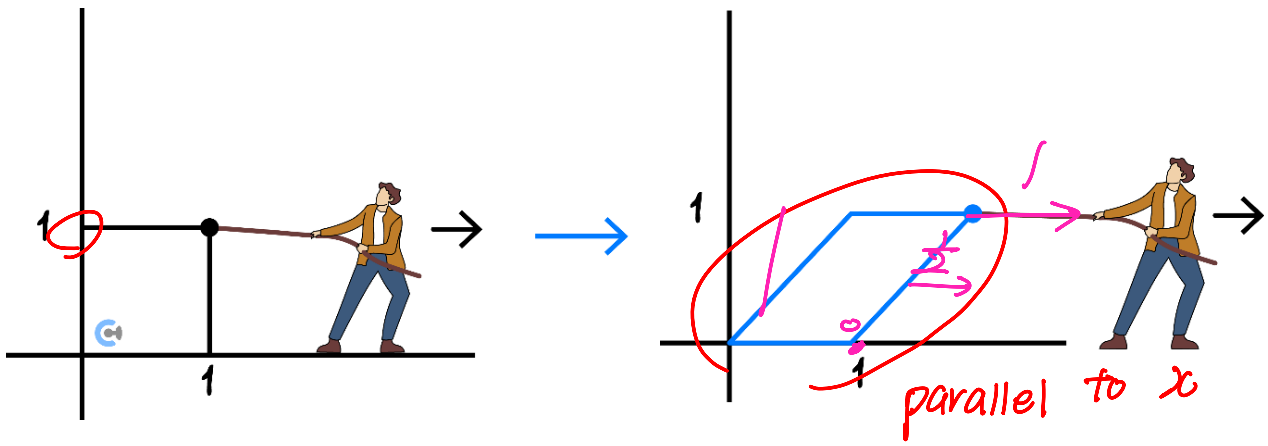
Sub-Section: Shear

What about "shear"?

Exploration: Understanding Shear Parallel to the x-axis

- Let's bring Krish back again.
- He ties a rope on the point $(1,1)$ of the "malleable" unit square and pulls it parallel to x-axis.



- Which variable (x or y) would change?

x-values

- Would all the points move the same distance parallel to the x-axis?

No

- Does the point move more if they are further from the x-axis or closer?

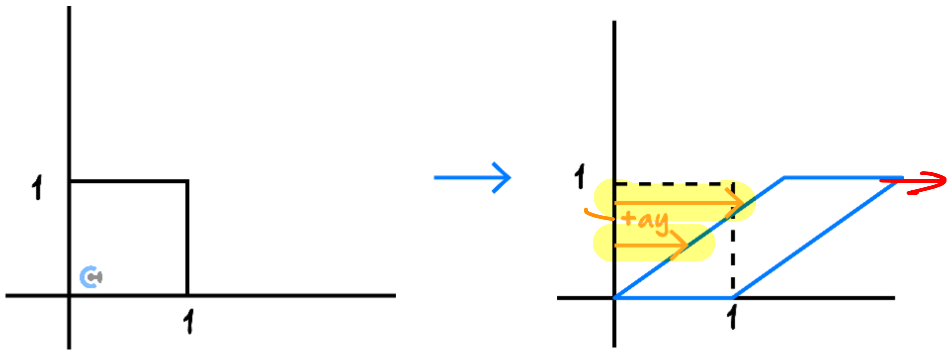
further (higher)

- Therefore, what does the change in x-value correspond to?

↳ y-value

NOTE: The x-value changes with respect to how big their y-value is.

Shear Parallel to the x-axis



Shear of a factor a parallel to the x-axis.

- Shear parallel to x-axis changes the x-values by a multiple of the y

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

← change in x
affected by y

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Question 9

a. State the transformation matrix for the shear of a factor 3 parallel to the x-axis.

$$\begin{bmatrix} 1 & \boxed{3} \\ 0 & 1 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ y \end{bmatrix}$$

$(x+3y, y)$

NOTE: The x-value is added by tripling the y-value.

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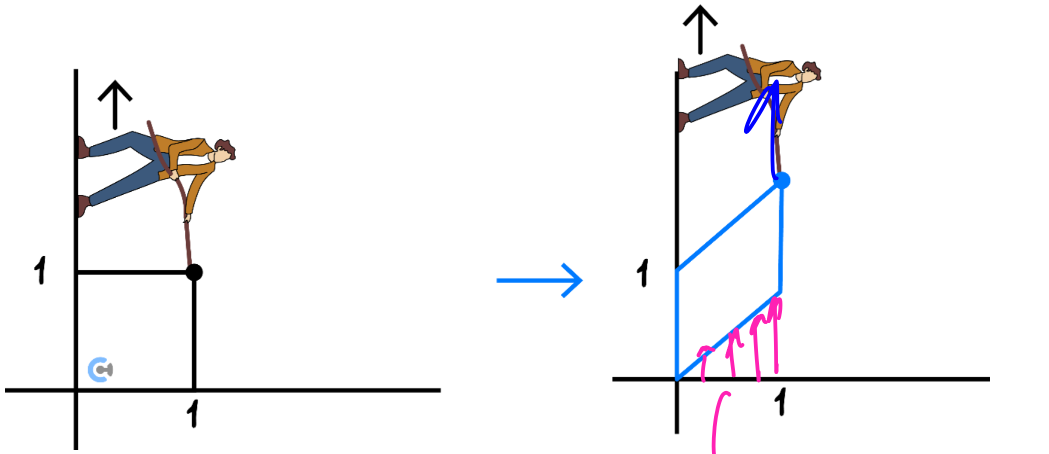
What about in the direction of the y-axis?

Exploration: Understanding Shear Parallel to the y-axis

- Let's bring Krish back again $\times$ 2.
- He ties a rope on the point $(1, 1)$ of the "malleable" unit square and pulls it parallel to y-axis.



affected by x

- Which variable (x or y) would change?

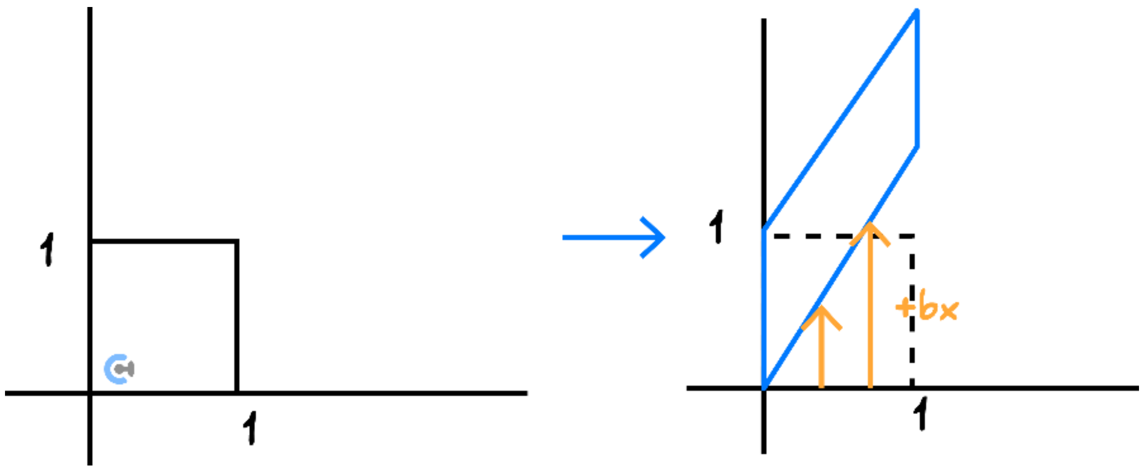
y

- Would all the points move the same distance parallel to the y-axis?
- Does the point move more if they are further from the y-axis or closer?
- Therefore, what does the change in y-value correspond to?

NOTE: The y-value changes with respect to how big their x-value is.

Definition

Shear Parallel to the y-axis



Shear of a factor b parallel to the y-axis.

- Shear parallel to y-axis changes the y value affected x.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

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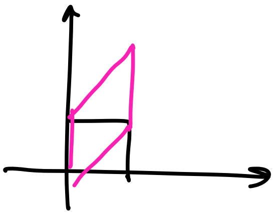
Question 10

a. State the transformation matrix for the shear of a factor 2 parallel to the y-axis.

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$



b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x+y \end{bmatrix}$$

NOTE: The y-value is added by doubling the x-value.

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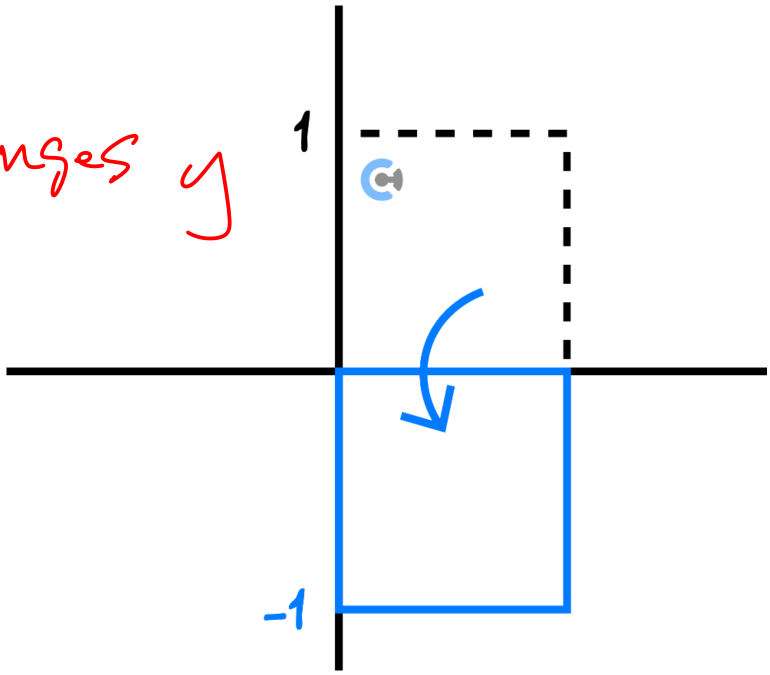
Sub-Section: Reflections around x and y-axis

Discussion: If you reflect something around the x-axis, what would happen? What about the y-axis?

Reflection around x-axis

changes y



Reflection in the x-axis.

- Reflection in the x-axis changes the y.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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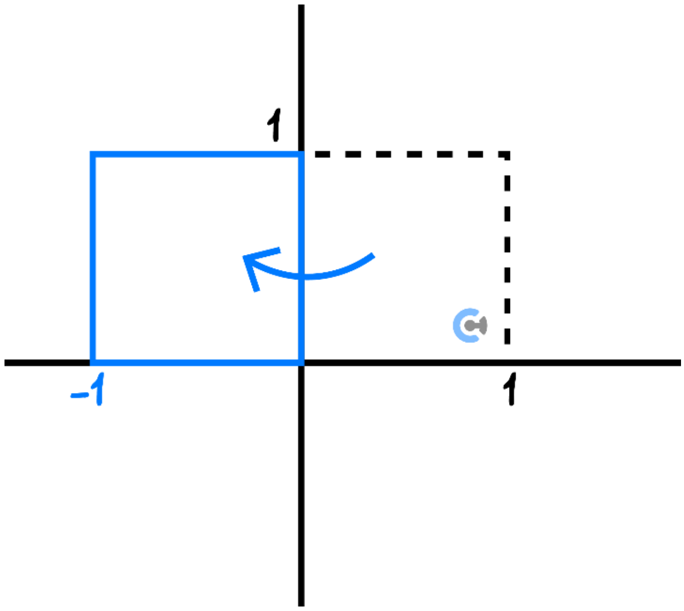
Now around y-axis.

Reflection around y-axis



Reflection in the y-axis

- Reflection in the y-axis changes the x.

$$Transformation\ Matrix = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Question 11

a. State the transformation matrix for reflection in both x and y-axis.

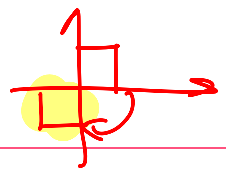
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

Discussion: Consider the size of the determinant of the reflection transformation matrix. Does it make sense?

$$|\det(\quad)| = 1 - 0 = 1$$



size doesn't change

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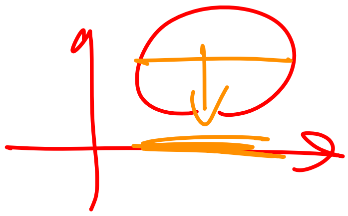
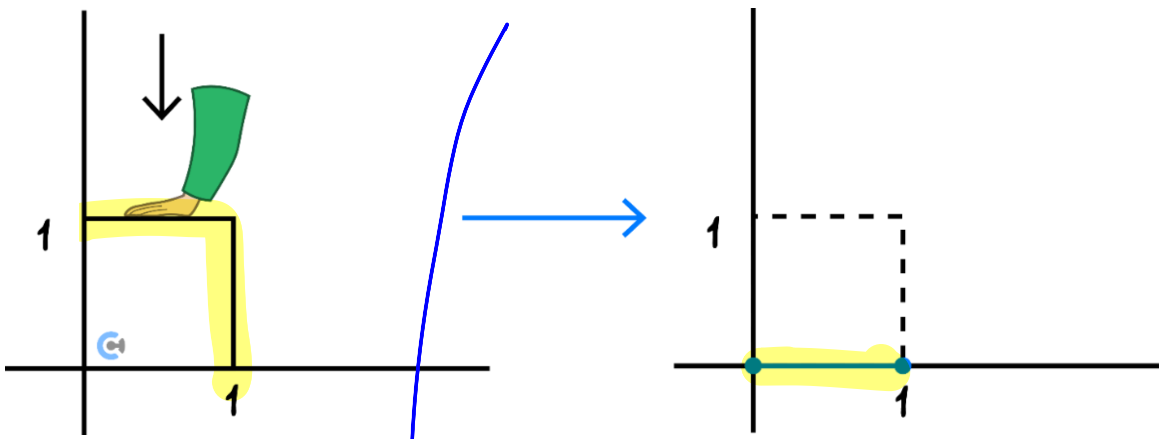
Sub-Section: Projections

What about "projections"?

Exploration: Understanding Projection



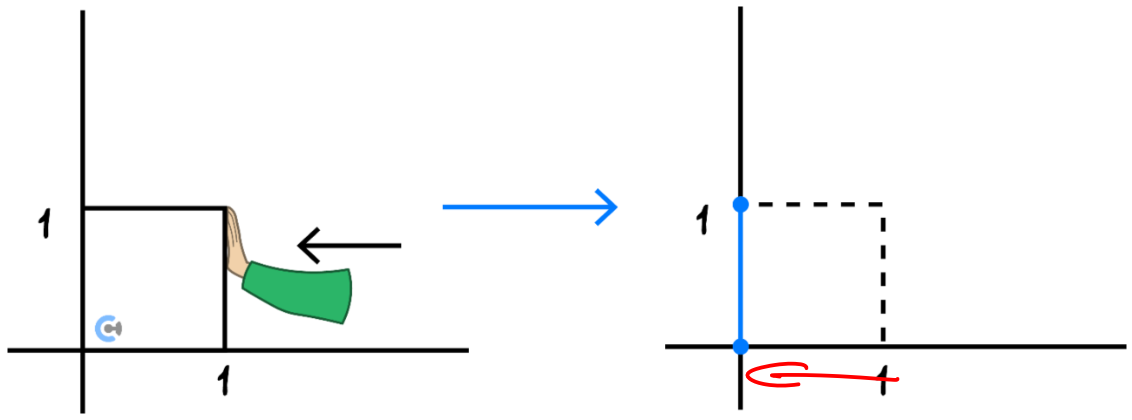
- Bringing Krish back again × 3.
- He wants to squish the unit square onto the x-axis.



- What would happen?
- Would this be a "projection" onto the x-axis or y-axis?

y - changes

$y = 0$



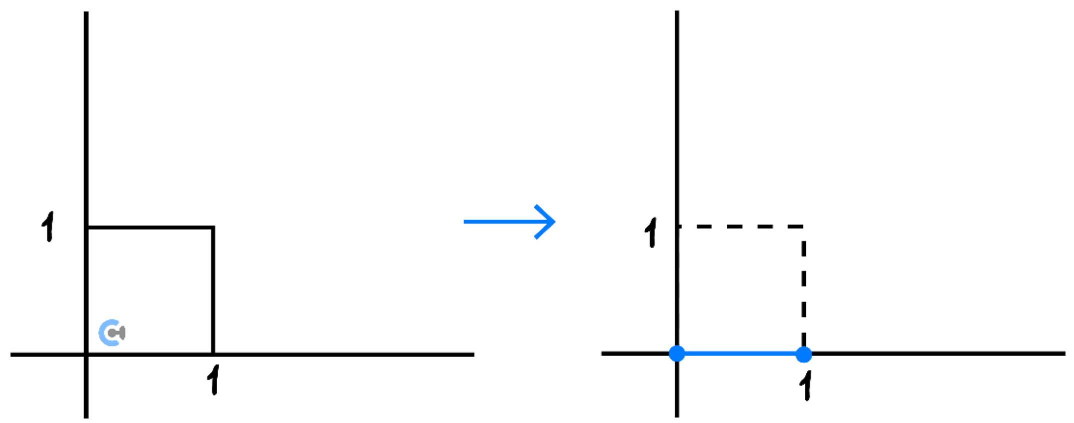
- How about now?

- What would happen?

$x = 0$

- Would this be a "projection" onto the x-axis or y-axis?

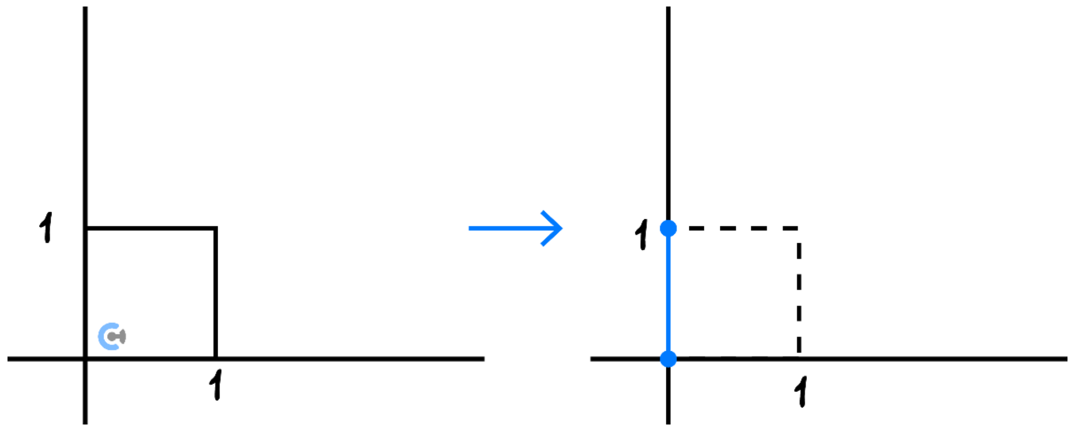
Projections



Projection onto the x-axis:

- The y becomes 0.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto the y-axis:

- The x becomes 0.

$$\textit{Transformation Matrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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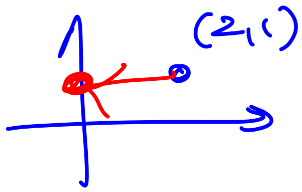
Question 12

a. State the transformation matrix for projection onto y-axis.

$x = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



b. Find the image of $(2, 1)$ after the transformation projection onto y-axis.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

NOTE: Projection onto y-axis only keeps the y-value.

Discussion: **Consider the determinant of the projection transformation matrix. Does it make sense?**

0

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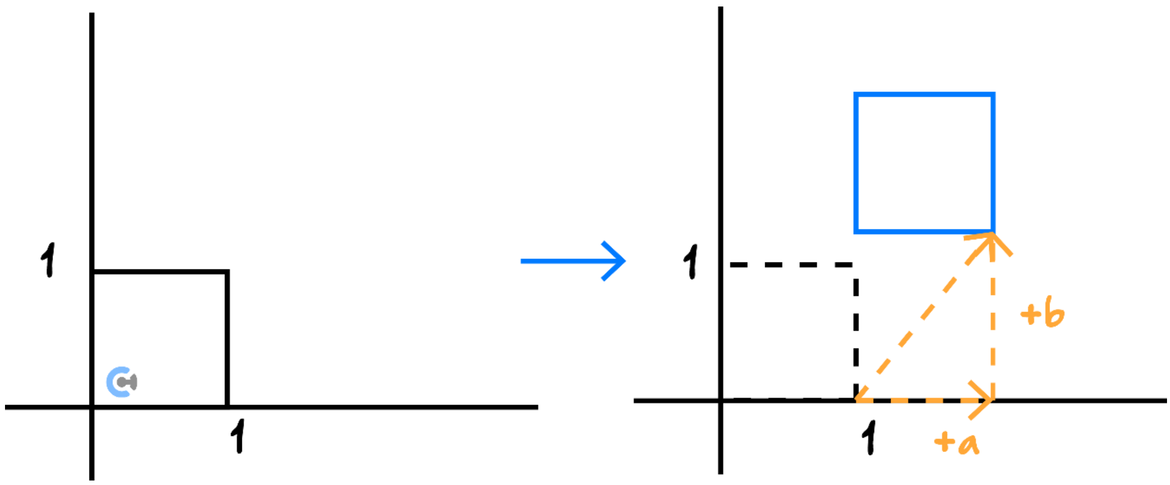
Sub-Section: Translations

Now translations!

Translation → we can't multiply



- Translation simply moves the point.

Translation a units in the positive direction of the x-axis.

Translation b units in the positive direction of the y-axis.

- We simply add/subtract the translation value to x and y.

$$Transformation: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

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Question 13

Consider the point $(4, 1)$.

The point has been translated 2 units in the positive direction of the x-axis and translated 3 units in the negative direction of the y-axis.

Find the image using matrices.

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

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Section C: Inverse Transformations

Sub-Section: Reversing Transformations

REMINDER: Inverse of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Inverse only exists for a square matrix.
- Matrix that has an inverse is called invertible.

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Question 14

Consider a transformation matrix given by $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$.

a. Find the image of $(2, 3)$ after applying transformation A.

$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

b. Find the inverse matrix of A.

$$A^{-1} = \frac{1}{-2-3}\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$$

c. Find the image of $(7, 3)$ after applying transformation A^{-1}.

$A \cdot X = X'$

$A^{-1} \cdot$ image $=$ pre-image

$$-\frac{1}{5}\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -10 \\ -15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

TIP: Take the factor out and multiply it afterwards.

Discussion: From the previous question, what do we do to reverse a transformation?

$A^{-1} \times$ image $=$ pre-image

Let's also prove this using matrix algebra!

Exploration: Algebraic Proof of Inverse Transformation

- Consider:

image, transformation matrix, pre-image

$$X' = AX$$

- Multiply A^{-1} on both sides.

NOTE: We always multiply the matrices on the LHS.

$A^{-1}X' = A^{-1}AX$

$A^{-1}X' = X$

inverse tranf. image pre-image

- What does AA^{-1} equal to?

- What does IA always equal to?

- We can multiply the inverse transformation matrix by the image to go back to the pre-image.

Inverse Transformation

$$\textbf{If } X' = AX \textbf{ then } A^{-1}X' = X.$$

- Multiply the inverse transformation matrix to the image to go back to the pre-image.

$$(x,y) \xrightarrow{A} (2,1) \qquad (2,1) \xrightarrow{A^{-1}} (x,y)$$

Question 15

A point (x, y) has been transformed by $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and the image was given by $(2, 1)$.

a. Find A^{-1}.

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} = \frac{1}{2-12}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

b. Hence, find the point (x, y).

$$A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow -\frac{1}{10}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{10}\begin{bmatrix} 2(2)-3(1) \\ -4(2)+1(1) \end{bmatrix}$$

$$= -\frac{1}{10}\begin{bmatrix} 1 \\ -7 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ \frac{7}{10} \end{bmatrix}$$

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Sub-Section: Validity of Inverse Transformations

Discussion: Do all matrices have an inverse?

No.

REMINDER: Determinant of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$ $= 0 \leftarrow$ not invertible

- If the determinant equals 0, then A does not have an inverse.
- A is not invertible.

$\frac{1}{0} =$ undef.

Discussion: If a transformation matrix A does not have an inverse A^{-1}, how can we reverse the transformation under A?

↳ we can't reverse

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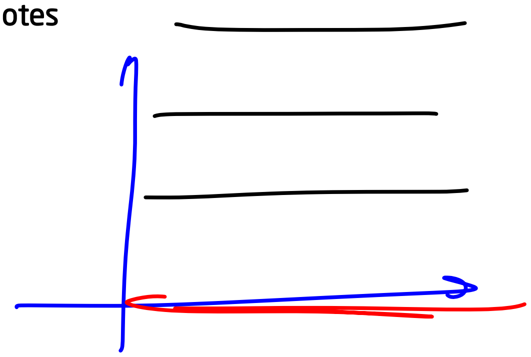

Why can't some transformations be reversed?

Question 16

Consider a transformation matrix given by $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

a. Find the det (A).

$\det(A) = 2 - 2 = 0 \Rightarrow$ No A^{-1}

↳ No reverse transformation

b. Find the image of $(3, 4)$ under the transformation given by A.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1(3)+1(4) \\ 2(3)+2(4) \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

c. Find the image of $(2, 5)$ under the transformation given by A.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1(2)+1(5) \\ 2(2)+2(5) \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$(3, 4) \longrightarrow (7, 14)$

$(2, 5) \nearrow$

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Discussion: Looking at the question above, how can we reverse the transformation from the image: $(7, 14)$?

We can't

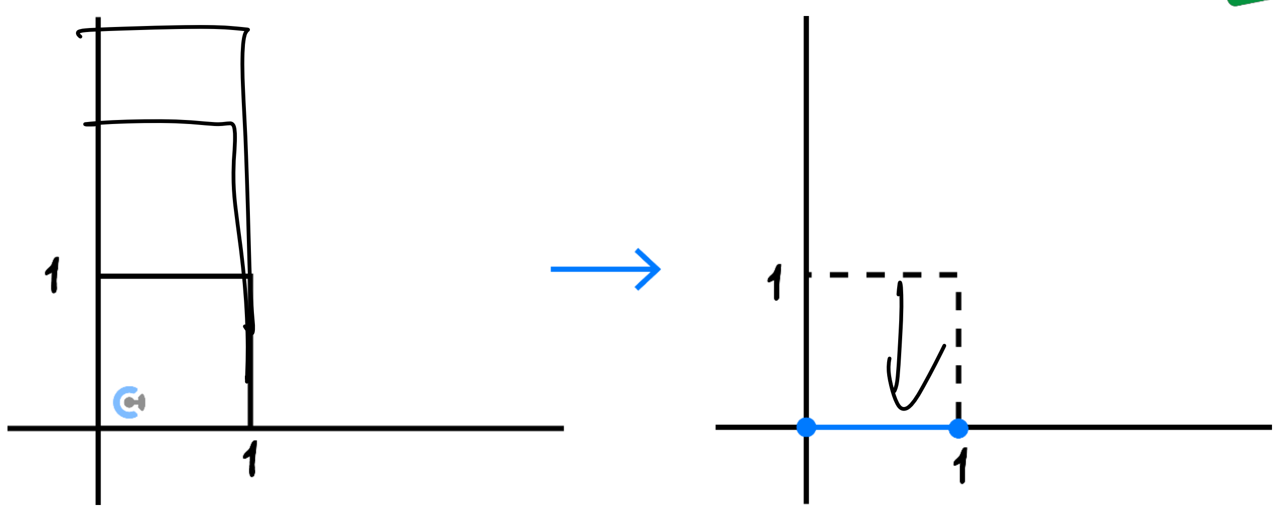
Non-Invertible Matrix and Inverse Transformations

$$X' = AX$$

$$\text{If } \det(A) = 0, \text{then } X \text{ cannot be solved as } A^{-1} \text{ is undefined.}$$

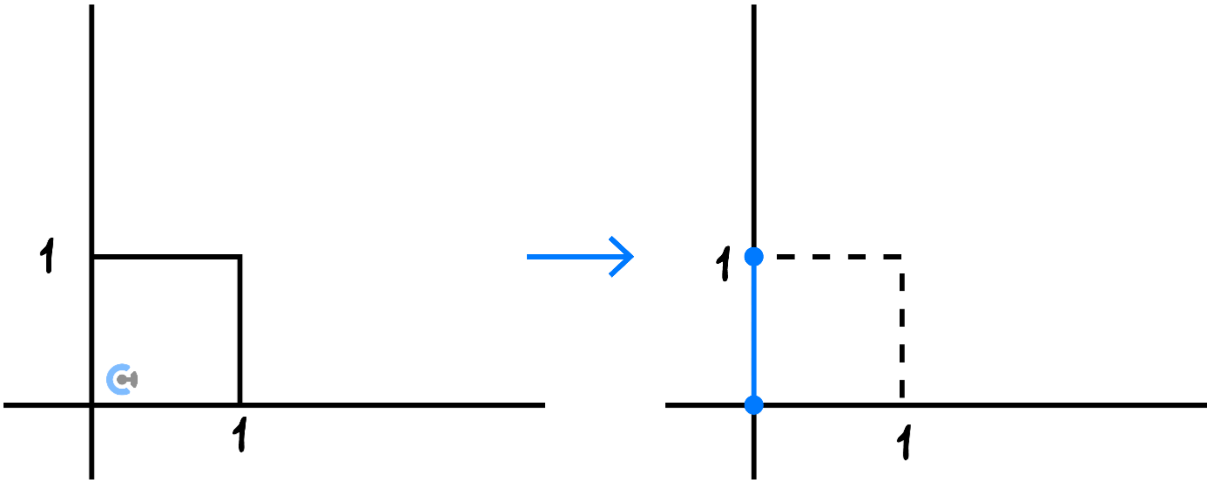
- The original point cannot be solved if the inverse matrix does not exist.
- The transformation cannot be reverse when det = 0.
- It happens as the image can be achieved from multiple pre-images.

Active Recall: Projection



Projection onto x-axis:

$$\text{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto y-axis:

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_$$

Discussion: Consider the determinant of the projection transformation matrix. Can any projection transformation be reversed? Does that make sense?

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Section D: Composite Transformations

Sub-Section: Composite Transformations

Discussion: How do we do multiple transformations?

$$\therefore \sim \overset{T_3}{\begin{bmatrix} \quad \end{bmatrix}}\overset{T_2}{\begin{bmatrix} \quad \end{bmatrix}}\overset{T_1}{\begin{bmatrix} \quad \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}$$

Composite Transformations

- For transformation under A and B respectively,

$$X' = BAX$$

- Always multiply the next transformation matrix on the front. (LHS)

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Question 17 Walkthrough.

a. State the transformation matrix for dilation by factor 2 from the x-axis and reflection in the x-axis.

1 2

2 1

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

b. Apply the transformation matrix found in **part a.** to the coordinate $(3, 1)$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

image

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Your turn!

Question 18

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

a. State the transformation matrix for dilation by factor 3 from the x-axis, shear of factor 3 parallel to the y-axis and reflection in the y-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 3 & 3 \end{bmatrix}$$

b. Hence, apply the transformation "dilation by factor 3 from the x-axis, shear of factor 3 parallel to the y-axis and reflection in the y-axis" to the coordinate $(-2, 5)$.

$$\begin{bmatrix} -1 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1(-2)+0(5) \\ 3(-2)+3(5) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

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Contour Check

Learning Objective: [4.2.1] - Using matrices for linear transformations

Key Takeaways

- **Linear Transformations:**

$$(x, y) \to (ax + by, cx + dy) = (x', y')$$

  - The (x', y') represents the new points and is called an image
  - Original point (x, y) is called the pre-image.

- **Matrices for Linear Transformations:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \text{transformation matrix}$$

- **Determinant of Transformation Matrix:**

  - Given that $A =$ Transformation matrix.

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

  - Determinant could be negative hence we put the modulus.

Learning Objective: [4.2.2] – Dilations, reflections, translations, shears and projections

Key Takeaways

- Dilation from the y-axis:

Dilation by a factor a from the y-axis.

- Dilation from the y-axis changes the x-value.

$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

- Dilation from the x-axis:

Dilation by a factor b from the x-axis.

- Dilation from the x-axis changes the y-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

- Dilation and its Transformation Matrix:

Dilation by a factor a from the y-axis.

Dilation by a factor b from the x-axis.

$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

- Shear Parallel to the x-axis:

Shear of a factor a parallel to the x-axis.

- Shear parallel to x-axis changes the x-value by a multiple of the y-value.

$$Transformation\ Matrix = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

- Shear Parallel to the y-axis:

Shear of a factor b parallel to the y-axis.

- Shear parallel to y-axis changes the y by a multiple of x.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

- Reflection around x-axis:

Reflection in the x-axis:

- Reflection in the x-axis changes the y.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Reflection around y-axis:

Reflection in the y-axis:

- Reflection in the y-axis changes the x.

$$\textit{Transformation Matrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Projections:

Projection onto the x-axis:

- The y becomes 0.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Projection onto the y-axis:

- The x becomes 0.

$$\textit{Transformation Matrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Translation:**
  - Translation simply moves the point.

Translation a units in the positive direction of the x-axis.

Translation b units in the positive direction of the y-axis.

  - We simply add/subtract the translation value to x and y.

$$Transformation: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Learning Objective: [4.2.3] – Inverse transformations

Key Takeaways

- **Inverse Transformation:**

A^{-1}

If $X' = AX$ then $X = \underline{A^{-1}X'}$.

  - Multiply the inverse transformation matrix to the image to go back to the pre-image.

- **Non-Invertible Matrix and Inverse Transformations:**

$$X' = AX$$

$$if\ \det(A) = 0, then\ X\ cannot\ be\ solved\ as\ A^{-1}\ is\ undefined.$$

  - The original point cannot be solved if the inverse matrix does not exist.
  - The transformation cannot be reversed when det = 0.
  - It happens as the image can be achieved from multiple pre-images.

Learning Objective: [4.2.4] – Composite transformations

Key Takeaways

- For transformation under A and B respectively:

$$X' = BAX$$

- Always multiply the next transformation matrix on the LHS.

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Transformations I [4.2]
Test

24.5 Marks. 1 Minute Reading. 25 Minutes Writing

Results:

Test Questions	________ / 24.5

Section A: Test Questions (24.5 Marks)

Question 1 (4.5 marks)

Tick whether the following statements are **true** or **false**.

Statement	**True**	**False**
a. A linear transformation is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.		
b. You can find the first column of a linear transformation by finding the coordinates that $(1, 0)$ maps to, and the second column by finding the coordinates that $(0, 1)$ maps to.		
c. The point $(2, -1)$ under the transformation $(x, y) \to (2x - 3y, -x + 4y)$ has image $(1, -6)$.		
d. If a triangle T has area $k\ units^2$, it then undergoes a transformation represented by matrix A, where $\mid det(A) \mid = c$, then the area of the transformation triangle T' has an area $\frac{k}{c}\ units^2$.		
e. A dilation by a factor 2 from the x-axis is always the same as a dilation by a factor $\frac{1}{2}$ from the y-axis.		
f. For a shear parallel to the x-axis, the points further away from the x-axis vertically shift further horizontally than the points closer to the x-axis.		
g. A projection of any point onto the y-axis will reduce the y-coordinate to 0.		
h. In order for a transformation to be reversed, the determinant of the transformation matrix must not be 0 and there must be only one pre-image for the image.		
i. If there is a sequence of transformations T_1, T_2, T_3 applied in that order, then the resulting transformation will be given by $T_R = T_1 T_2 T_3$, where each T_i is a 2×2 matrix.		

Question 2 (6 marks)

For the following questions, state the composite transformation matrix and find the image of the point $(1, -2)$.

a. Reflection in the y-axis and projection onto x-axis. (2 marks)

b. Shear of factor 4 parallel to the x-axis and reflection in the x-axis. (2 marks)

c. Dilation of factor 2 from y-axis and shear of factor 3 parallel to the y-axis. (2 marks)

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Question 3 (9 marks)

$$\boldsymbol{A} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

a. Find $det\,\boldsymbol{A}$. (1 mark)

b. Find $\boldsymbol{A}^{-1}$. (2 marks)

The triangle R is transformed into the triangle S by the matrix $\boldsymbol{A}$.

c. Given that the area of triangle S is 72 square units, find the area of triangle R. (2 marks)

The triangle S has vertices at the points $(0, 4)$, $(8, 16)$ and $(12, 4)$.

d. Find the coordinates of the vertices of R. (4 marks)

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Question 4 (5 marks)

Consider the transformation given by $A = \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}$.

a. Find the image of $(1,1)$ under the transformation A. (1 mark)

$$\begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

b. Find the inverse matrix A^{-1}. (2 marks)

$$A^{-1} = \frac{1}{-5-6}\begin{bmatrix} 5 & -3 \\ -2 & -1 \end{bmatrix}$$

$$= -\frac{1}{11}\begin{bmatrix} 5 & -3 \\ -2 & -1 \end{bmatrix}$$

It is known that (a, b) under the transformation A was $(11, 33)$.

A

A^{-1}

c. Find the values of a and b. (2 marks)

$$A^{-1} \cdot \begin{bmatrix} 11 \\ 33 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} 5 & -3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 11 \\ 33 \end{bmatrix}$$

$$= -\frac{1}{11}\begin{bmatrix} -44 \\ -55 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad a=4 \quad b=5$$

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