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VCE Specialist Mathematics ½
Transformations I [4.2]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [4.2.1]** – Using Matrices for Linear Transformations
- ☐ **SM12 [4.2.2]** – Dilations, Reflections, Translations, Shears and Projections
- ☐ **SM12 [4.2.3]** – Inverse Transformations
- ☐ **SM12 [4.2.4]** – Composite Transformations

Section A: Linear Transformations

Sub-Section: Introduction to Linear Transformations

Context: Linear Transformations

- Consider a point $(1, 4)$.
- What would the new x-value be if it's triple the current x-values plus double the current y-value?
- What would the new y-value be if it's double the current x-values minus half the current y-value?
- Hence, what would the new point be?

Linear Transformations

$$(x, y) \rightarrow (ax + by, cx + dy) = (x', y')$$

- The (x', y') represents the new points and is called an ________.
- Original point (x, y) is called the __________.

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Question 1

Find the image of the point $(2, 1)$ under the transformation with rule $(x, y) \to (3x - 5y, 2x - 4y)$.

REMINDER: Matrix Multiplication

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + (-1) \times 1 \\ 1 \times 3 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$Number\ of\ Columns\ of\ 1^{st}\ Matrix = Number\ of\ Rows\ of\ 2^{nd}$$

- The answer will always be a matrix.

How can we represent the transformation using matrices?

Exploration: Matrices for Linear Transformations

- Consider the following matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Evaluate the answer for the above multiplication!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = Transformation\ Matrix$$

Question 2 Walkthrough.

Consider a point (x, y) which is represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

Find the image given by $\begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

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Question 3

Consider a point (x, y) which is represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

Find the transformed point given by $\begin{bmatrix} 3 & 2 \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

NOTE: $\begin{bmatrix} 3 & 2 \\ 2 & -\frac{1}{2} \end{bmatrix}$ is called a transformation matrix.

<u>Discussion:</u> Considering the answer from above, why is it called linear transformation?

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Question 4

a. Find the matrix of the linear transformation with the rule $(x, y) \rightarrow (x - 2y, 3x + y)$.

b. Use the matrix to find the image of the point $(2, 3)$ under the transformation.

c. The image of a point (c, d) under the linear transformation is $(2, 3)$. Find c and d.

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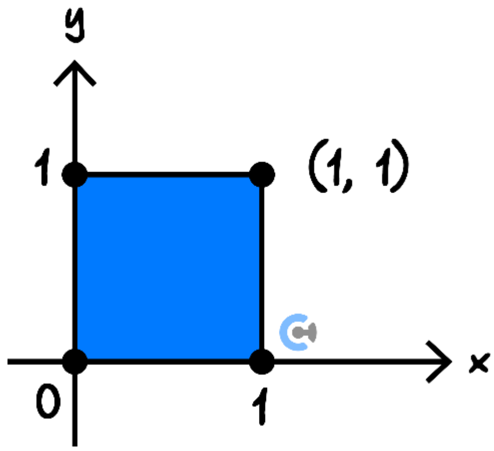
Sub-Section: Unit Square

What would be the best way to visualise the linear transformations?

Transforming the Unit Square

Unit Square has a side length of 1.



- Unit square has a coordinate ________________________________.
- Apply the transformation to $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ to see the effect of the transformations.

NOTE: We use unit squares to visualise how the transformation affects different points.

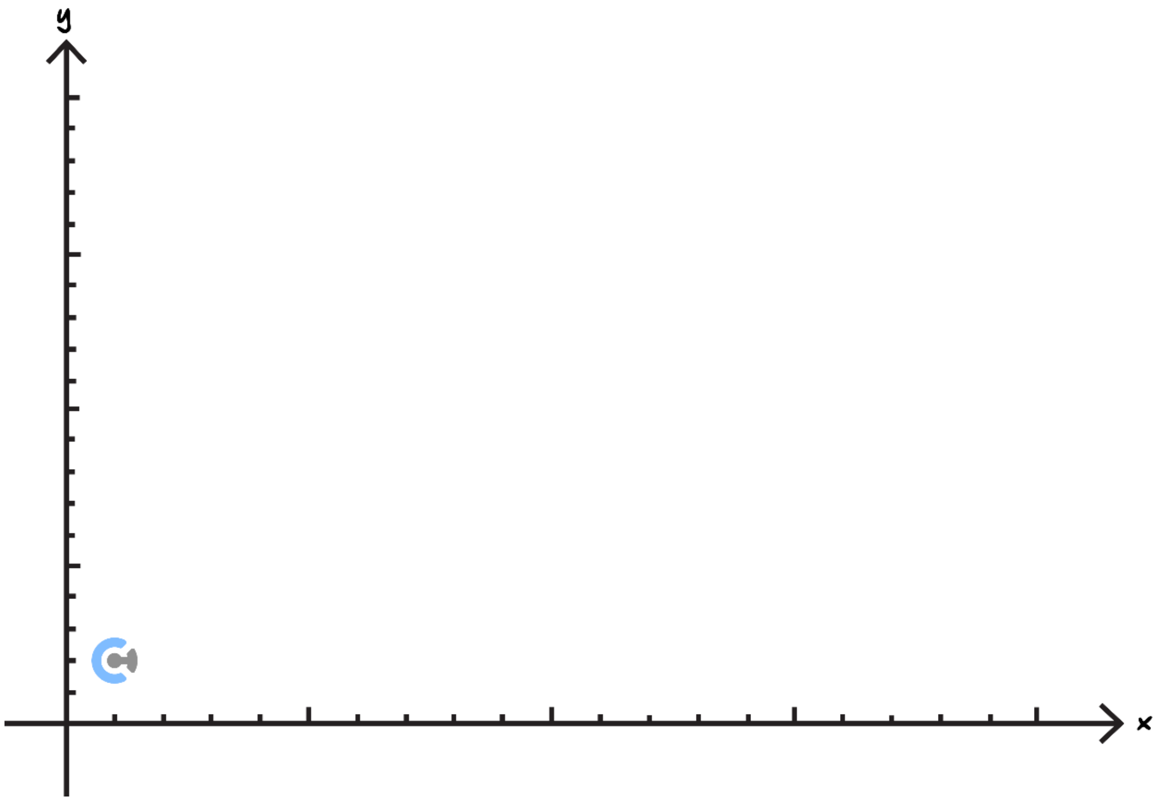
Discussion: Does it have to be a square then?

Question 5 Walkthrough.

A linear transformation is represented by the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$.

a. Find the image of the points of the unit square $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ under this transformation and write the image points as column vectors.

b. Sketch the unit square and its image on the axes below.



NOTE: Unit square simply helps us to understand how the transformation affects the points.

Discussion: How could we have done the linear transformations for $(\mathbf{0}, \mathbf{0})$, $(\mathbf{1}, \mathbf{0})$, $(\mathbf{0}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{1})$ using one matrix multiplication?

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Sub-Section: Determinant and Area of Unit Square

REMINDER: Determinant of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

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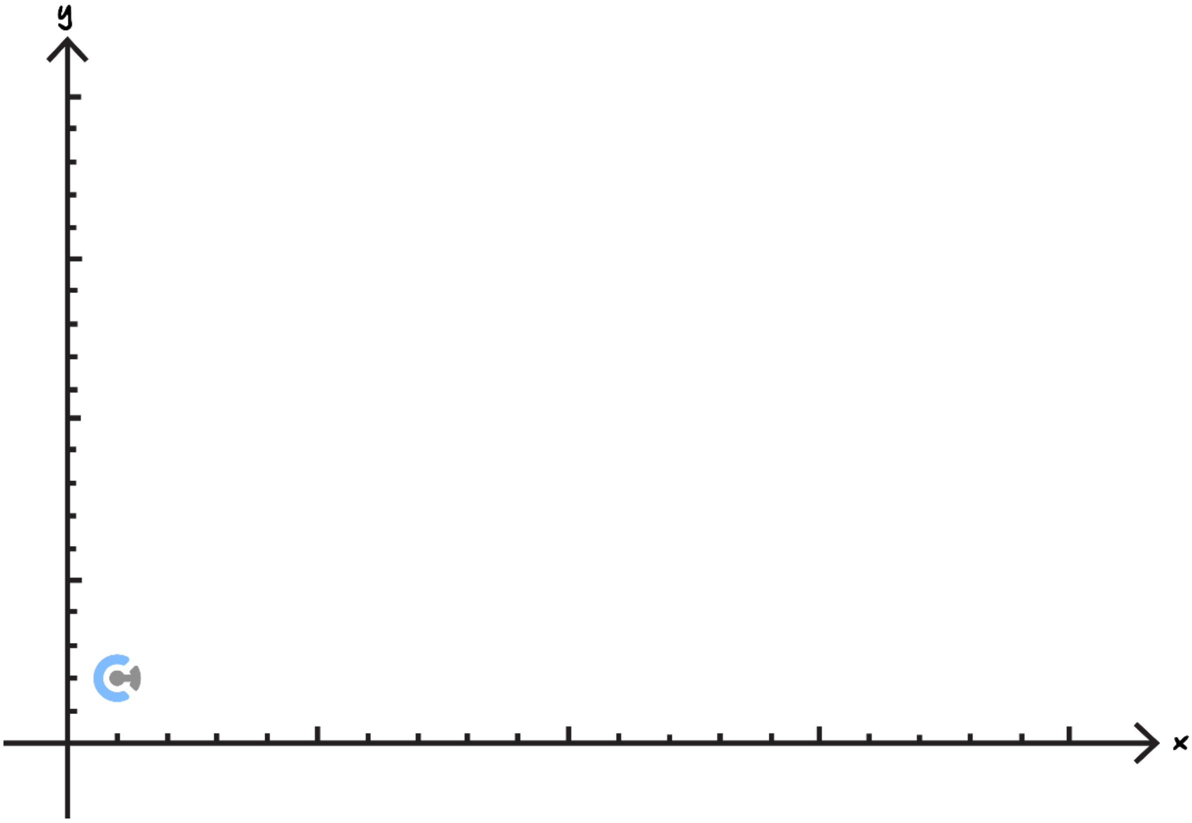
Question 6 Walkthrough.

A linear transformation is represented by the transformation matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

a. Find the image of the points of the unit square $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ under this transformation and write the image points as column vectors.

b. Sketch the unit square and its image on the axes below.



c. State the area of the unit square and its image.

d. Find the determinant of the transformation matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Discussion: What do you notice? What does the determinant of the transformation matrix tell us?

Determinant of Transformation Matrix

- Given that $A =$ Transformation matrix.

$$\textit{Area of the image} = |\det(A)| \times \textit{Area of the pre image}$$

- Determinant could be ____________ hence we put the modulus.

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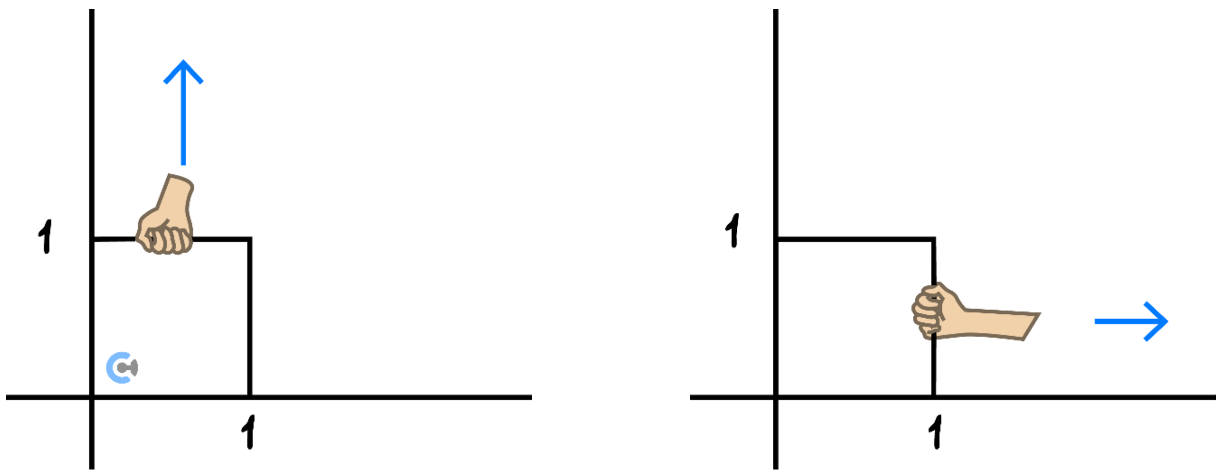
Section B: Types of Transformations

Sub-Section: Dilations

What do dilations do?

Exploration: Understanding Dilations

- Let's say Krish is bored that the unit square has a length of 1, and decides to stretch the unit square from the x and the y-axis.



- From the diagram above, state which one is dilation from the x-axis and y-axis.

- Which variable (x or y) does the dilation from the x-axis change?

- Which variable (x or y) does the dilation from the y-axis change?

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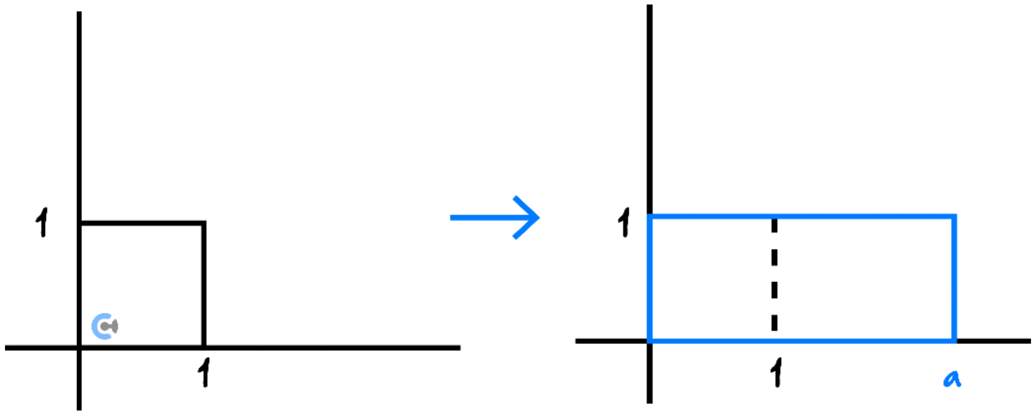
Definition

<u>Dilation from the y-axis</u>



Dilation by a factor a from the y-axis.

- Dilation from the y-axis changes the ___________.

$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

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Question 7

a. State the transformation matrix for dilation by a factor of 3 from the y-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

NOTE: The x-value is tripled for dilation by a factor 3 from the y-axis.

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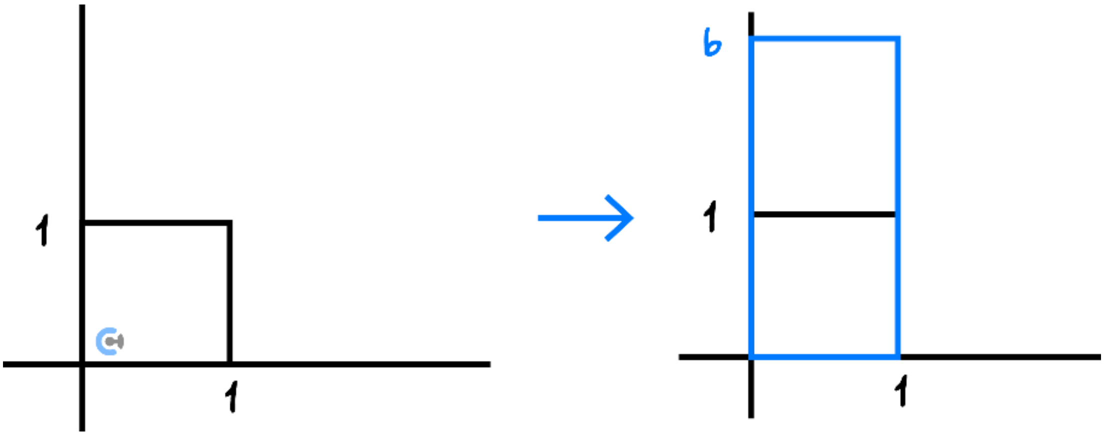
Definition

Dilation from the x-axis



Dilation by a factor b from the x-axis.

- Dilation from the x-axis changes the ________.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

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Question 8

a. State the transformation matrix for dilation by factor 2 from the x-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

NOTE: The y-value is doubled for dilation by a factor 2 from the x-axis.

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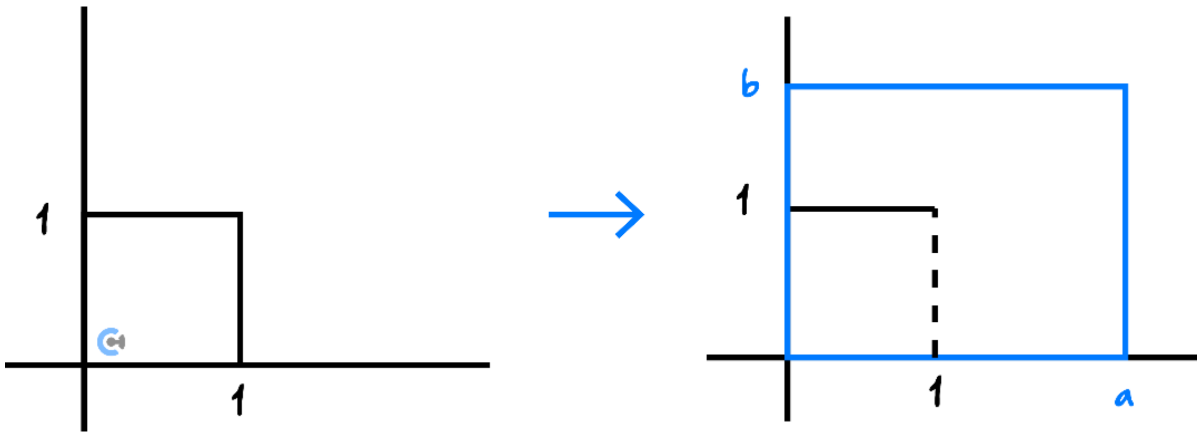
Let's combine.

Definition

Dilation and its Transformation Matrix



Dilation by a factor a from the y-axis.

Dilation by a factor b from the x-axis.

$$Transformation\ Matrix = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Discussion: Find the determinant of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Does it make sense?

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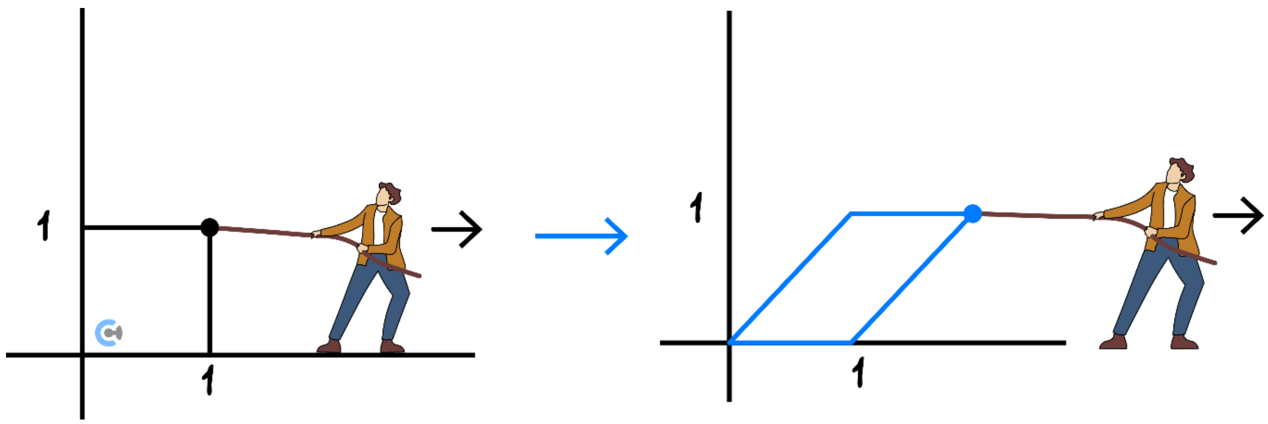
Sub-Section: Shear

What about "shear"?

Exploration: Understanding Shear Parallel to the x-axis

- Let's bring Krish back again.
- He ties a rope on the point $(1,1)$ of the "malleable" unit square and pulls it parallel to x-axis.



- Which variable (x or y) would change?

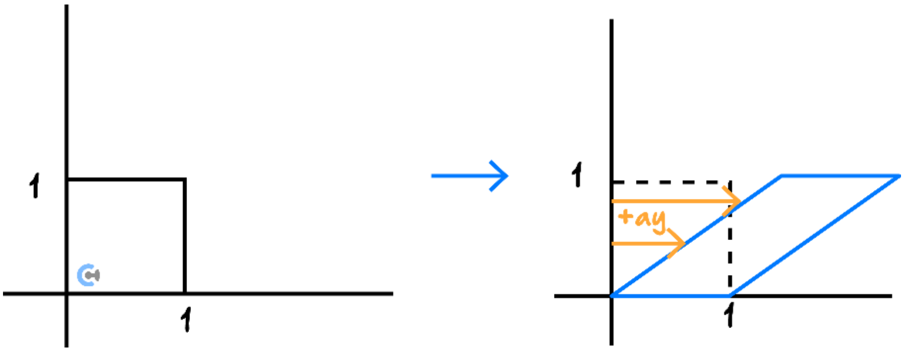
- Would all the points move the same distance parallel to the x-axis?

- Does the point move more if they are further from the x-axis or closer?

- Therefore, what does the change in x-value correspond to?

NOTE: The x-value changes with respect to how big their y-value is.

<u>Shear Parallel to the x-axis</u>



Shear of a factor a parallel to the x-axis.

- Shear parallel to x-axis changes the ______________________________.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

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Question 9

a. State the transformation matrix for the shear of a factor 3 parallel to the x-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

NOTE: The x-value is added by tripling the y-value.

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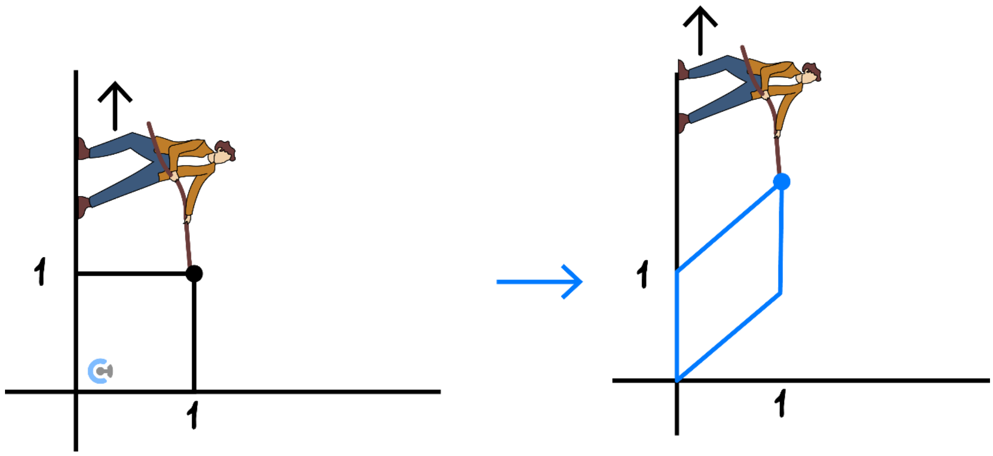
What about in the direction of the y-axis?

<u>Exploration</u>: Understanding Shear Parallel to the y-axis

- Let's bring Krish back again $\times$ 2.
- He ties a rope on the point $(1, 1)$ of the "malleable" unit square and pulls it parallel to y-axis.

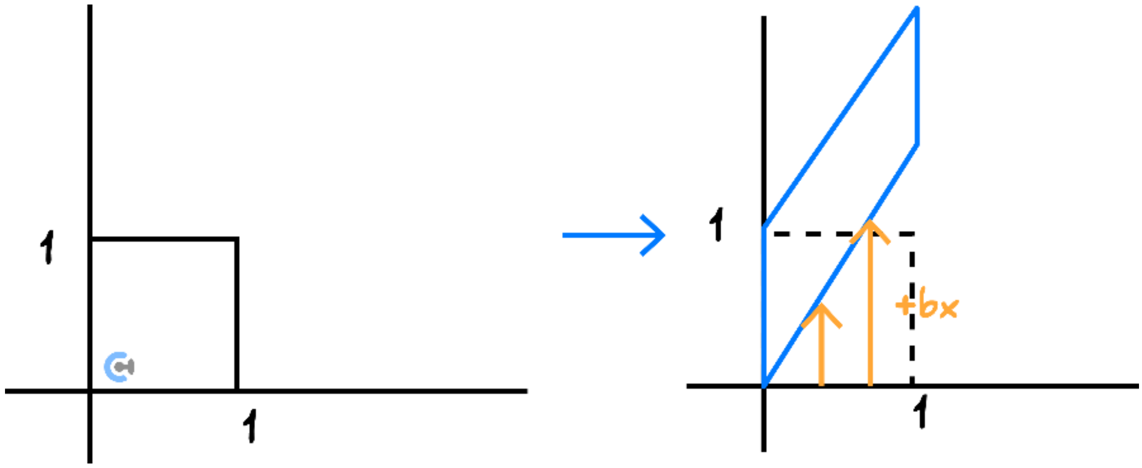


- Which variable (x or y) would change?
- Would all the points move the same distance parallel to the y-axis?
- Does the point move more if they are further from the y-axis or closer?
- Therefore, what does the change in y-value correspond to?

NOTE: The y-value changes with respect to how big their x-value is.

Definition

<u>Shear Parallel to the y-axis</u>



Shear of a factor b parallel to the y-axis.

- Shear parallel to y-axis changes the ______________________________.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

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Question 10

a. State the transformation matrix for the shear of a factor 2 parallel to the y-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

NOTE: The y-value is added by doubling the x-value.

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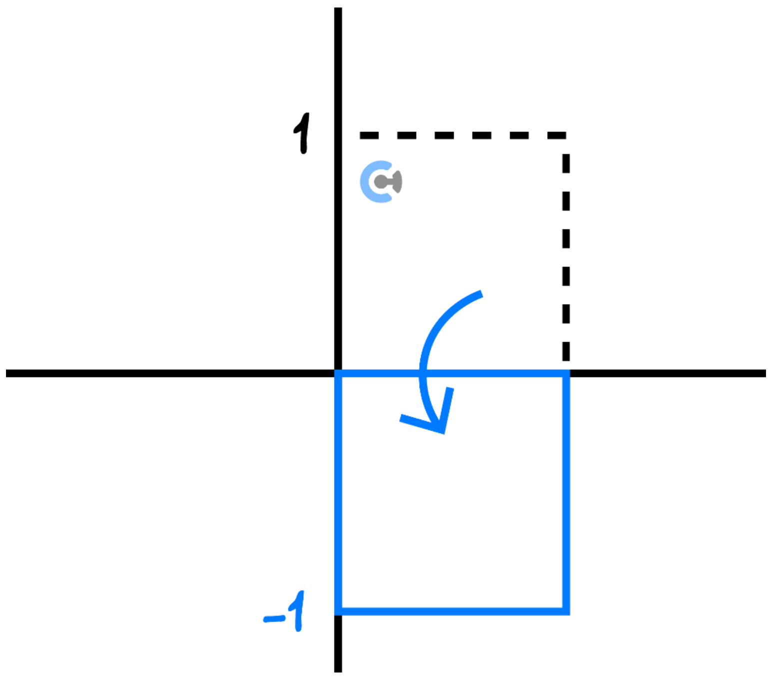
Sub-Section: Reflections around x and y-axis

Discussion: If you reflect something around the x-axis, what would happen? What about the y-axis?

Reflection around x-axis



Reflection in the x-axis.

- Reflection in the x-axis changes the ________.

$$\textit{Transformation Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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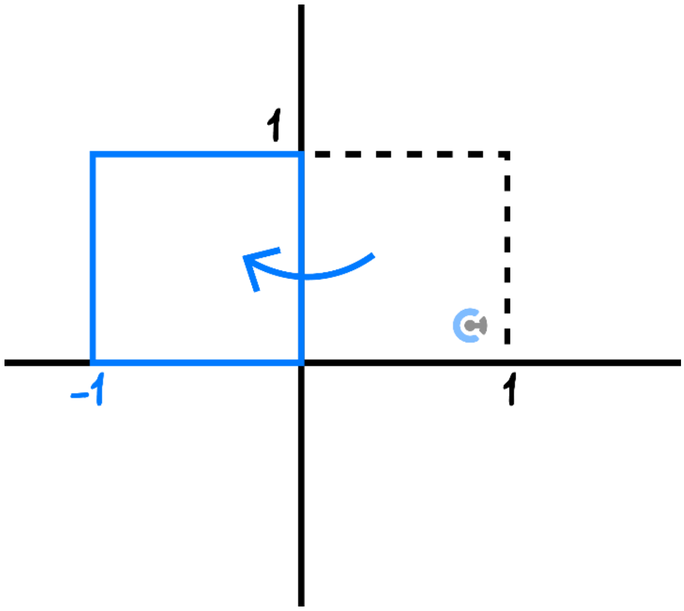
Now around y-axis.

Reflection around y-axis



Reflection in the y-axis

- Reflection in the y-axis changes the __________.

$$\textit{Transformation Matrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Question 11

a. State the transformation matrix for reflection in both x and y-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate (x, y).

Discussion: Consider the size of the determinant of the reflection transformation matrix. Does it make sense?

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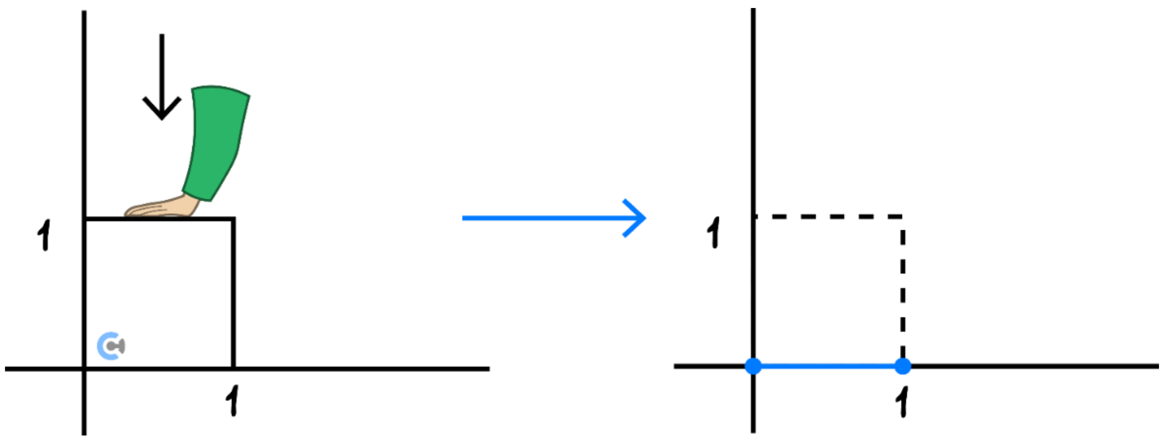
Sub-Section: Projections

What about "projections"?

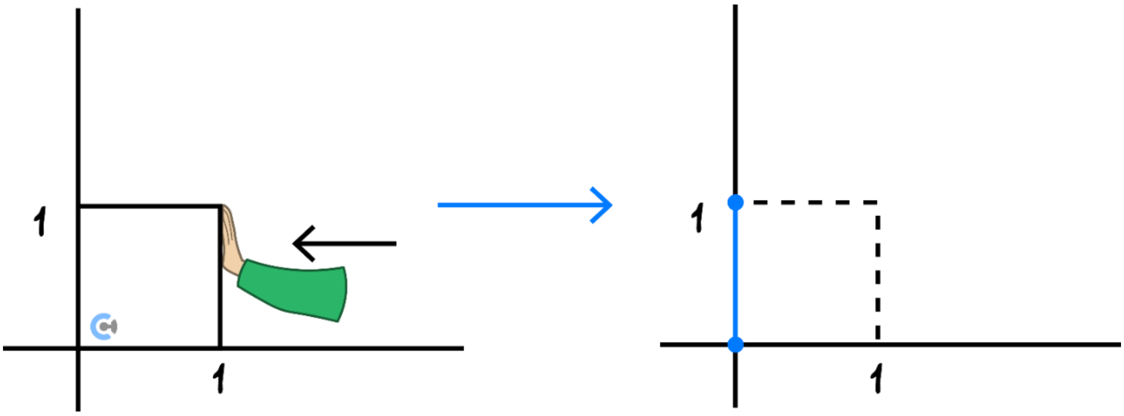
Exploration: Understanding Projection

- Bringing Krish back again $\times$ 3.
- He wants to squish the unit square onto the x-axis.



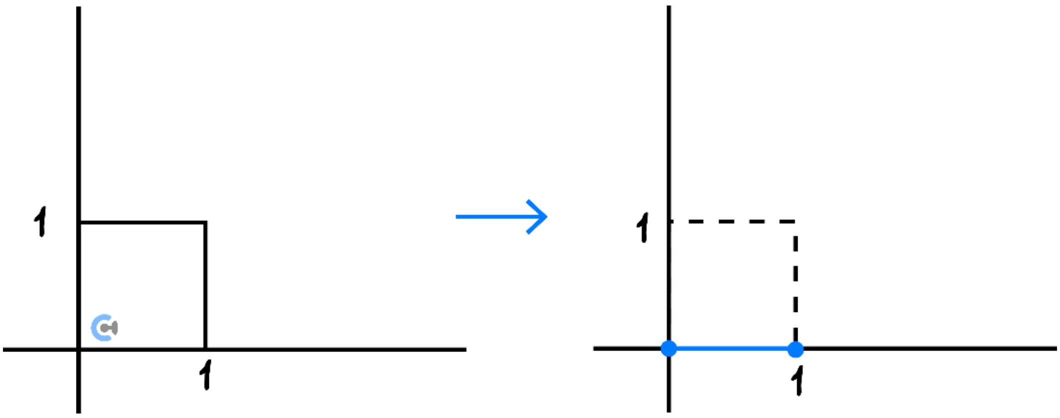
- What would happen?

- Would this be a "projection" onto the x-axis or y-axis?



- How about now?
- What would happen?
- Would this be a "projection" onto the x-axis or y-axis?

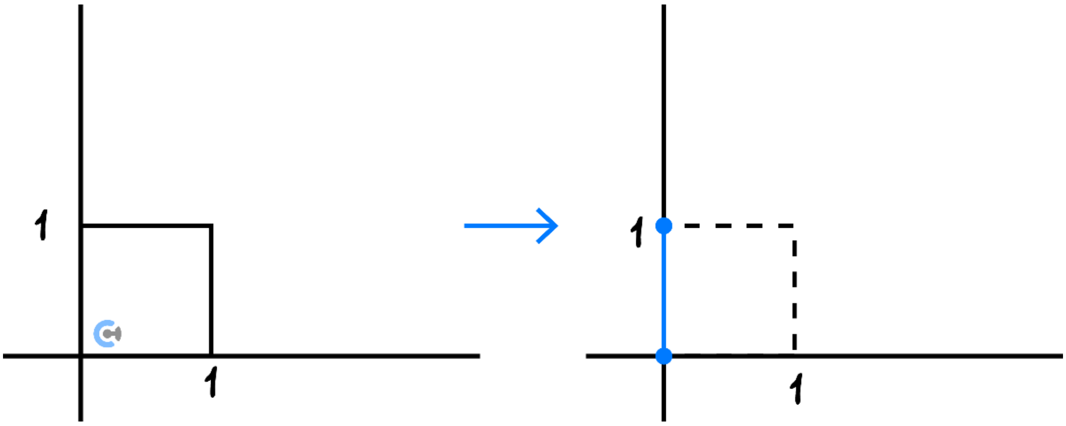
Projections



Projection onto the x-axis:

- The ________ becomes 0.

$$Transformation\ Matrix = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto the y-axis:

- The __________ becomes 0.

$$\textbf{Transformation Matrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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Question 12

a. State the transformation matrix for projection onto y-axis.

b. Find the image of $(2, 1)$ after the transformation projection onto y-axis.

NOTE: Projection onto y-axis only keeps the y-value.

Discussion: **Consider the determinant of the projection transformation matrix. Does it make sense?**

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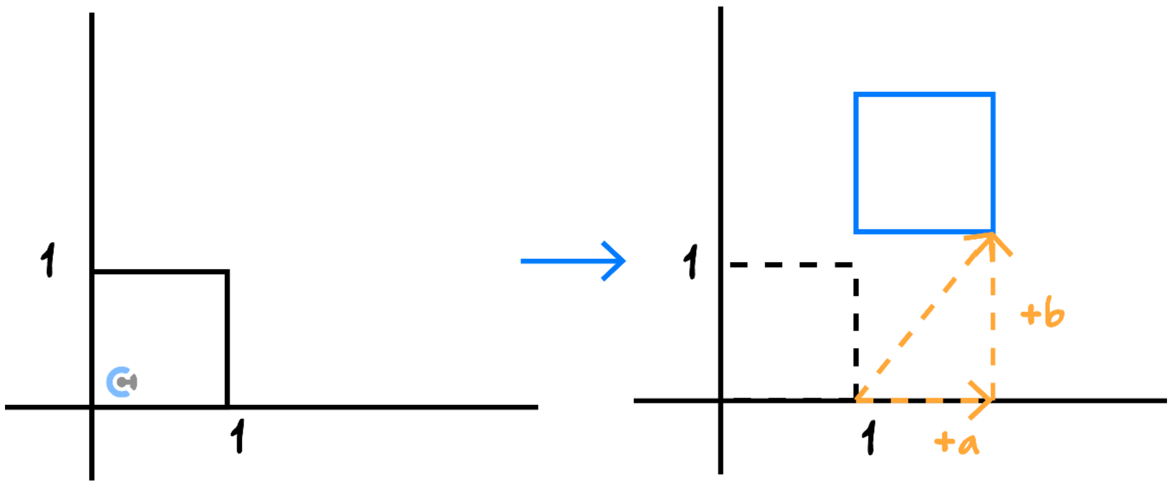
Sub-Section: Translations

Now translations!

Translation



- Translation simply moves the point.

Translation a units in the positive direction of the x-axis.

Translation b units in the positive direction of the y-axis.

- We simply add/subtract the translation value to x and y.

$$\boldsymbol{Transformation}: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Space for Personal Notes

Question 13

Consider the point $(4, 1)$.

The point has been translated 2 units in the positive direction of the x-axis and translated 3 units in the negative direction of the y-axis.

Find the image using matrices.

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Section C: Inverse Transformations

Sub-Section: Reversing Transformations

REMINDER: Inverse of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Inverse only exists for a square matrix.
- Matrix that has an inverse is called ______________.

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Question 14

Consider a transformation matrix given by $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$.

a. Find the image of $(2, 3)$ after applying transformation A.

b. Find the inverse matrix of A.

c. Find the image of $(7, 3)$ after applying transformation A^{-1}.

TIP: Take the factor out and multiply it afterwards.

Discussion: From the previous question, what do we do to reverse a transformation?

Let's also prove this using matrix algebra!

Exploration: Algebraic Proof of Inverse Transformation

- Consider:

$$X' = AX$$

- Multiply A^{-1} on both sides.

  NOTE: We always multiply the matrices on the LHS.

- What does AA^{-1} equal to?

- What does IA always equal to?

- We can multiply the inverse transformation matrix by the image to go back to the pre-image.

Inverse Transformation

$$\text{If } X' = AX \text{ then } A^{-1}X' = X.$$

- Multiply the inverse transformation matrix to the image to go back to the pre-image.

Question 15

A point (x, y) has been transformed by $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and the image was given by $(2, 1)$.

a. Find A^{-1}.

b. Hence, find the point (x, y).

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Sub-Section: Validity of Inverse Transformations

Discussion: Do all matrices have an inverse?

REMINDER: Determinant of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

- If the determinant equals ___________, then A does not have an inverse.
- A is not ______________________.

Discussion: If a transformation matrix A does not have an inverse A^{-1}, how can we reverse the transformation under A?

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Why can't some transformations be reversed?

Question 16

Consider a transformation matrix given by $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

a. Find the det (A).

b. Find the image of $(3, 4)$ under the transformation given by A.

c. Find the image of $(2, 5)$ under the transformation given by A.

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<u>Discussion:</u> Looking at the question above, how can we reverse the transformation from the image: $(7, 14)$?

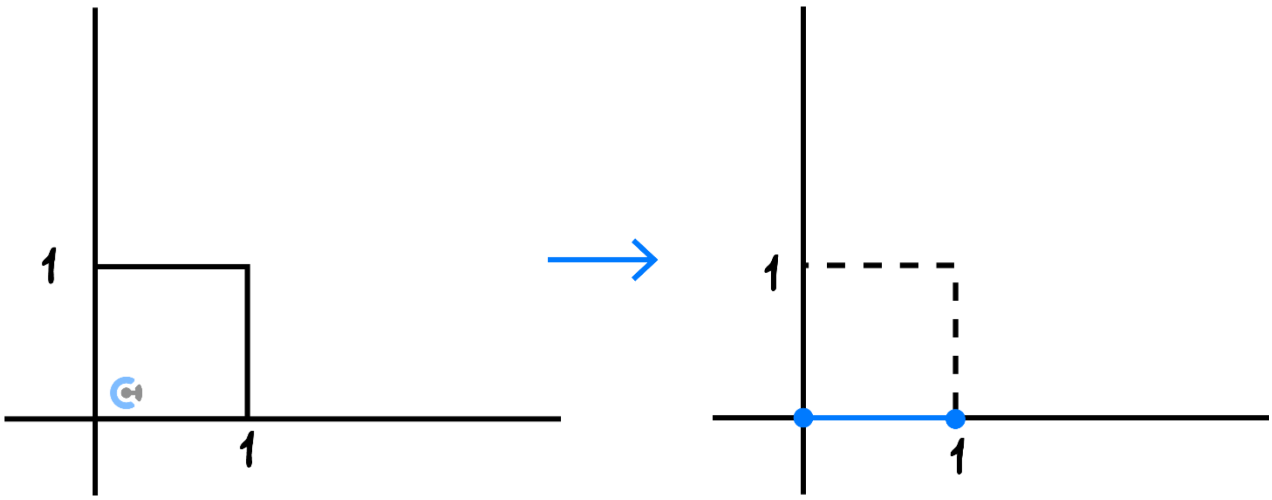
<u>Non-Invertible Matrix and Inverse Transformations</u>

$$X' = AX$$

$$If\ \det(A) = 0, then\ X\ cannot\ be\ solved\ as\ A^{-1}\ is\ undefined.$$

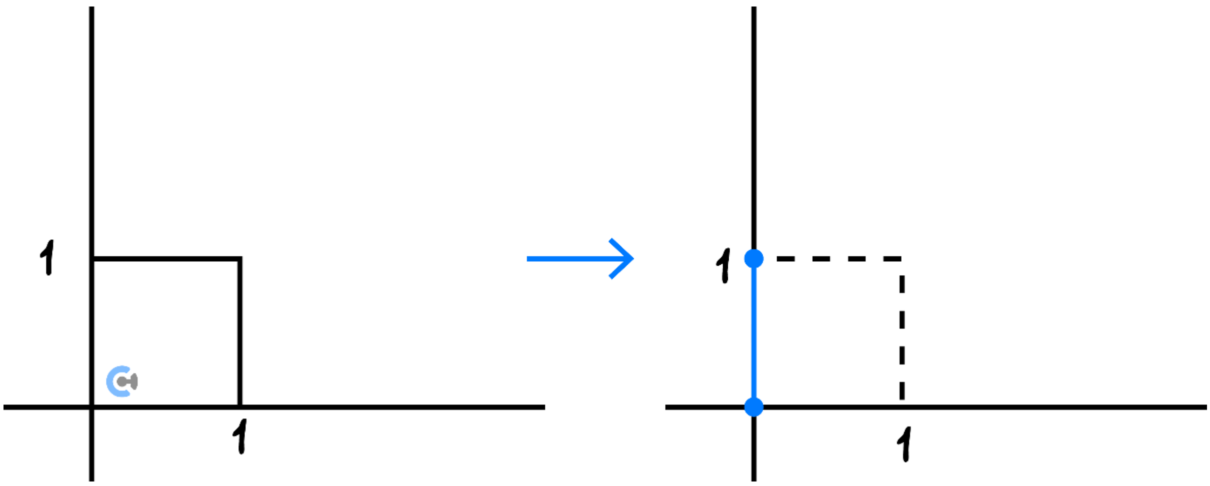
- The original point cannot be solved if the inverse matrix does not exist.
- The transformation cannot be ____________when ___________.
- It happens as the image can be achieved from multiple pre-images.

<u>Active Recall:</u> Projection



Projection onto x-axis:

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_$$



Projection onto y-axis:

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_$$

<u>Discussion:</u> Consider the determinant of the projection transformation matrix. Can any projection transformation be reversed? Does that make sense?

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Section D: Composite Transformations

Sub-Section: Composite Transformations

Discussion: How do we do multiple transformations?

Composite Transformations

- For transformation under A and B respectively,

$$X' = BAX$$

- Always multiply the next transformation matrix on the ________.

Space for Personal Notes

Question 17 Walkthrough.

a. State the transformation matrix for dilation by factor 2 from the x-axis and reflection in the x-axis.

b. Apply the transformation matrix found in **part a.** to the coordinate $(3, 1)$.

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Your turn!

Question 18

a. State the transformation matrix for dilation by factor 3 from the x-axis, shear of factor 3 parallel to the y-axis and reflection in the y-axis.

b. Hence, apply the transformation "dilation by factor 3 from the x-axis, shear of factor 3 parallel to the y-axis and reflection in the y-axis" to the coordinate $(-2, 5)$.

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Contour Check

Learning Objective: [4.2.1] - Using matrices for linear transformations

Key Takeaways

- **Linear Transformations:**

$$(x, y) \rightarrow (ax + by, cx + dy) = (x', y')$$

  - The (x', y') represents the new points and is called an ________.
  - Original point (x, y) is called the ____________.

- **Matrices for Linear Transformations:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- **Determinant of Transformation Matrix:**
  - Given that $A =$ Transformation matrix.

$$Area\ of\ the\ image = |\det(A)| \times Area\ of\ the\ pre\ image$$

  - Determinant could be __________ hence we put the modulus.

- **Learning Objective: [4.2.2] – Dilations, reflections, translations, shears and projections**

Key Takeaways

- Dilation from the y-axis:

Dilation by a factor a from the y-axis.

  - Dilation from the y-axis changes the _____________.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_$$

- Dilation from the x-axis:

Dilation by a factor b from the x-axis.

  - Dilation from the x-axis changes the _________.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_$$

- Dilation and its Transformation Matrix:

Dilation by a factor a from the y-axis.

Dilation by a factor b from the x-axis.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_$$

- Shear Parallel to the x-axis:

Shear of a factor a parallel to the x-axis.

  - Shear parallel to x-axis changes the ______________________________.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_$$

- Shear Parallel to the y-axis:

Shear of a factor b parallel to the y-axis.

- Shear parallel to y-axis changes the ______________________________.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Reflection around x-axis:

Reflection in the x-axis:

- Reflection in the x-axis changes the __________.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Reflection around y-axis:

Reflection in the y-axis:

- Reflection in the y-axis changes the ____________.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Projections:

Projection onto the x-axis:

- The _________ becomes 0.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_\_\_$$

Projection onto the y-axis:

- The __________ becomes 0.

$$Transformation\ Matrix = \_\_\_\_\_\_\_\_\_\_\_\_$$

- **Translation:**
  - Translation simply moves the point.

Translation a units in the positive direction of the x-axis.

Translation b units in the positive direction of the y-axis.

  - We simply add/subtract the translation value to x and y.

$$Transformation: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Learning Objective: [4.2.3] – Inverse transformations

Key Takeaways

- **Inverse Transformation:**

If $X' = AX$ then $X =$ __________.

  - Multiply the inverse transformation matrix to the image to go back to the ____________.

- **Non-Invertible Matrix and Inverse Transformations:**

$$X' = AX$$

$$if\ \det(A) = 0, then\ X\ cannot\ be\ solved\ as\ A^{-1}\ is\ undefined.$$

  - The original point cannot be solved if the inverse matrix does not exist.
  - The transformation cannot be _________ when ________.
  - It happens as the image can be achieved from multiple pre-images.

☐ Learning Objective: [4.2.4] – Composite transformations

Key Takeaways

- ☐ For transformation under A and B respectively:

$$X' = BAX$$

- ☐ Always multiply the next transformation matrix on the ________.