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VCE Specialist Mathematics ½
Transformations I [4.2]
Homework Solutions

Admin Info & Homework Outline:

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Questions You Need Help For	
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Section A: Compulsory Questions

Sub-Section [4.2.1]: Using Matrices for Linear Transformations

Question 1

For each of the following, write the transformation matrix for the given mapping.

a. $(x, y) \to (2x + 3y, 4x + 5y)$

The transformation can be represented using the matrix:

$$T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

b. $(x, y) \to (-x + 4y, 6x - 2y)$

The transformation matrix is:

$$T = \begin{bmatrix} -1 & 4 \\ 6 & -2 \end{bmatrix}.$$

c. $(x, y) \to (7x - 3y, -5x + 8y)$

The transformation matrix is:

$$T = \begin{bmatrix} 7 & -3 \\ -5 & 8 \end{bmatrix}.$$

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Question 2

a. Find the image of the point $(3,-2)$ under the transformation matrix:

$$T = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$$

The image of $(3,-2)$ is found by multiplying the transformation matrix by the point vector:

$$T\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Computing:

$$\begin{bmatrix} (4)(3)+(1)(-2) \\ (-2)(3)+(3)(-2) \end{bmatrix} = \begin{bmatrix} 12-2 \\ -6-6 \end{bmatrix} = \begin{bmatrix} 10 \\ -12 \end{bmatrix}.$$

Thus, the image of $(3,-2)$ is $(10,-12)$.

b. Find the image of the point $(-1,5)$ under the transformation matrix:

$$T = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$$

The image of $(-1,5)$ is:

$$T\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

Computing:

$$\begin{bmatrix} (-2)(-1)+(3)(5) \\ (1)(-1)+(4)(5) \end{bmatrix} = \begin{bmatrix} 2+15 \\ -1+20 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \end{bmatrix}$$

Thus, the image of $(-1,5)$ is $(17,19)$.

c. Find the image of the point $(4,3)$ under the transformation matrix:

$$T = \begin{bmatrix} 5 & -1 \\ 2 & 6 \end{bmatrix}$$

The image of $(4,3)$ is:

$$T\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Computing:

$$\begin{bmatrix} (5)(4)+(-1)(3) \\ (2)(4)+(6)(3) \end{bmatrix} = \begin{bmatrix} 20-3 \\ 8+18 \end{bmatrix} = \begin{bmatrix} 17 \\ 26 \end{bmatrix}.$$

Thus, the image of $(4,3)$ is $(17,26)$.

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Question 3

a. A unit square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ is transformed into a parallelogram with vertices $(0,0)$, $(4,1)$, $(2,3)$ and $(6,4)$. Find a possible transformation matrix T.

The transformation matrix T must satisfy:

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix},\quad T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}.$$

Therefore, T is:

$$T=\begin{bmatrix}4&2\\1&3\end{bmatrix}.$$

Could also get

$$T=\begin{bmatrix}2&4\\3&1\end{bmatrix}.$$

b. Verify that the transformation matrix correctly maps $(1,1)$ to $(6,4)$.

Applying T to $(1,1)$:

$$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4&2\\1&3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4+2\\1+3\end{bmatrix}=\begin{bmatrix}6\\4\end{bmatrix}.$$

Since the result matches the given transformed point, the transformation matrix is verified.

c. Compute the area of the transformed parallelogram.

The area of the transformed parallelogram is given by the absolute value of the determinant of T:

$$\text{Area}=|\det(T)|=|(4)(3)-(2)(1)|=|12-2|=|10|=10.$$

Thus, the area of the transformed parallelogram is 10 square units.

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Question 4 Tech-Active.

Find the image of the point $(2, -4)$ under the transformation matrix:

$$T = \begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$$

$(-14, -12)$.

TI:	Mathematica:	Casio:
Define $t=\begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$ Done	In[169]:= {{-3, 2}, {2, 4}}.{2, -4}	$\begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow t$
$t \cdot \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $\begin{bmatrix} -14 \\ -12 \end{bmatrix}$	Out[169]= {-14, -12}	$\begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$
	In[171]:= $\begin{pmatrix} -3 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ // MatrixForm	$t \times \begin{bmatrix} 2 \\ -4 \end{bmatrix}$
	Out[171]//MatrixForm= $\begin{pmatrix} -14 \\ -12 \end{pmatrix}$	$\begin{bmatrix} -14 \\ -12 \end{bmatrix}$

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Sub-Section [4.2.2]: Dilations, Reflections, Shears and Projections

Question 5

a. Write the transformation matrix for a dilation by a factor of $k = 3$ in both the x and y-directions.

A dilation transformation scales both coordinates by a factor of k. The transformation matrix is:

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

b. Write the transformation matrix for a reflection in the x-axis.

A reflection in the x-axis negates the y-coordinate while leaving the x-coordinate unchanged. The transformation matrix is:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

c. Write the transformation matrix for a shear in the x-direction with a shear factor of $k = 2$.

A shear in the x-direction moves points horizontally in proportion to their y-coordinate. The transformation matrix is:

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

d. Write the transformation matrix for a projection onto the x-axis.

A projection onto the x-axis eliminates the y-component of any point. The transformation matrix is:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

e. Use matrices to find the rule for a translation 3 units to the right and 2 units down.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+3 \\ y-2 \end{bmatrix}$$

Question 6

a. Apply a reflection in the y-axis to the point $(3, -2)$.

The transformation matrix for a reflection in the y-axis is:

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying this transformation:

$$T\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

Thus, the image of $(3, -2)$ is $(-3, -2)$.

b. Apply a shear in the x-direction with shear factor $k = 2$ to the point $(4,1)$.

The transformation matrix for a shear in the x-direction with shear factor $k = 2$ is:

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Applying this transformation:

$$T\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+2(1) \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$$

Thus, the image of $(4, 1)$ is $(6, 1)$.

c. Apply a projection onto the x-axis to the point $(-2,5)$.

The transformation matrix for a projection onto the x-axis is:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Applying this transformation:

$$T\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Thus, the image of $(-2, 5)$ is $(-2, 0)$.

Question 7

a. The point $(4,2)$ is mapped to $(-4,2)$. Find a transformation matrix that achieves this.

The x-coordinate is negated while the y-coordinate remains unchanged. This corresponds to a reflection in the y-axis, given by the transformation matrix:

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b. The point $(3,-1)$ is mapped to $(7,-1)$. Find a transformation matrix that achieves this.

The x-coordinate is increased by 4 while the y-coordinate remains unchanged. This corresponds to a horizontal shear in the x-direction with shear factor $k = 4$, given by the transformation matrix:

$$T = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

c. The point $(-2,5)$ is mapped to $(-2,0)$. Find a transformation matrix that achieves this.

The y-coordinate is reduced to 0 while the x-coordinate remains unchanged. This corresponds to a projection onto the x-axis, given by the transformation matrix:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

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Sub-Section [4.2.3]: Inverse Transformations

Question 8

a. Find the inverse transformation matrix for the mapping:

$$(x, y) \to (2x + 3y, 4x + 5y)$$

The transformation matrix is:

$$T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

The inverse transformation is given by T^{-1}, computed as:

$$T^{-1} = \frac{1}{(2)(5) - (3)(4)} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \frac{1}{10 - 12} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}.$$

Thus, the inverse transformation matrix is:

$$\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}.$$

b. Find the inverse transformation matrix for the mapping:

$$(x, y) \to (-x + 4y, 6x - 2y)$$

The transformation matrix is:

$$T = \begin{bmatrix} -1 & 4 \\ 6 & -2 \end{bmatrix}.$$

Computing the inverse:

$$T^{-1} = \frac{1}{(-1)(-2) - (4)(6)} \begin{bmatrix} -2 & -4 \\ -6 & -1 \end{bmatrix} = \frac{1}{2 - 24} \begin{bmatrix} -2 & -4 \\ -6 & -1 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} -2 & -4 \\ -6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{22} & \frac{4}{22} \\ \frac{6}{22} & \frac{1}{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ \frac{3}{11} & \frac{1}{22} \end{bmatrix}.$$

Thus, the inverse transformation matrix is:

$$\begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ \frac{3}{11} & \frac{1}{22} \end{bmatrix}.$$

c. Find the inverse transformation matrix for the mapping:

$$(x, y) \to (7x - 3y, -5x + 8y)$$

The transformation matrix is:

$$T = \begin{bmatrix} 7 & -3 \\ -5 & 8 \end{bmatrix}.$$

Computing the inverse:

$$T^{-1} = \frac{1}{(7)(8) - (-3)(-5)} \begin{bmatrix} 8 & 3 \\ 5 & 7 \end{bmatrix} = \frac{1}{56 - 15} \begin{bmatrix} 8 & 3 \\ 5 & 7 \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 8 & 3 \\ 5 & 7 \end{bmatrix}$$

Thus, the inverse transformation matrix is:

$$\begin{bmatrix} \frac{8}{41} & \frac{3}{41} \\ \frac{5}{41} & \frac{7}{41} \end{bmatrix}.$$

Question 9

a. The transformation matrix:

$$T = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Maps a point (x, y) to $(10,6)$. Find (x, y).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 28 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Thus, the original point is:

$$\left(\frac{14}{5}, \frac{4}{5}\right).$$

b. The transformation matrix:

$$T = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}$$

Maps a point (x, y) to $(4, 11)$. Find (x, y).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 11 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 61 \\ 47 \end{bmatrix} = \begin{bmatrix} \frac{61}{41} \\ \frac{47}{41} \end{bmatrix}.$$

Thus, the original point is:

$$\left(\frac{61}{41}, \frac{47}{41}\right).$$

c. The transformation matrix:

$$T = \begin{bmatrix} 6 & 2 \\ -3 & 5 \end{bmatrix}$$

Maps a point (x, y) to $(12,7)$. Find (x, y).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 5 & -2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 46 \\ 78 \end{bmatrix} = \begin{bmatrix} \frac{23}{18} \\ \frac{13}{6} \end{bmatrix}.$$

Thus, the original point is:

$$\left(\frac{23}{18}, \frac{13}{6}\right).$$

Question 10

a. A transformation maps the points:

$$(1,2) \to (5,4), \quad (3,-1) \to (7,6)$$

Find the transformation matrix T.

The transformation matrix T satisfies:

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad T\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}.$$

Writing this as a matrix equation:

$$T\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}.$$

Solving for T:

$$T = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}^{-1}.$$

Computing the inverse:

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{(1)(-1)-(3)(2)}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}.$$

Computing:

$$T = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix} = \begin{bmatrix} \frac{19}{7} & \frac{8}{7} \\ \frac{16}{7} & \frac{6}{7} \end{bmatrix}.$$

Thus, the transformation matrix is:

$$\begin{bmatrix} \frac{19}{7} & \frac{8}{7} \\ \frac{16}{7} & \frac{6}{7} \end{bmatrix}.$$

b. A transformation maps the points:

$$(2,1) \to (6,5), \qquad (1,3) \to (4,2)$$

Find the transformation matrix T.

Following the same method, we set up:

$$T\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 5 & 2 \end{bmatrix}.$$

Compute the inverse:

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{(2)(3)-(-1)(1)}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

Compute T:

$$T = \begin{bmatrix} 6 & 4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} \frac{14}{7} & \frac{14}{7} \\ \frac{13}{7} & \frac{9}{7} \end{bmatrix}.$$

Thus, the transformation matrix is:

$$\begin{bmatrix} 2 & 2 \\ \frac{13}{7} & \frac{9}{7} \end{bmatrix}.$$

c. A transformation maps the points:

$$(3,2) \to (9,7), \qquad (0,5) \to (2,10)$$

Find the transformation matrix T.

Set up the equation:

$$T\begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 7 & 10 \end{bmatrix}.$$

Compute the inverse:

$$\begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{(3)(5)-(0)(2)}\begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}$$

Compute T:

$$T = \begin{bmatrix} 9 & 2 \\ 7 & 10 \end{bmatrix}\begin{bmatrix} \frac{5}{15} & 0 \\ -\frac{2}{15} & \frac{3}{15} \end{bmatrix} = \begin{bmatrix} \frac{41}{15} & \frac{2}{5} \\ 1 & 2 \end{bmatrix}.$$

Thus, the transformation matrix is:

$$\begin{bmatrix} \frac{41}{15} & \frac{2}{5} \\ 1 & 2 \end{bmatrix}.$$

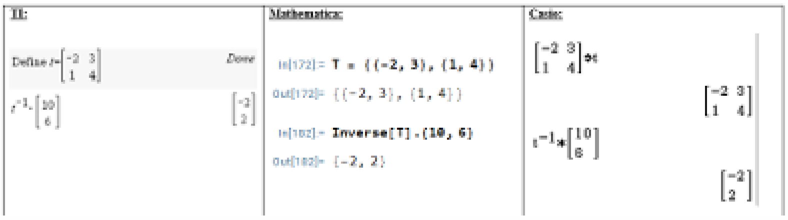
Question 11 Tech-Active.

A transformation matrix:

$$T = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$$

Maps a point (x, y) to $(10,6)$. Find (x, y).

$(2, -2)$.



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Sub-Section [4.2.4]: Composite Transformations

Question 12

A transformation consists of:

- A reflection in the x-axis.
- A shear in the x-direction with shear factor $k = 3$.

a. Find the composite transformation matrix.

The reflection in the x-axis is given by:

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The shear transformation in the x-direction with $k = 3$ is:

$$S_x = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

The composite transformation matrix is:

$$T = S_x R_x = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}.$$

b. Find the image of $(4, 2)$ under this transformation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 - 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

Thus, the image of $(4, 2)$ is $(-2, -2)$.

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Question 13

A transformation consists of:

- A dilation by a factor of 2 from both the x and y-axes.
- A reflection in the y-axis.

a. Find the composite transformation matrix.

The dilation by a factor of 2 is:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The reflection in the y-axis is:

$$R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The composite transformation matrix is:

$$T = R_y D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

b. Find the image of $(-3, 5)$ under this transformation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

Thus, the image of $(-3, 5)$ is $(6, 10)$.

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Question 14

A transformation consists of:

- A reflection in the x-axis.
- A shear in the y-direction with shear factor $k = 2$.
- A projection onto the x-axis.

a. Find the composite transformation matrix.

The reflection in the x-axis is:

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The shear in the y-direction with $k = 2$ is:

$$S_y = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

The projection onto the x-axis is:

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The composite transformation matrix is:

$$T = P_x S_y R_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

b. Find the image of $(3, -2)$ under this transformation, followed by a translation 1 unit right and 3 units down.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Thus, the image of $(3, -2)$ is $(4, -3)$.

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Question 15 Tech-Active.

The point $(3, 4)$ is the image of a point (x, y) after the transformations T and then S have been applied in that order. The transformation matrices are:

$$T = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \text{ and } S = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

Find the original point (x, y).

The overall transformation matrix is $Q = \begin{bmatrix} -5 & 0 \\ 3 & 11 \end{bmatrix}$

We find the original point is then $\left(-\frac{3}{5}, \frac{29}{55}\right)$.

TI:	Mathematica:	Casio:
Define $t=\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$ Done	In[189]:= T = {{-1, 3}, {1, 2}} Out[189]= {{-1, 3}, {1, 2}}	$\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \Rightarrow t$ $\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$
Define $s=\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ Done	In[190]:= S = {{2, -3}, {1, 4}} Out[190]= {{2, -3}, {1, 4}}	$\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \Rightarrow s$ $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$
$(s \cdot t)^{-1} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -\frac{3}{5} \\ \frac{29}{55} \end{bmatrix}$	In[191]:= S.T // MatrixForm Out[191]//MatrixForm= $\begin{pmatrix} -5 & 0 \\ 3 & 11 \end{pmatrix}$	$(s \times t)^{-1} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -\frac{3}{5} \\ \frac{29}{55} \end{bmatrix}$
	In[192]:= Inverse[S.T].{3, 4} Out[192]= $\{-\frac{3}{5}, \frac{29}{55}\}$	

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Sub-Section: The 'Final Boss'

Question 16

A company is designing a robotic arm that manipulates objects in a 2D workspace. The workspace is represented by the xy-plane, and transformations are applied using matrices. The arm operates on a unit square with initial vertices at $A(0,0)$, $B(1,0)$, $C(0,1)$, and $D(1,1)$.

a. The robotic arm first applies:

- A shear in the x-direction with shear factor $k = 2$.
- A reflection in the y-axis.

Find the composite transformation matrix T.

The shear in the x-direction with shear factor $k = 2$ is:

$$S_x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The reflection in the y-axis is:

$$R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The composite transformation matrix is:

$$T = R_y S_x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

b. Apply the transformation matrix T to the original unit square. Find the coordinates of the transformed points A', B', C', D'.

Applying T to each vertex:

$$A' = T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$B' = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

$$C' = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$$D' = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Thus, the transformed unit square has vertices at $A'(0,0)$, $B'(-1,0)$, $C'(-2,1)$, and $D'(-3,1)$.

c. Find the area of the transformed shape. Does the transformation preserve area?

The area of the transformed parallelogram is given by:

$$|\det(T)| = |(-1)(1) - (-2)(0)| = |-1| = 1.$$

Since the original unit square has an area of 1, the transformation preserves area.

d. Inverse transformation.
The robotic arm must reverse the transformation to return the shape to its original position. Find the inverse transformation matrix T^{-1}, if it exists.

The inverse transformation is given by:

$$T^{-1} = \frac{1}{\det(T)} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

Since $\det(T) = -1$,

$$T^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}.$$

e. Alternative transformations.
Instead of the previous transformation, the arm now applies:

- A dilation by a factor of 3 in both the x- and y-directions.
- A projection onto the x-axis.

Find the new composite transformation matrix T'.

The dilation by a factor of 3 is:

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

The projection onto the x-axis is:

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The composite transformation matrix is:

$$T' = P_x D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

f. Calculate the area of the transformed shape under this new transformation. Compare it to the original and explain the result.

The area of the transformed parallelogram is given by:

$$|\det(T')| = |(3)(0) - (0)(0)| = 0.$$

Since the determinant is zero, the transformation collapses the unit square onto the x-axis, reducing its area to zero. This shows that the transformation is not invertible, and information about the original shape is lost.

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Section B: Supplementary Questions

Sub-Section [4.2.1]: Using Matrices For Linear Transformations

Question 17

For each of the following, write the transformation matrix for the given mapping:

a. $(x, y) \rightarrow (3x - 2y, 5x + 4y)$

The transformation can be represented using the matrix:

$$T = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}.$$

b. $(x, y) \rightarrow (6x + y, -x + 3y)$

The transformation matrix is:

$$T = \begin{bmatrix} 6 & 1 \\ -1 & 3 \end{bmatrix}.$$

c. $(x, y) \rightarrow (-2x + 4y, 7x - 5y)$

The transformation matrix is:

$$T = \begin{bmatrix} -2 & 4 \\ 7 & -5 \end{bmatrix}.$$

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Question 18

a. Find the image of the point $(2,-3)$ under the transformation matrix:

$$T = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -18 \end{bmatrix}$$

Thus, the image of $(2,-3)$ is $(4,-18)$.

b. Find the image of the point $(-2,4)$ under the transformation matrix:

$$T = \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$T\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$$

Thus, the image of $(-2,4)$ is $(14,16)$.

c. Find the image of the point $(3,2)$ under the transformation matrix:

$$T = \begin{bmatrix} 4 & -2 \\ 3 & 6 \end{bmatrix}$$

$$T\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 21 \end{bmatrix}$$

Thus, the image of $(3,2)$ is $(8,21)$.

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Question 19

a. A unit square with vertices (0,0), (1,0), (0,1), and (1,1) is transformed to a parallelogram with vertices (0,0), (3,2), (4,1), and (7,3). Find a possible transformation matrix T.

The transformation matrix T must satisfy:

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}, \quad T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$$

Therefore, T is:

$$T=\begin{bmatrix}3 & 4\\2 & 1\end{bmatrix}.$$

Could also get

$$T=\begin{bmatrix}4 & 3\\1 & 2\end{bmatrix}.$$

b. Verify that the transformation matrix correctly maps (1,1) to (7,3).

Applying T to $(1,1)$:

$$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3 & 4\\2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3+4\\2+1\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}.$$

Since the result matches the given transformed point, the transformation matrix is verified.

c. Compute the area of the transformed parallelogram.

The area of the transformed parallelogram is given by the absolute value of the determinant of T:

$$\text{Area}=|\det(T)|=|(3)(1)-(4)(2)|=|3-8|=|-5|=5.$$

Thus, the area of the transformed parallelogram is 5 square units.

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Sub-Section [4.2.2]: Dilations, Reflections, Shears, and Projections

Question 20

a. Write the transformation matrix for a dilation by a factor of $k = 4$ in both the x- and y-directions.

A dilation transformation scales both coordinates by a factor of k. The transformation matrix is:

$$T = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

b. Write the transformation matrix for a reflection in the y-axis.

A reflection in the y-axis negates the x-coordinate while leaving the y-coordinate unchanged. The transformation matrix is:

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

c. Write the transformation matrix for a shear in the y-direction with shear factor $k = 3$.

A shear in the y-direction moves points vertically in proportion to their x-coordinate. The transformation matrix is:

$$T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

d. Write the transformation matrix for a projection onto the y-axis.

A projection onto the y-axis eliminates the x-component of any point. The transformation matrix is:

$$T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

e. Use matrices to find the rule for a translation 5 units left and 4 units up.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x-5 \\ y+4 \end{bmatrix}$$

Question 21

a. Apply a reflection in the x-axis to the point $(-5,3)$.

The transformation matrix for a reflection in the x-axis is:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Applying this transformation:

$$T\begin{bmatrix} -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

Thus, the image of $(-5,3)$ is $(-5,-3)$.

b. Apply a shear in the y-direction with shear factor $k = 2$ to the point $(2,3)$.

The transformation matrix for a shear in the y-direction with shear factor $k = 2$ is:

$$T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Applying this transformation:

$$T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

Thus, the image of $(2,3)$ is $(2,7)$.

c. Apply a projection onto the y-axis to the point $(6,-4)$.

The transformation matrix for a projection onto the y-axis is:

$$T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying this transformation:

$$T\begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}.$$

Thus, the image of $(6,-4)$ is $(0,-4)$.

Question 22

a. The point $(-3,2)$ is mapped to $(3,2)$. Find a transformation matrix that achieves this.

The x-coordinate is negated while the y-coordinate remains unchanged. This corresponds to a reflection in the y-axis, given by the transformation matrix:

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b. The point $(1,4)$ is mapped to $(1,8)$. Find a transformation matrix that achieves this.

The y-coordinate is doubled while the x-coordinate remains unchanged. This corresponds to a vertical dilation with scale factor $k = 2$, given by the transformation matrix:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

c. The point $(7,-2)$ is mapped to $(7,0)$. Find a transformation matrix that achieves this.

The y-coordinate is set to 0 while the x-coordinate remains unchanged. This corresponds to a projection onto the x-axis, given by the transformation matrix:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

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Sub-Section [4.2.3]: Inverse Transformations

Question 23

a. Find the inverse transformation matrix for the mapping:

$$(x,y) \to (4x + 5y, 7x + 2y)$$

The transformation matrix is:

$$T = \begin{bmatrix} 4 & 5 \\ 7 & 2 \end{bmatrix}.$$

The inverse transformation is given by T^{-1}, computed as:

$$T^{-1} = \frac{1}{(4)(2)-(5)(7)} \begin{bmatrix} 2 & -5 \\ -7 & 4 \end{bmatrix} = \frac{1}{8-35} \begin{bmatrix} 2 & -5 \\ -7 & 4 \end{bmatrix} = \frac{1}{-27} \begin{bmatrix} 2 & -5 \\ -7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{27} & \frac{5}{27} \\ \frac{7}{27} & -\frac{4}{27} \end{bmatrix}.$$

b. Find the inverse transformation matrix for the mapping:

$$(x,y) \to (3x - 4y, 6x + 5y)$$

The transformation matrix is:

$$T = \begin{bmatrix} 3 & -4 \\ 6 & 5 \end{bmatrix}.$$

Computing the inverse:

$$T^{-1} = \frac{1}{(3)(5)-(-4)(6)} \begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix} = \frac{1}{15+24} \begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix} = \frac{1}{39} \begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{39} & \frac{4}{39} \\ -\frac{2}{13} & \frac{1}{13} \end{bmatrix}.$$

c. Find the inverse transformation matrix for the mapping:

$$(x,y) \to (2x + 6y, 5x - 3y)$$

The transformation matrix is:

$$T = \begin{bmatrix} 2 & 6 \\ 5 & -3 \end{bmatrix}.$$

Computing the inverse:

$$T^{-1} = \frac{1}{(2)(-3)-(6)(5)} \begin{bmatrix} -3 & -6 \\ -5 & 2 \end{bmatrix} = \frac{1}{-6-30} \begin{bmatrix} -3 & -6 \\ -5 & 2 \end{bmatrix} = \frac{1}{-36} \begin{bmatrix} -3 & -6 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{5}{36} & -\frac{1}{18} \end{bmatrix}.$$

Question 24

a. The transformation matrix:

$$T = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Maps a point (x, y) to $(8,7)$. Find (x, y).

$$\begin{bmatrix} x \\ y \end{bmatrix} = T^{-1} \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$

Computing T^{-1} and multiplying:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{23} & -\frac{4}{23} \\ \frac{2}{23} & \frac{3}{23} \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{40}{23} - \frac{28}{23} \\ \frac{16}{23} + \frac{21}{23} \end{bmatrix} = \begin{bmatrix} \frac{12}{23} \\ \frac{37}{23} \end{bmatrix}$$

b. The transformation matrix:

$$T = \begin{bmatrix} 5 & -3 \\ 2 & 6 \end{bmatrix}$$

Maps a point (x, y) to $(11, 4)$. Find (x, y).

$$\begin{bmatrix} x \\ y \end{bmatrix} = T^{-1} \begin{bmatrix} 11 \\ 4 \end{bmatrix}.$$

Computing:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{6}{36} & \frac{3}{36} \\ -\frac{2}{36} & \frac{5}{36} \end{bmatrix} \begin{bmatrix} 11 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{66}{36} + \frac{12}{36} \\ -\frac{22}{36} + \frac{20}{36} \end{bmatrix} = \begin{bmatrix} \frac{13}{6} \\ -\frac{1}{18} \end{bmatrix}$$

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Question 25

a. A transformation maps the points:

$$(3,2) \to (7,5), \qquad (1,-1) \to (4,2)$$

Find the transformation matrix T.

The transformation matrix T satisfies:

$$T\begin{bmatrix}3 & 1\\2 & -1\end{bmatrix} = \begin{bmatrix}7 & 4\\5 & 2\end{bmatrix}.$$

Computing T by inverting the matrix:

$$T = \begin{bmatrix}7 & 4\\5 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\2 & -1\end{bmatrix}^{-1}.$$

Computing the inverse:

$$\begin{bmatrix}3 & 1\\2 & -1\end{bmatrix}^{-1} = \frac{1}{(3)(-1)-(1)(2)}\begin{bmatrix}-1 & -1\\-2 & 3\end{bmatrix} = \frac{1}{-3-2}\begin{bmatrix}-1 & -1\\-2 & 3\end{bmatrix} = \frac{1}{-5}\begin{bmatrix}-1 & -1\\-2 & 3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\\frac{2}{5} & -\frac{3}{5}\end{bmatrix}.$$

Computing T:

$$T = \begin{bmatrix}7 & 4\\5 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\\frac{2}{5} & -\frac{3}{5}\end{bmatrix} = \begin{bmatrix}\frac{7}{5}+\frac{8}{5} & \frac{7}{5}-\frac{12}{5}\\\frac{5}{5}+\frac{4}{5} & \frac{5}{5}-\frac{6}{5}\end{bmatrix} = \begin{bmatrix}3 & -1\\\frac{9}{5} & -\frac{1}{5}\end{bmatrix}.$$

b. A transformation maps the points:

$$(2,4) \to (10,3), \qquad (-3,1) \to (5,7)$$

Find the transformation matrix T.

The transformation matrix T satisfies:

$$T\begin{bmatrix}2 & -3\\4 & 1\end{bmatrix} = \begin{bmatrix}10 & 5\\3 & 7\end{bmatrix}.$$

Computing T by inverting the matrix:

$$T = \begin{bmatrix}10 & 5\\3 & 7\end{bmatrix}\begin{bmatrix}2 & -3\\4 & 1\end{bmatrix}^{-1}.$$

Computing the inverse:

$$\begin{bmatrix}2 & -3\\4 & 1\end{bmatrix}^{-1} = \frac{1}{(2)(1)-(-3)(4)}\begin{bmatrix}1 & 3\\-4 & 2\end{bmatrix} = \frac{1}{2+12}\begin{bmatrix}1 & 3\\-4 & 2\end{bmatrix} = \frac{1}{14}\begin{bmatrix}1 & 3\\-4 & 2\end{bmatrix}.$$

Computing T:

$$T = \begin{bmatrix}10 & 5\\3 & 7\end{bmatrix}\begin{bmatrix}\frac{1}{14} & \frac{3}{14}\\-\frac{4}{14} & \frac{2}{14}\end{bmatrix} = \begin{bmatrix}-\frac{5}{7} & \frac{20}{7}\\-\frac{25}{14} & \frac{23}{14}\end{bmatrix}.$$

c. A transformation maps the points:

$$(1,3) \to (8,5), \qquad (2,-2) \to (4,6)$$

Find the transformation matrix T.

The transformation matrix T satisfies:

$$T\begin{bmatrix}1 & 2\\3 & -2\end{bmatrix} = \begin{bmatrix}8 & 4\\5 & 6\end{bmatrix}.$$

Computing T by inverting the matrix:

$$T = \begin{bmatrix}8 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\3 & -2\end{bmatrix}^{-1}.$$

Computing the inverse:

$$\begin{bmatrix}1 & 2\\3 & -2\end{bmatrix}^{-1} = \frac{1}{(1)(-2)-(2)(3)}\begin{bmatrix}-2 & -2\\-3 & 1\end{bmatrix} = \frac{1}{-2-6}\begin{bmatrix}-2 & -2\\-3 & 1\end{bmatrix} = \frac{1}{-8}\begin{bmatrix}-2 & -2\\-3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\\frac{3}{8} & -\frac{1}{8}\end{bmatrix}.$$

Computing T:

$$T = \begin{bmatrix}8 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\\frac{3}{8} & -\frac{1}{8}\end{bmatrix} = \begin{bmatrix}\frac{7}{2} & \frac{3}{2}\\\frac{7}{2} & \frac{1}{2}\end{bmatrix}.$$

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Sub-Section [4.2.4]: Composite Transformations

Question 26

A transformation consists of:

- A reflection in the x-axis.
- A shear in the y-direction with shear factor $k = 3$.

a. Find the composite transformation matrix.

The reflection in the x-axis is:

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The shear in the y-direction with $k = 3$ is:

$$S_y = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

The composite transformation matrix is:

$$T = S_y R_x = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

b. Find the image of $(5, -2)$ under this transformation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 + 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 17 \end{bmatrix}$$

Thus, the image of $(5, -2)$ is $(5, 17)$.

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Question 27

A transformation consists of:

- A shear in the y-direction with shear factor $k = 4$.
- A reflection in the x-axis.

a. Find the composite transformation matrix

The shear in the y-direction with $k = 4$ is:

$$S_y = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}.$$

The reflection in the x-axis is:

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The composite transformation matrix is:

$$T = R_x S_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & -1 \end{bmatrix}.$$

b. Find the image of $(-1,2)$ under this transformation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2+4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Thus, the image of $(-1, 2)$ is $(-1, 2)$.

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Question 28

A transformation consists of:

- A reflection in the y-axis.
- A shear in the x-direction with shear factor $k = 2$.
- A dilation by a factor of 3 in both directions.

a. Find the composite transformation matrix.

The reflection in the y-axis is:

$$R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The shear in the x-direction with $k = 2$ is:

$$S_x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The dilation by a factor of 3 is:

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

The composite transformation matrix is:

$$T = DS_xR_y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 0 & 3 \end{bmatrix}$$

b. Find the image of $(4, -2)$ under this transformation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 - 12 \\ 0 - 6 \end{bmatrix} = \begin{bmatrix} -24 \\ -6 \end{bmatrix}$$

Thus, the image of $(4, -2)$ is $(-24, -6)$.

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