4.1

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VCE Specialist Mathematics ½
Matrices [4.1]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [4.1.1]** - Basics of Matrices and Identifying Types of Matrices. Calculate the Transpose and Trace of a Matrix
- ☐ **SM12 [4.1.2]** - Perform Matrix Addition, Scalar Multiplication, and Matrix Multiplication
- ☐ **SM12 [4.1.3]** - Calculate the Inverse of a Matrix and Determine its Determinant
- ☐ **SM12 [4.1.4]** - Apply Matrix Operations to Solve Systems of Linear Equations

Section A: Introduction to Matrices

Sub-Section: Understanding Matrices

Context

- Why matrices?
  - Consider the system of n many linear equations.

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ \vdots \\ a_nx + b_ny + c_nz = d_n \end{cases}$$

$2x+y-z=6$
$10x-15y+7z=\pi$
$9x-y-z=e^{10}$

  - Imagine solving them!
  - That would be very tedious.

What if there was a way to solve it as one equation?

$$A \cdot Var = D$$

  - We can do that using matrices!

Space for Personal Notes

Let's have a look at what a matrix is!

Matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- A matrix is an array of numbers.
- It has rows and columns.
- Dimension (similar to size) of a matrix is given by rows × columns.

2 × 2

$$\textbf{Dimension} = \textbf{Rows} \times \textbf{Columns}$$

- Each number is called an element.

Discussion: For a 2×2 matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ how can we represent elements in the 1^{st} row, 2^{nd} column?

3

$a_{1,2} = 3$

row column

Labelling Elements

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$a_{i,j} = i^{th}\ row, \qquad j^{th} column$$

- Elements of matrix A is given by lower case a with row and column number.

Question 1

Consider the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 3 & 0 \end{bmatrix}$$

a. State the dimension of the matrix.

rows × columns

$2 \times 3 \cancel{= 6}$

b. $a_{1,1} + a_{2,1}$

$1 + (-2) = -1$

c. $a_{2,1} + a_{2,3}$

$-2 + 0 = -2$

TIP: Remember the order ARC! A matrix with R row and C column.

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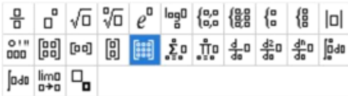
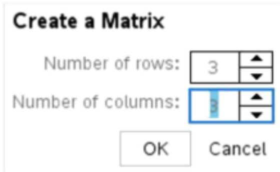
Calculator Commands: Defining Matrices on Technology

Mathematica

- We use round brackets.
- Control Comma: Makes new columns.
- Control Enter: Makes new rows.

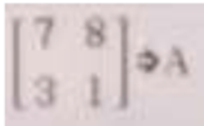
TI-Nspire





- Press the button **next to** the book.

Casio Classpad



- Under **maths 2.**
- Press it multiple times to change the dimension.
- Use the arrow to save the matrix.

Question 2 Tech-Active.

On your technology, save the following matrix.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & -5 \end{bmatrix}$$

Sub-Section: Types of Matrices

Let's have a look at different types of matrices!

Row and Column Matrix

Row Matrix

- A matrix that has only one row.

$$[1]$$
$$[1 \quad 2]$$
$$[1 \quad 2 \quad 3]$$
$$[1 \quad 2 \quad 3 \quad 4]$$
$$[1 \quad 2 \quad 3 \quad 4 \quad 5]$$

Column Matrix

- A matrix that has only one column.

$$[1] \quad \begin{bmatrix}1\\2\end{bmatrix} \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \quad \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$$

Square Matrix

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}, \begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}$$

- Matrices that have the same number of rows and columns.

$$\boldsymbol{Dimension} = \boldsymbol{n} \times \boldsymbol{n}$$

Zero Matrix

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

- Matrix with all the elements equaling to zero.

Sub-Section: Transpose and Trace

What does it mean to transpose a matrix?

Transpose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

- Simply swap the row and the column.
  - Eg: 1^{st} row becomes the 1^{st} column.
- You may also see the transpose written as A'.

Question 3 Walkthrough.

Consider the matrix below.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Find A^T.

$$A^T = \begin{bmatrix} 1 & 0 \\ -1 & 4 \\ 3 & 5 \end{bmatrix}$$

Question 4

For each of the following, find A^T.

a. $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$

$A^T = \begin{bmatrix} 7 & 6 & 5 \end{bmatrix}$

c. $A = \begin{bmatrix} 5 & 1 \\ 0 & 7 \\ 9 & 8 \end{bmatrix}$

$A^T = \begin{bmatrix} 5 & 0 & 9 \\ 1 & 7 & 8 \end{bmatrix}$

Let's now look at the trace of a square matrix!

Trace

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$tr(A) = a + d$$

- Find the sum of all the diagonal elements.
- We can only find the trace of a square matrix.

Question 5

For each of the following square matrix, calculate $tr(A)$.

a. $A = \begin{bmatrix} 1 & 1 & 3 \\ \frac{1}{2} & -7 & 0 \\ e & \pi & 1 \end{bmatrix}$

$tr(A) = 1 - 7 + 1$

$= -5$

b. $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$tr(A) = 1 - 1 = 0$

c. $A = [\pi]$

$tr(A) = \pi$

Section B: Operations of Matrices

Sub-Section: Addition and Subtraction

Discussion: Could we add two matrices of different size?

$[\ \ ] + [\ \ \ \ ] =$ NO!

Addition and Subtraction of Matrices

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

- Add the elements in the same position of each matrix.
- We can only add/subtract matrices of the same size.

Question 6

Let $X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, Y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Find $X - Y$.

$$X - Y = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Sub-Section: Scalar Multiplication

Discussion: What will happen to the matrix if we multiply it by two?

$$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Scalar Multiplication

$$k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- Multiply all the elements by k.

Space for Personal Notes

Question 7

Let $X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $Y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$, find:

a. $2X - 3Y$

$$2X-3Y = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 9 \\ 18 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$$

b. $5A + \frac{1}{2}B$

$$5A + \frac{1}{2}B = 5\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} \frac{5}{2} & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 25/2 & 0 \\ -6 & 12 \end{bmatrix}$$

c. $4Y + X + B$

$$\underbrace{4\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}}_{\text{undefined}}$$

Space for Personal Notes

Sub-Section: Matrix Multiplication

How do we multiply two matrices?

Exploration: Matrix Multiplication

- Let's take a look at two matrices A and B.

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- What would the multiplication look like?

$$A \times B = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\times 3 + 2\times 1 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$$

- We simply take the row of the 1st and column of the 2nd matrix and multiply them.

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac + bd \end{bmatrix}$$

- Let's take a look at two matrices A and B.

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- What would the multiplication look like?

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\times 3 + -1\times 1 \\ 1\times 3 + 2\times 1 \end{bmatrix}$$

- What does the multiplication of two matrices always give us?

$[\ ] \times [\ ] = [\ ]$

Discussion: For $A \times B$, what does the number of columns for A has to be the same as?

rows of B

Discussion: What happens if we multiply 2×3 and 3×1 matrix?

$(2 \times 3) \times (3 \times 1)$

ans: → 2×1

Matrix Multiplication

Definition

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + (-1) \times 1 \\ 1 \times 3 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Number of Columns of 1^{st} Matrix = Number of Rows of 2^{nd}

- The answer will always be a matrix.

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Question 8 Walkthrough.

For $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, find AB.

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2\times4+3\times1+4\times0 & 2\times0+3\times2+4\times3 \\ 5\times4+6\times1+7\times0 & 5\times0+6\times2+7\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix}$$

NOTE: $m \times n$ matrix multiplied with $n \times k$ gives us $m \times k$ matrix.

Space for Personal Notes

Question 9

Find AB for the following set of matrices.

a. $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times3+1\times2 \\ 0\times3+5\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

b. $A = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 & 1 \\ 5 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0\times2+2\times0+1\times-1 & 0\times4+2\times1+1\times1 \\ 5\times2+3\times0+4\times-1 & 5\times4+3\times1+4\times1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 6 & 27 \end{bmatrix}$$

Discussion: Is $A \times B$ the same as $B \times A$?

$$\left.\begin{matrix} 2\times3 \\ 3\times2 \end{matrix}\right\} \checkmark \qquad \left.\begin{matrix} A\times B \\ B\times A \end{matrix}\right\} \times$$

Sub-Section: Identity Matrix

What matrix if you multiply doesn't change the other matrix? Similar to 1 in numbers?

Question 10

Consider $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find $A \times B$.

$$A \times B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Space for Personal Notes

We call B in the previous question an identity matrix!

Identity Matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3$$

- Identity matrix I_n has a size $n \times n$.
- Its diagonal elements are equal to one
- Identity matrices are always a square matrix

$$AI = IA = A$$

Discussion: What would $tr(I_n)$ always equal to?

$tr(I_n) = n$

Properties of Matrix Multiplication

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

$$A(BC) = (AB)C$$

$$AI = IA = A$$

$$A0 = 0A = 0$$

Section C: Inverse Matrices and Determinants

Sub-Section: Inverse Matrix

Inverse of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Inverse only exists for a square matrix.
- Matrix that has an inverse is called invertible.

Question 11 Walkthrough.

Consider $A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$

Find A^{-1}.

$$A^{-1} = \frac{1}{-12 - 2}\begin{bmatrix} -3 & -1 \\ -2 & 4 \end{bmatrix}$$

f^{-1}

$$= -\frac{1}{14}\begin{bmatrix} -3 & -1 \\ -2 & 4 \end{bmatrix}$$

Space for Personal Notes

Question 12

Consider $A = \begin{bmatrix} 3 & -3 \\ -2 & 1 \end{bmatrix}$

a. Find A^{-1}.

$$A^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & -1 \\ -\frac{2}{3} & -1 \end{bmatrix}$$

b. Find AA^{-1}.

$$AA^{-1} = \begin{bmatrix} 3 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & -1 \\ -\frac{2}{3} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow I$$

Discussion: For the previous question, what do you notice when you multiply a matrix by its inverse?

$$AA^{-1} = I$$

Multiplication of Inverse Matrices

$A \times B \neq B \times A$

$$AA^{-1} = I_n = A^{-1}A$$ ✓

- The order does not matter.

Sub-Section: Determinant

Discussion: How can we tell if a square matrix does not have an inverse?

- Consider for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$ad - bc = 0$

Determinant of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$

- If determinant equals to 0, then A does not have an inverse.
- A is not invertible.
- Notation for the determinant also includes using straight lines | | around the matrix.

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Space for Personal Notes

Question 13

Calculate the determinant of the following matrices and state whether it is invertible.

a. $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$

$\det(A) = 3 - (-8)$

$= 11$ → Invertible

b. $A = \begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}$

$2 \times 4 - 1 \times 8 = 0$

Not invertible

How about 3 × 3 matrix?

<u>Determinant of 3 × 3 Matrix</u>

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(A) = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- For the first term, simply cover the first column and row and copy the 4 numbers.

$$\det(A) = a \times (ei - fh) - b \times (di - fg) + c \times (dh - eg)$$

Question 14 Walkthrough.

Calculate the determinant of the following matrix.

Find the $A = \begin{bmatrix} 1 & 3 & 4 \\ -3 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix}$

$= 1 \times \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} - 3 \begin{vmatrix} -3 & 5 \\ 0 & -1 \end{vmatrix} + 4 \begin{vmatrix} -3 & 2 \\ 0 & 3 \end{vmatrix}$

$= (-2-15) - 3(3) + 4(-9)$

$= -17 - 9 - 36$

$= -62$

Space for Personal Notes

Question 15

Calculate the determinant of the following matrices and state whether it is invertible.

a. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$\det(A) = 2 \times \begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 1 & 5 \end{vmatrix}$

$= 2(-1-15) - 0 + (1)$

$= -32 + 1 = -31$

b. $A = \begin{bmatrix} 0 & 2 & 5 \\ 7 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix}$

$= 0 \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - 2 \begin{vmatrix} 7 & 3 \\ -1 & 4 \end{vmatrix} + 5 \begin{vmatrix} 7 & 1 \\ -1 & 2 \end{vmatrix}$

$= 0 - 2(28+3) + 5(14+1)$

$= 13$

Calculator Commands: Finding Determinant on Technology

Mathematica

- Control comma/enter to make cells.

TI-Nspire

- **(Menu)** > **Matrix and Vector** > **Determinant:** $\boldsymbol{det}(A)$
- Where A is a matrix, you enter in the brackets or a previously defined matrix.

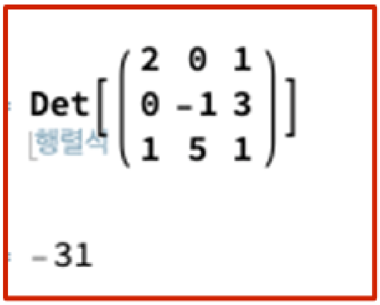
Casio Classpad

- **(Interactive)** > **Matrix** > **Calculation** > **det:** $\boldsymbol{det}(A)$
- Where A is a matrix you enter in the bracket, or a previously defined matrix.

Question 16 Tech-Active.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$$

Find the determinant of A.



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Section D: Systems of Linear Equations

Recall the context at the start of the class.

REMINDERS

- Why matrices?
  - Consider the system of n many linear equations.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$\vdots$$

$$a_nx + b_ny + c_nz = d_n$$

  - Imagine solving them!
  - That would be very tedious.

What if there was a way to solve it as one equation?

$$A \cdot Var = D$$

  - We can do that using matrices!

Space for Personal Notes

Question 17

Consider the two matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} x \\ y \end{bmatrix}$$

a. Find $A \times B$.

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 3x-2y \end{bmatrix}$$

b. Hence, solve $A \times B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2x+y \\ 3x-2y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$2x+y=1 \rightarrow y=1-2x$

$3x-2y=0 \Rightarrow 3x-2(1-2x)=0$

$7x=2$

$x = 2/7$

$y = 1 - \frac{4}{7}$

$= 3/7$

NOTE: We can use matrices to represent simultaneous equations.

Discussion: Hence, how can we represent $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ using matrices?

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Space for Personal Notes

Definition

System of Linear Equation Using Matrices

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- We can extend the idea to more than 2 equations.

Discussion: Previously we used matrices to only represent the equations. Is there any way to solve the equation using matrices?

$\boxed{A^{-1}}$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Exploration: Solving System of Linear Equation using Matrices

- Consider the system of linear equations.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- We can simplify the equation:

$$AX = C$$

$$AXA^{-1} = CA^{-1}$$

don't cancel

- Let's try to solve for our variables by solving our matrix X.
- What happens when you multiply both sides by A^{-1}.

$$\underbrace{A^{-1}A}X = A^{-1}C$$

$$IX = A^{-1}C$$

$$X = A^{-1}C$$

Definition

Solving System of Linear Equation Using Matrices

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$AX = C$$

$$X = A^{-1}C$$

Question 18 Walkthrough.

Convert the following system of linear equations into matrix equation and solve for x and y.

$$-x + 6y = 14$$

$$-5x + 4y = -8$$

$$A^{-1}AX = A^{-1}C$$

$$\begin{bmatrix} -1 & 6 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ -8 \end{bmatrix}$$

$$X = A^{-1}C$$

$$A^{-1} = \frac{1}{26}\begin{bmatrix} 4 & -6 \\ 5 & -1 \end{bmatrix}$$

$$X = \frac{1}{26}\begin{bmatrix} 4 & -6 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 14 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 56+48 \\ 70+8 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 104 \\ 78 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$x = 4, \; y = 3$

Question 19

Convert the following system of linear equations into matrix equation and solve for x and y.

a. $3x + 5y = 21$ and $6x - 2y = 6$.

$$\begin{bmatrix} 3 & 5 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-36}\begin{bmatrix} -2 & -5 \\ -6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{36}\begin{bmatrix} -2 & -5 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 21 \\ 6 \end{bmatrix} = -\frac{1}{36}\begin{bmatrix} -72 \\ -108 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

b. $-x + y = 2$
$x + y = 3$

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$$

Discussion: Okay seems like the inverse matrix is important. What if the matrix is not invertible?

No solutions or ∞ solutions

Definition

Determinants to Determine Number of Solutions

$$AX = C$$

$$X = A^{-1}C \quad \times$$

$$det(A) = 0 \Rightarrow \text{No or infinite solution}$$

- If the determinant equal to 0, the equation cannot be solved as there is no inverse of A.

$$det(A) \neq 0 \Rightarrow \text{Unique Solution}$$

- If the determinant is non-zero, equation can be solved as there is an inverse of A.

Question 20

Consider the system of linear equations. Determine whether the system of linear equations has a unique solution or not by using determinants.

a. $2x + y = 4$ and $-4x - 2y = 10$.

Either have 0 or ∞ soln.

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

$$det(A) = -4 - (-4) = 0$$

b. $2x + y + z = 1$, $-x - y + z = 10$ and $x + z = 4$.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad det(A) = 2\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}$$

One solution

$= 1$

Contour Check

Learning Objective: [4.1.1] – Basics of matrices and identifying types of matrices. Calculate the transpose and trace of a matrix

Key Takeaways

- **Matrices**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  - A matrix is an array of numbers.
  - It has rows and columns.
  - Dimension (similar to size) of a matrix is given by rows × columns.

$$Dimension = Rows \times Columns$$

  - Each number is called an element.

- **Row and Column Matrix**
  - **Row Matrix**
    - A matrix that has only one row.
  - **Column Matrix**
    - A matrix that has only one column.

- **Square Matrix:** Has dimension $n \times n$.

- **Zero Matrix:** All elements are equal to 0.

- Identity Matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Identity matrix I_n has a size $n \times n$.
  - Its diagonal elements are equal to 1.
  - Identity matrices are always a square matrix.

$$AI = IA = A$$

- Transpose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

- Trace

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$tr(A) = a + d$$

  - Find the sum of all the diagonal elements.
  - We can only find the trace of a square matrix.

Learning Objective: [4.1.2] – Perform matrix addition, scalar multiplication, and matrix multiplication

Key Takeaways

Addition and Subtraction of Matrices

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

- We can only add/subtract matrices of the same dimension
- Add the elements in the same position of each matrix.

Scalar Multiplication

$$k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- Multiply all the elements by k.

Matrix Multiplication

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + (-1) \times 1 \\ 1 \times 3 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Number of columns of 1st Matrix
= Number of rows of 2nd

- The answer will always be a matrix.

$(1 \times 2) \times (2 \times 1)$

- Properties of Matrix Multiplication

$$A(B+C) = \underline{AB + AC}$$

$$(A+B)C = \underline{AC + BC}$$

$$A(BC) = \underline{(AB)C}$$

$$AI = IA = \underline{A}$$

$$A0 = 0A = \underline{0}$$

- **Learning Objective: [4.1.3] – Calculate the inverse of a matrix and determine its determinant**

Key Takeaways

- Inverse of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  - Inverse only exists for a square matrix.
  - Matrix that has an inverse is called invertible.

- Multiplication of Inverse Matrices

$$AA^{-1} = \underline{I} = A^{-1}A$$

  - The order does not matter.

- **Determinant of a 2×2 Matrix**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

- If determinant equals to zero, then A does not have an inverse.
- A is not invertible.

- **Determinant of 3×3 Matrix**

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(A) = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Learning Objective: [4.1.4] – Apply matrix operations to solve systems of linear equations

Key Takeaways

- **Solving System of Linear Equation using Matrices**

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$AX = C$$

$$X = \underline{A^{-1}C}$$

- **Determinants to Determine Number of Solutions**

$$AX = C$$

$$X = A^{-1}C$$

$$\det(A) = 0 \Rightarrow \underline{\text{No. soln. or } \infty \text{ soln.}}$$

- If the determinant equal to 0, the equation cannot be solved as there is no inverse of A.

$$\det(A) \neq 0 \Rightarrow \underline{\text{unique solution}}$$

- If the determinant is non-zero, equation can be solved as there is an inverse of A.