CONTOUR EDUCATION

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VCE Specialist Mathematics ½
Matrices [4.1]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [4.1.1]** – Basics of Matrices and Identifying Types of Matrices. Calculate the Transpose and Trace of a Matrix
- ☐ **SM12 [4.1.2]** – Perform Matrix Addition, Scalar Multiplication, and Matrix Multiplication
- ☐ **SM12 [4.1.3]** – Calculate the Inverse of a Matrix and Determine its Determinant
- ☐ **SM12 [4.1.4]** – Apply Matrix Operations to Solve Systems of Linear Equations

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Section A: Introduction to Matrices

Sub-Section: Understanding Matrices

Context

- Why matrices?
  - Consider the system of n many linear equations.

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ \vdots \\ a_nx + b_ny + c_nz = d_n \end{cases}$$

  - Imagine solving them!
  - That would be very tedious.

What if there was a way to solve it as one equation?

$$A \cdot Var = D$$

  - We can do that using matrices!

Space for Personal Notes

Let's have a look at what a matrix is!

Matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- A matrix is an array of numbers.
- It has columns and rows.
- Dimension (similar to size) of a matrix is given by rows × columns.

$$\textit{Dimension} = \textit{Rows} \times \textit{Columns}$$

- Each number is called an element.

$\{1, 2, 3, 4\}$

Discussion: For a 2×2 matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ how can we represent elements in the 1^{st} row, 2^{nd} column?

Labelling Elements

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

row: 2
column: 2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$a_{ij} = i^{\text{th}}\ \textit{row}, \quad j^{\text{th}}\ \textit{column}$$

- Elements of matrix A is given by lower case with row and column number.

$b_{2,2}$

Question 1

Consider the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 3 & 0 \end{bmatrix}$$

a. State the dimension of the matrix.

2×3

b. $a_{1,1} + a_{2,1}$

$1 + -2 = -1$

c. $a_{2,1} + a_{2,3}$

$-2 + 0 = -2$

TIP: Remember the order ARC! A matrix with R row and C column.

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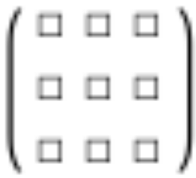
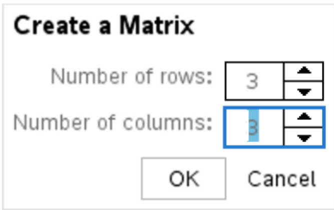
Calculator Commands: Defining Matrices on Technology

Mathematica



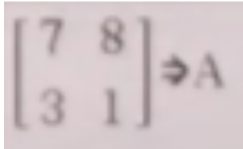
- We use round brackets.
- Control Comma: Makes new columns.
- Control Enter: Makes new rows.

TI-Nspire



- Press the button **next to** the book.

Casio Classpad



- Under **maths 2.**
- Press it multiple times to change the dimension.
- Use the arrow to save the matrix.

Question 2 Tech-Active.

On your technology, save the following matrix.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & -5 \end{bmatrix}$$

In[20]:= $A = \begin{pmatrix} 3 & 4 & 1 \\ 0 & 2 & -5 \end{pmatrix}$

Out[20]= {{3, 4, 1}, {0, 2, -5}}

In[21]:= A

Out[21]= {{3, 4, 1}, {0, 2, -5}}

Sub-Section: Types of Matrices

Let's have a look at different types of matrices!

Row and Column Matrix

- Row Matrix
  - A matrix that has only one row.

$$[1]$$
$$[1 \quad 2]$$
$$[1 \quad 2 \quad 3]$$
$$[1 \quad 2 \quad 3 \quad 4]$$
$$[1 \quad 2 \quad 3 \quad 4 \quad 5]$$

- Column Matrix
  - A matrix that has only one column.

$$[1] \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Square Matrix

2x2 3x3

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- Matrices that have the same number of rows and columns.

$$Dimension = n \times n$$

Zero Matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Matrix with all the elements equaling to zero.

Sub-Section: Transpose and Trace

What does it mean to transpose a matrix?

Transpose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

- Simply swap the row and the column.
  - Eg: 1^{st} row becomes the 1^{st} column.
- You may also see the transpose written as A'.

Question 3 Walkthrough.

Consider the matrix below.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Find A^T.

$$A^T = \begin{bmatrix} 1 & 0 \\ -1 & 4 \\ 3 & 5 \end{bmatrix}$$

Question 4

For each of the following, find A^T.

a. $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$$

b. $A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$

$$A^T = \begin{bmatrix} 7 & 6 & 5 \end{bmatrix}$$

c. $A = \begin{bmatrix} 5 & 1 \\ 0 & 7 \\ 9 & 8 \end{bmatrix}$

$$A^T = \begin{bmatrix} 5 & 0 & 9 \\ 1 & 7 & 8 \end{bmatrix}$$

Let's now look at the trace of a square matrix!

<u>Trace</u>

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$tr(A) = a + d$$

for a $n \times n$ matrix

$$tr(A) = \sum_{k=1}^{n} a_{kk}$$

- Find the sum of all the diagonals.
- We can only find the trace of a square matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{11} + a_{22} + a_{33}$

Question 5

For each of the following square matrix, calculate $tr(A)$.

a. $A = \begin{bmatrix} 1 & 1 & 3 \\ \frac{1}{2} & -7 & 0 \\ e & \pi & 1 \end{bmatrix}$

$tr(A) = 1 - 7 + 1$

$= -5$

b. $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$tr(A) = 1 - 1$

$= 0$

c. $A = [\pi]$

$tr(A) = \pi$

Section B: Operations of Matrices

Sub-Section: Addition and Subtraction

Discussion: Could we add two matrices of different size?

$[0]+\begin{bmatrix} & \end{bmatrix}=$ u.d.

$1+2=3$

$$\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}2\\6\end{bmatrix}=\begin{bmatrix}3\\9\end{bmatrix}$$

Addition and Subtraction of Matrices

$$\begin{bmatrix}a_1 & b_1\\c_1 & d_1\end{bmatrix}+\begin{bmatrix}a_2 & b_2\\c_2 & d_2\end{bmatrix}=\begin{bmatrix}a_1+a_2 & b_1+b_2\\c_1+c_2 & d_1+d_2\end{bmatrix}$$

- Add the elements in the same position of each matrix.
- We can only add/subtract matrices of the same size.

Question 6

Let $X=\begin{bmatrix}2\\4\end{bmatrix}$, $Y=\begin{bmatrix}3\\6\end{bmatrix}$

Find $X-Y$.

$$X-Y=\begin{bmatrix}2\\4\end{bmatrix}-\begin{bmatrix}3\\6\end{bmatrix}$$

$$=\begin{bmatrix}2-3\\4-6\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

Sub-Section: Scalar Multiplication

number

Discussion: What will happen to the matrix if we multiply it by two?

$$2\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Scalar Multiplication

$$k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- Multiply all the elements by k.

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Question 7

Let $X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $Y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$, find:

a. $2X - 3Y$

$$= 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 9 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -10 \end{bmatrix}$$

b. $5A + \frac{1}{2}B$

$$= 5\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} \frac{5}{2} & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{25}{2} & 0 \\ -6 & 12 \end{bmatrix}$$

c. $4Y + X + B$

$$= 4\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \underbrace{\begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}}_{2\times2}$$

$= \text{undefined}$

$$\frac{1}{0} + 5 + 6$$

$$= \frac{1}{0} + 11$$

Space for Personal Notes

Sub-Section: Matrix Multiplication

How do we multiply two matrices?

Exploration: Matrix Multiplication

- Let's take a look at two matrices A and B.

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- What would the multiplication look like?

$$A \times B = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(1) \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$$

- We simply take the row of the 1^{st} and column of the 2^{nd} matrix and multiply them.

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac + bd \end{bmatrix}$$

- Let's take a look at two matrices A and B.

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- What would the multiplication look like?

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(3) - 1(1) \\ 1(3) + 2(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

2×2 $\quad$ 2×1 $\quad$ 2×1

- What does the multiplication of two matrices always give us?

$$\underset{n \times m}{\overset{A}{[\quad]}} \times \underset{m \times k}{[\quad]} = \underset{n \times k}{[\quad]}$$

Discussion: For $A \times B$, what does the number of columns for A has to be the same as?

= row B

Discussion: What happens if we multiply 2×3 and 3×1 matrix?

$$= \underset{2 \times 1}{[\quad]}$$

Matrix Multiplication

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + (-1) \times 1 \\ 1 \times 3 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Number of Columns of 1^{st} Matrix = Number of Rows of 2^{nd}

- The answer will always be a matrix.

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Question 8 Walkthrough.

For $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, find AB.

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \times \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2(4)+3(1)+4(0) & 2(0)+3(2)+4(3) \\ 5(4)+6(1)+7(0) & 5(0)+6(2)+7(3) \end{bmatrix}$$

2×3 $\quad 3 \times 2$

2×2

$$= \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix}$$

NOTE: $m \times n$ matrix multiplied with $n \times k$ gives us $m \times k$ matrix.

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Question 9

Find AB for the following set of matrices.

a. $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(3)+1(2) \\ 0(3)+5(2) \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

b. $A = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0(2)+2(0)+1(-1) & 0(4)+2(1)+1(1) \\ 5(2)+3(0)+4(-1) & 5(4)+3(1)+4(1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 6 & 27 \end{bmatrix}$$

]:= $\begin{pmatrix} 0 & 2 & 1 \\ 5 & 3 & 4 \end{pmatrix} . \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$

]= {{-1, 3}, {6, 27}}

<u>Discussion:</u> Is $A \times B$ the same as $B \times A$?

Rows

Columns

No

Sub-Section: Identity Matrix

I

What matrix if you multiply doesn't change the other matrix? Similar to 1 in numbers?

Question 10

Consider $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find $A \times B$.

$:= \begin{pmatrix} 0 & 2 & 1 \\ 5 & 3 & 4 \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

]= {{3, 3}, {12, 12}}

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+3(0) & 2(0)+3(1) \\ 1(1)+4(0) & 1(0)+4(1) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Space for Personal Notes

We call B in the previous question an identity matrix!

Identity Matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Identity matrix I_n has a size $n \times n$.
- Its diagonal elements are equal to 1. everything is 0
- Identity matrices are always a square matrix.

$$AI = IA = A$$

Discussion: What would $tr(I_n)$ always equal to?

$tr(I_n) = n$

Properties of Matrix Multiplication

$$A(B + C) = AB + AC \neq BA + CA$$

Left

$$(A + B)C = AC + BC$$

Right

$$A(BC) = (AB)C$$

$$AI = IA = A$$

identity

$$A0 = 0A = 0$$

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Section C: Inverse Matrices and Determinants

Sub-Section: Inverse Matrix

Inverse of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Inverse only exists for a square matrix.
- Matrix that has an inverse is called invertible.

Question 11 Walkthrough.

Consider $A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Find A^{-1}.

$$= \frac{1}{-12-2} \begin{bmatrix} -3 & -1 \\ -2 & 4 \end{bmatrix} = -\frac{1}{14} \begin{bmatrix} -3 & -1 \\ -2 & 4 \end{bmatrix}$$

Space for Personal Notes

Question 12

Consider $A = \begin{bmatrix} 3 & -3 \\ -2 & 1 \end{bmatrix}$

a. Find A^{-1}.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & -1 \\ -\frac{2}{3} & -1 \end{bmatrix}$$

b. Find AA^{-1}.

$$AA^{-1} = \begin{bmatrix} 3 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & -1 \\ -\frac{2}{3} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-\frac{1}{3})-3(-\frac{2}{3}) & 3(-1)-3(-1) \\ -2(-\frac{1}{3})+1(-\frac{2}{3}) & -2(-1)+1(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Discussion:</u> For the previous question, what do you notice when you multiply a matrix by its inverse?

$$AA^{-1} = I$$

$$A^{-1}A = I$$

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x$$

<u>Multiplication of Inverse Matrices</u>

$$AA^{-1} = I_n = A^{-1}A$$

- The order does not matter.

Sub-Section: Determinant

Discussion: How can we tell if a square matrix does not have an inverse?

- Consider for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \boxed{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$. = undefined

won't

ad − bc = 0

Determinant of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

- If determinant equals to "0", then A does not have an inverse.
- "A is not invertible".
- Notation for the determinant also includes using straight lines | | around the matrix.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Space for Personal Notes

Question 13

Calculate the determinant of the following matrices and state whether it is invertible.

a. $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$

$\det(A) = ad - bc$

$\det(A) = 3(1) - 4(-2) = 3+8$
$= 11$

→ invertible

b. $A = \begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}$

$\det(A) = 2(4) - 8(1) = 0$

→ not invertible

How about 3×3 matrix?

Definition

__Determinant of 3×3 Matrix__ ← cofactor expansion

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \end{array}$$

$$\det(A) = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- For the first term, simply cover the first column and row and copy the 4 numbers.

$$\det(A) = a \times (ei - fh) - b \times (di - fg) + c \times (dh - eg)$$

Question 14 Walkthrough.

Calculate the determinant of the following matrix.

Find the $A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 1 & 3 & 4 \\ -3 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix} \end{array} = 1 \times \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} - 3 \begin{vmatrix} -3 & 5 \\ 0 & -1 \end{vmatrix} + 4 \begin{vmatrix} -3 & 2 \\ 0 & 3 \end{vmatrix}$

$= 1\left(2(-1) - 3(5)\right) - 3\left(-3(-1) - 5(0)\right) + 4\left(-3(3) + 2(0)\right)$

$= -17 - 9 - 36$

$= -62$

↳ invertible

(det $\neq 0$)

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Question 15

Calculate the determinant of the following matrices and state whether it is invertible.

a. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$$|A| = 2\begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 1 & 5 \end{vmatrix}$$

$$= 2(-1-15) - 0 + (0+1)$$

$$= -32 + 1$$

$$= -31$$

b. $A = \begin{bmatrix} 0 & 2 & 5 \\ 7 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix}$

$$|A| = 0\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 7 & 3 \\ -1 & 4 \end{vmatrix} + 5\begin{vmatrix} 7 & 1 \\ -1 & 2 \end{vmatrix}$$

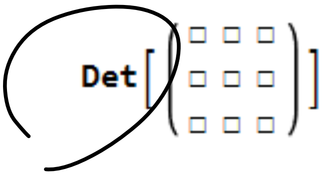
$$= 0 - 2(28+3) + 5(14+1)$$

$$= -62 + 75$$

$$= 13$$

Calculator Commands: Finding Determinant on Technology

- **Mathematica**
  - Control comma/enter to make cells.



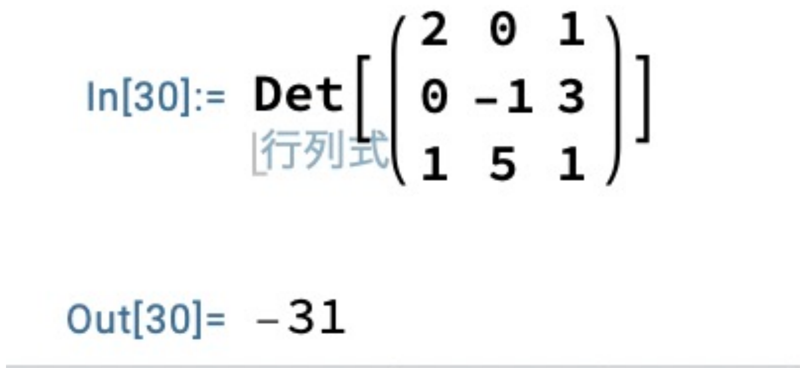
- **TI-Nspire**
  - **(Menu) > Matrix and Vector > Determinant: $\boldsymbol{det}(\boldsymbol{A})$**
  - Where A is a matrix, you enter in the brackets or a previously defined matrix.

- **Casio Classpad**
  - **(Interactive) > Matrix > Calculation > det: $\boldsymbol{det}(\boldsymbol{A})$**
  - Where A is a matrix you enter in the bracket, or a previously defined matrix.

Question 16 Tech-Active.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$$

Find the determinant of A.



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Section D: Systems of Linear Equations

Recall the context at the start of the class.

REMINDERS

- Why matrices?
  - Consider the system of n many linear equations.

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ \vdots \\ a_nx + b_ny + c_nz = d_n \end{cases}$$

  - Imagine solving them!
  - That would be very tedious.

What if there was a way to solve it as one equation?

$$A \cdot Var = D$$

  - We can do that using matrices!

Space for Personal Notes

Question 17

Consider the two matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} x \\ y \end{bmatrix}$$

a. Find $A \times B$.

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 3x-2y \end{bmatrix}$$

b. Hence, solve $A \times B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2x+y \\ 3x-2y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x+y=1 \Rightarrow y=-2x+1 \\ 3x-2y=0 \end{cases}$$

sub

$3x-2(-2x+1)=0$

$3x+4x-2=0$

$7x=2$

$x=\frac{2}{7}$

$y=-2\left(\frac{2}{7}\right)+1$

$=-\frac{4}{7}+1$

$=\frac{3}{7}$

NOTE: We can use matrices to represent simultaneous equations.

<u>Discussion:</u> Hence, how can we represent $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ using matrices?

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Space for Personal Notes

Definition

System of Linear Equation Using Matrices

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- We can extend the idea to more than 2 equations.

Discussion: Previously we used matrices to only represent the equations. Is there any way to solve the equation using matrices?

$$\begin{bmatrix} x \\ y \end{bmatrix} = ?$$

$$A^{-1}A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}C$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}C$$

Exploration: Solving System of Linear Equation using Matrices

- Consider the system of linear equations.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- We can simplify the equation:

$$AX = C$$

- Let's try to solve for our variables by solving our matrix X.
- What happens when you multiply both sides by A^{-1}.

$$A^{-1}AX = A^{-1}C$$

$$\underline{\quad} \qquad =$$

$$IX$$

$$X$$

Definition

Solving System of Linear Equation Using Matrices

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

make eq.

$$AX = C$$

$$X = A^{-1}C$$

Question 18 Walkthrough.

Convert the following system of linear equations into matrix equation and solve for x and y.

$$-x + 6y = 14$$

$$-5x + 4y = -8$$

A, C

$$\begin{bmatrix} -1 & 6 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-4+30}\begin{bmatrix} 4 & -6 \\ 5 & -1 \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix} 4 & -6 \\ 5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 14 \\ -8 \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix} 4 & -6 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 14 \\ -8 \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix} 4(14) + -6(-8) \\ 5(14) + -1(-8) \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix} 56+48 \\ 70+8 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 104 \\ 78 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$x = 4$
$y = 3$

Question 19

Convert the following system of linear equations into matrix equation and solve for x and y.

a. $3x + 5y = 21$ and $6x - 2y = 6$.

$A^{-1} = \frac{1}{-36}\begin{bmatrix} -2 & -5 \\ -6 & 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & 5 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{36}\begin{bmatrix} -2 & -5 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 21 \\ 6 \end{bmatrix}$$

$$= -\frac{1}{36}\begin{bmatrix} -72 \\ -108 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$x = 2$
$y = 3$

b. $-x + y = 2$
$x + y = 3$

$A^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

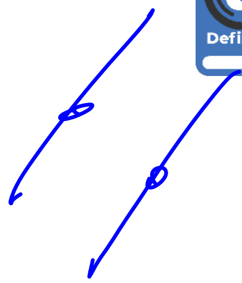
$x = \frac{1}{2}$ $y = \frac{5}{2}$

Discussion: Okay seems like the inverse matrix is important. What if the matrix is not invertible?

no solutions
OR
infinite solution

Definition

Determinants to Determine Number of Solutions



$$AX = C$$

$$X = A^{-1}C$$

$$det(A) = 0 \Rightarrow \text{No or infinite solution}$$

- If the determinant equal to 0, the equation cannot be solved as there is no inverse of A.

$$det(A) \neq 0 \Rightarrow \text{Unique Solution}$$

- If the determinant is non-zero, equation can be solved as there is an inverse of A.

Question 20

Consider the system of linear equations. Determine whether the system of linear equations has a unique solution or not by using determinants.

$x=0 \Rightarrow y=4$

a. $2x + y = 4$ and $-4x - 2y = 10$.

$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

$det(A) = 2(-2) - 1(-4)$
$= -4 + 4$
$= 0 \Rightarrow$ None or infinite

b. $2x + y + z = 1,\ -x - y + z = 10$ and $x + z = 4$.

$det(A) = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}$

$= 2(-1) - (-1-1) + (0+1)$
$= -2 + 2 + 1$
$= 1 \rightarrow$ unique sol

Contour Check

Learning Objective: [4.1.1] - Basics of matrices and identifying types of matrices. Calculate the transpose and trace of a matrix

Key Takeaways

- Matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  - A matrix is an array of numbers.
  - It has rows and columns
  - Dimension (similar to size) of a matrix is given by rows $\times$ columns.

$$Dimension = Rows \times Columns$$

  - Each number is called an element.

- Row and Column Matrix
  - Row Matrix
    - A matrix that has only one row.
  - Column Matrix
    - A matrix that has only one column.

- **Square Matrix:** Has dimension $n \times n$
- **Zero Matrix:** All elements are equal to 0.

Identity Matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Identity matrix I_n has a size $n \times n$.
- Its diagonal elements are equal to 1.
- Identity matrices are always a square matrix.

$$AI = IA = A$$

Transpose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

rows ⟺ col.

$$A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Trace

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$tr(A) = a + d$$

- Find the sum of all the diagonals.
- We can only find the trace of a square matrix.

Learning Objective: [4.1.2] – Perform matrix addition, scalar multiplication, and matrix multiplication

Key Takeaways

Addition and Subtraction of Matrices

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & \end{bmatrix}$$

- We can only add/subtract matrices of the same dimensions.
- Add the elements in the same position of each matrix.

Scalar Multiplication

$$k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \_\_\_\_\_\_\_\_\_\_$$

- Multiply all the elements by k.

Matrix Multiplication

check

2×2 2×1 2×1

$$A \times B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 3 + (-1) \times 1 \\ 1 \times 3 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Number of columns of 1st Matrix
= Number of rows of 2nd

- The answer will always be a matrix.

- Properties of Matrix Multiplication

$$A(B+C) = \underline{AB+AC}$$

$$(A+B)C = \underline{AC+BC}$$

$$A(BC) = \underline{(AB)C}$$

$$AI = IA = \underline{A}$$

$$A0 = 0A = \underline{0}$$

- Learning Objective: [4.1.3] – Calculate the inverse of a matrix and determine its determinant

Key Takeaways

- Inverse of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  - Inverse only exists for a square matrix.
  - Matrix that has an inverse is called invertible.

- Multiplication of Inverse Matrices

$$AA^{-1} = \underline{I} = A^{-1}A$$

  - The order does not matter.

- **Determinant of a 2×2 Matrix**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$

- If determinant equals to 0, then A does not have an inverse.
- A is not invertible.

- **Determinant of 3×3 Matrix**

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

ci-fh

$$det(A) = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$=$

Learning Objective: [4.1.4] – Apply matrix operations to solve systems of linear equations

Key Takeaways

- Solving System of Linear Equation using Matrices

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$AX = C$$

$$X = \underline{A^{-1}C}$$

- Determinants to Determine Number of Solutions

$$AX = C$$

$$X = A^{-1}C$$

$$det(A) = 0 \Rightarrow \underline{\boxed{\text{No or infinite solution}}}$$

- If the determinant equal to 0, the equation cannot be solved as there is no inverse of A.

$$det(A) \neq 0 \Rightarrow \underline{\boxed{\text{Unique Solution}}}$$

- If the determinant is non-zero, equation can be solved as there is an inverse of A.

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