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VCE Specialist Mathematics ½
Matrices [4.1]
Homework Solutions

Admin Info & Homework Outline:

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Section A: Compulsory Questions

Sub-Section [3.5.1]: Basics of Matrices and Identifying Types of Matrices. Calculate the Transpose and Trace of a Matrix

Question 1

a. What does it mean for a matrix to be a **square matrix**? Give an example.

A matrix is called a **square matrix** if it has the same number of rows and columns, i.e., its dimensions are $n \times n$.
Example:

$$S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is a 2×2 square matrix.

b. Given the matrix:

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$$

Find $a_{1,1} + a_{1,2}$.

We have:

$$a_{1,1} = 3, \quad a_{1,2} = 5.$$

Adding these values:

$$a_{1,1} + a_{1,2} = 3 + 5 = 8.$$

c. Compute the **trace** of matrix A.

The trace of a square matrix is the sum of its diagonal elements.

$$\text{tr}(A) = 3 + 4 = 7.$$

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Question 2

a. Consider the matrix:

$$B = [7 \quad -2 \quad 5]$$

What type of matrix is B?

The matrix B has only one row, so it is called a **row matrix** (or row vector).

b. Compute the **transpose** of the matrix:

$$C = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}$$

The transpose of a matrix is obtained by swapping its rows and columns:

$$C^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}.$$

c. Find the **trace** of:

$$D = \begin{bmatrix} 6 & -1 & 2 \\ 4 & 3 & -5 \\ 0 & 7 & 8 \end{bmatrix}$$

The trace of a matrix is the sum of its diagonal elements:

$$\text{tr}(D) = 6 + 3 + 8 = 17.$$

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Question 3

a. If a matrix E is such that $E^T = E$, what can we conclude about the matrix's dimensions? Give an example of a 3×3 matrix satisfying this condition.

A matrix satisfying $E^T = E$ is called a **symmetric matrix**. For a matrix to be symmetric, it must be square, meaning it has the same number of rows and columns.
Example:

$$E = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 5 \\ 1 & 5 & 6 \end{bmatrix}$$

is a 3×3 symmetric matrix since $E^T = E$.

b. Evaluate the **transpose** of:

$$F = \begin{bmatrix} 2 & -3 & 1 & 0 \\ 4 & 5 & -2 & 6 \\ -1 & 0 & 3 & -4 \end{bmatrix}$$

The transpose of F is obtained by swapping its rows and columns:

$$F^T = \begin{bmatrix} 2 & 4 & -1 \\ -3 & 5 & 0 \\ 1 & -2 & 3 \\ 0 & 6 & -4 \end{bmatrix}.$$

c. Explain why the trace of matrix F cannot be found.

The trace of a matrix is only defined for **square matrices**, which have the same number of rows and columns.
Since F is a 3×4 matrix (not square), it does not have a trace.

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Sub-Section [3.5.2]: Perform Matrix Addition, Scalar Multiplication, and Matrix Multiplication

Question 4

a. Given the matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$$

Evaluate $A + B$.

Matrix addition is performed element-wise:

$$A + B = \begin{bmatrix} 3+2 & 1+(-3) \\ -2+5 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 5 \end{bmatrix}$$

b. Given the matrix:

$$C = \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$$

Find $3C$.

Scalar multiplication is applied to each element:

$$3C = 3 \times \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 0 & 12 \end{bmatrix}.$$

c. Given the matrices:

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Evaluate $D \times E$.

Matrix multiplication is computed as follows:

$$D \times E = \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} = \begin{bmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

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Question 5

a. Matrices F and G are given as follows:

$$F = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 3 & -5 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 5 \end{bmatrix}$$

Evaluate $2F - 4G$.

$$2F = \begin{bmatrix} 8 & -2 & 4 \\ 0 & 6 & -10 \end{bmatrix}, \quad 4G = \begin{bmatrix} 4 & 8 & -12 \\ 16 & 0 & 20 \end{bmatrix}$$

Subtracting:

$$2F - 4G = \begin{bmatrix} 8-4 & -2-8 & 4-(-12) \\ 0-16 & 6-0 & -10-20 \end{bmatrix} = \begin{bmatrix} 4 & -10 & 16 \\ -16 & 6 & -30 \end{bmatrix}$$

b. Consider the matrices:

$$H = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$$

Find $H + Q$, if possible.

Matrix addition is only possible if both matrices have the same dimensions. H is a 2×3 matrix, and Q is a 2×2 matrix. Since their dimensions do not match, $H + Q$ is **not possible**.

c. Given:

$$J = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}, \quad k = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

Compute $J \times K$.

$$J \times K = \begin{bmatrix} (1)(2) + (-1)(-1) & (1)(3) + (-1)(5) \\ (2)(2) + (0)(-1) & (2)(3) + (0)(5) \\ (3)(2) + (4)(-1) & (3)(3) + (4)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 3-5 \\ 4+0 & 6+0 \\ 6-4 & 9+20 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 6 \\ 2 & 29 \end{bmatrix}.$$

Question 6

a. Consider matrices L and M given by:

$$L = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 2 \\ 5 & 0 \end{bmatrix}$$

Evaluate $L(L + M)$.

$$L + M = \begin{bmatrix} 2+(-1) & 4+2 \\ -3+5 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 1 \end{bmatrix}.$$

$$L(L+M) = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 6 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+4(2) & 2(6)+4(1) \\ -3(1)+1(2) & -3(6)+1(1) \end{bmatrix} = \begin{bmatrix} 10 & 16 \\ -1 & -17 \end{bmatrix}.$$

b. Consider the matrices:

$$N = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}$$

Determine whether $N^2 - P^2 = (N + P)(N - P)$.

First, compute N^2 and P^2:

$$N^2 = N \times N = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & 3+12 \\ -2-8 & -6+16 \end{bmatrix} = \begin{bmatrix} -5 & 15 \\ -10 & 10 \end{bmatrix}.$$

$$P^2 = P \times P = \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0-2 & 0-5 \\ 0+10 & -2+25 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 10 & 23 \end{bmatrix}.$$

$$N^2 - P^2 = \begin{bmatrix} -5 & 15 \\ -10 & 10 \end{bmatrix} - \begin{bmatrix} -2 & -5 \\ 10 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 20 \\ -20 & -13 \end{bmatrix}.$$

Now, compute $N + P$ and $N - P$:

$$N + P = \begin{bmatrix} 1+0 & 3+(-1) \\ -2+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}.$$

$$N - P = \begin{bmatrix} 1-0 & 3-(-1) \\ -2-2 & 4-5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -4 & -1 \end{bmatrix}.$$

$$(N+P)(N-P) = \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8 & 4-2 \\ 0-36 & 0-9 \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ -36 & -9 \end{bmatrix}.$$

Since $N^2 - P^2 \neq (N+P)(N-P)$, the matrix identity does not hold.

c. Consider the matrices:

$$Q = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 \\ x & 4 \\ 5 & y \end{bmatrix}$$

Find the value of x and y if $QR = \begin{bmatrix} 20 & 6 \\ -6 & 4 \end{bmatrix}$.

Compute the matrix product $Q \times R$:

$$QR = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ x & 4 \\ 5 & y \end{bmatrix}$$

$$= \begin{bmatrix} 17 - x & 3y \\ -6 & 8 - 2y \end{bmatrix}.$$

We are given that:

$$\begin{bmatrix} 17 - x & 3y \\ -6 & 8 - 2y \end{bmatrix} = \begin{bmatrix} 20 & 6 \\ -6 & 4 \end{bmatrix}.$$

Equating corresponding elements yields, $x = -3, y = 2$

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Sub-Section [3.5.3]: Calculate the Inverse of a Matrix and Determine Its Determinant

Question 7

a. Consider det (A) for:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$\det(A) = ad - bc.$$

Substituting the given values:

$$\det(A) = (3)(4) - (2)(1) = 12 - 2 = 10.$$

b. Is A **invertible**? Justify your answer.

A matrix is invertible if and only if its determinant is nonzero. Since:

$$\det(A) = 10 \neq 0,$$

the matrix A is invertible.

c. If A is invertible, find A^{-1}.

The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Substituting the values:

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}$$

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Question 8

a. Compute $\det(B)$ for:

$$B = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix}$$

Using the determinant formula:

$$\det(B) = (5)(1) - (-3)(2) = 5 + 6 = 11.$$

b. If $BX = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find X.

We solve for X using $X = B^{-1} \times \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$.

We find that

$$B^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}.$$

Then evaluating $X = B^{-1} \times \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, we get

$$X = \begin{bmatrix} \frac{6}{11} & 2 \\ -\frac{1}{11} & 2 \end{bmatrix}$$

c. Determine the value of x is the matrix $\begin{bmatrix} x & 4 \\ 3 & x+4 \end{bmatrix}$ does not have an inverse.

$\text{Det} = x^2 + 4x - 12 = (x+6)(x-2)$.
$\text{Det} = 0 \implies x = -6, 2$. Thus the matrix does not have an inverse if $x = -6, 2$.

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Question 9

a. Compute $\det(D)$ for:

$$D = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 5 \\ -2 & 1 & 6 \end{bmatrix}$$

The determinant of a 3×3 matrix may be calculated using:

$$\det(D) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Expanding along the first row:

$$\det(D) = 2(0 \cdot 6 - 5 \cdot 1) - (-1)(4 \cdot 6 - 5 \cdot (-2)) + 3(4 \cdot 1 - 0 \cdot (-2)).$$

$$= 2(0 - 5) + 1(24 + 10) + 3(4 - 0).$$

$$= 2(-5) + 1(34) + 3(4).$$

$$= -10 + 34 + 12 = 36.$$

b. Is D invertible? Justify your answer.

Since $\det(D) = 36 \neq 0$, the matrix D is invertible.

c. Find the determinant of D^{-1} given that D is invertible.

The determinant of the inverse of a matrix is given by:

$$\det(D^{-1}) = \frac{1}{\det(D)}.$$

(see the Supplementary section for a proof)
Substituting the known determinant:

$$\det(D^{-1}) = \frac{1}{36}.$$

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Sub-Section [3.5.4]: Apply Matrix Operations to Solve Systems of Linear Equations

Question 10

a. Write the system of equations:

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 7 \end{cases}$$

in the form $AX = C$.

The system of equations can be written in matrix form as:

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

b. Compute A^{-1} for:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$\det(A) = 10$. Thus

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$$

c. If $AX = C$ and $A^{-1} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$, find X given:

$$C = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

The solution to $AX = C$ is given by:

$$X = A^{-1}C.$$

Multiplying:

$$X = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

$$= \begin{bmatrix} (4)(6) + (2)(5) \\ (3)(6) + (1)(5) \end{bmatrix} = \begin{bmatrix} 24 + 10 \\ 18 + 5 \end{bmatrix} = \begin{bmatrix} 34 \\ 23 \end{bmatrix}.$$

Question 11

a. Solve for X in:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} X = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

using $X = A^{-1}C$.

$$X = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

b. Determine whether the system:

$$\begin{cases} 4x + y = 2 \\ 8x + 2y = 5 \end{cases}$$

Has a unique solution, no solution, or infinitely many solutions.

Det $= 0$, further inspection shows that system has no solutions.

c. Determine the value(s) of k for which the system:

$$\begin{cases} 4x + 2ky = 5 \\ 2kx + 4y = 5 \end{cases}$$

Has a unique solution, no solution, or infinitely many solutions.

Det $= 16 - 4k^2 = 0 \implies k = \pm 2$.
Then we see that there is infinite solutions if $k = 2$, no solutions if $k = -2$ and unique solution for $k \in \mathbb{R} \setminus \{-2, 2\}$.

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Question 12

a. Write the following system in matrix form and solve using the inverse matrix method.

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 7 \end{cases}$$

The system can be rewritten as:

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{4}{14} & -\frac{2}{14} \end{bmatrix}.$$

$$X = A^{-1}C = \begin{bmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{4}{14} & -\frac{2}{14} \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

Thus,

$$x = \frac{1}{14}(5) + \frac{3}{14}(7) = \frac{5}{14} + \frac{21}{14} = \frac{26}{14} = \frac{13}{7}.$$

$$y = \frac{4}{14}(5) + \left(-\frac{2}{14}(7)\right) = \frac{20}{14} - \frac{14}{14} = \frac{6}{14} = \frac{3}{7}.$$

The solution is:

$$x = \frac{13}{7}, \quad y = \frac{3}{7}.$$

b. Write the following system in matrix form and solve using the inverse matrix method.

$$\begin{cases} x - 2y = 4 \\ 3x + 6y = 2 \end{cases}$$

The system can be rewritten as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 6 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{12} & \frac{2}{12} \\ \frac{-3}{12} & \frac{1}{12} \end{bmatrix}.$$

$$X = A^{-1}C = \begin{bmatrix} \frac{6}{12} & \frac{2}{12} \\ \frac{-3}{12} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Thus,

$$x = \frac{6}{12}(4) + \frac{2}{12}(2) = \frac{24}{12} + \frac{4}{12} = \frac{28}{12} = \frac{7}{3}.$$

$$y = \frac{-3}{12}(4) + \frac{1}{12}(2) = \frac{-12}{12} + \frac{2}{12} = \frac{-10}{12} = -\frac{5}{6}.$$

The solution is:

$$x = \frac{7}{3}, \quad y = -\frac{5}{6}.$$

c. Write the following system in matrix form and solve using the inverse matrix method.

$$\begin{cases} 5x + 2y = 3 \\ x - 4y = 6 \end{cases}$$

The system can be rewritten as:

$$\begin{bmatrix} 5 & 2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 5 & 2 \\ 1 & -4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{-22} \begin{bmatrix} -4 & -2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{22} & \frac{2}{22} \\ \frac{1}{22} & \frac{-5}{22} \end{bmatrix}.$$

$$X = A^{-1}C = \begin{bmatrix} \frac{4}{22} & \frac{2}{22} \\ \frac{1}{22} & \frac{-5}{22} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Thus,

$$x = \frac{4}{22}(3) + \frac{2}{22}(6) = \frac{12}{22} + \frac{12}{22} = \frac{24}{22} = \frac{12}{11}.$$

$$y = \frac{1}{22}(3) + \frac{-5}{22}(6) = \frac{3}{22} - \frac{30}{22} = -\frac{27}{22}.$$

The solution is:

$$x = \frac{12}{11}, \quad y = -\frac{27}{22}.$$

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Sub-Section: Final Boss

Question 13

A company tracks the sales of two different products, P_1 and P_2, across two stores, Store A and Store B. The data collected over a week is represented using matrices.

a. The weekly sales data (in units sold) for both products at each store is given as:

$$S = \begin{bmatrix} 40 & 25 \\ 30 & 50 \end{bmatrix}$$

Where the rows correspond to Store A and Store B, and the columns correspond to Product P_1 and Product P_2.

i. Identify what type of matrix S is.

The matrix S is a square matrix since it has an equal number of rows and columns (2×2).

ii. Compute the trace of S.

The trace of a matrix is the sum of its diagonal elements:

$$\text{tr}(S) = 40 + 50 = 90.$$

iii. Compute the transpose of S and interpret it in this context.

The transpose of S is obtained by swapping rows and columns:

$$S^T = \begin{bmatrix} 40 & 30 \\ 25 & 50 \end{bmatrix}.$$

The significance of S^T is that it reorganizes the data, switching the roles of stores and products.

b. Each unit of Product P_1 generates \$5 in revenue, while each unit of Product P_2 generates \$8.

i. Represent the price per unit as a column matrix.

The price matrix is:

$$P = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

ii. Compute the total revenue per store using matrix multiplication.

The revenue per store is given by:

$$R = SP = \begin{bmatrix} 40 & 25 \\ 30 & 50 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Computing:

$$R = \begin{bmatrix} (40 \times 5) + (25 \times 8) \\ (30 \times 5) + (50 \times 8) \end{bmatrix} = \begin{bmatrix} 200 + 200 \\ 150 + 400 \end{bmatrix} = \begin{bmatrix} 400 \\ 550 \end{bmatrix}.$$

Store A generates \$400, and Store B generates \$550.

iii. Compute the overall total revenue.

The total revenue is the sum of all store revenues:

$$400 + 550 = 950.$$

Thus, the total revenue is \$950.

c. If the company wants both stores to generate \$500 in weekly revenue, and the sales matrix remains the same, determine how the products P_1 and P_2 should be priced.

The revenue equation is given by:

$$R = SP = \begin{bmatrix} 500 \\ 500 \end{bmatrix}.$$

$$P = S^{-1} \begin{bmatrix} 500 \\ 500 \end{bmatrix} = \frac{1}{1250} \begin{bmatrix} 50 & -25 \\ -30 & 40 \end{bmatrix} \begin{bmatrix} 500 \\ 500 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

Thus P_1 should be \$10 and P_2 should be \$4

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Section B: Supplementary Questions

Sub-Section [3.5.1]: Basics of Matrices and Identifying Types of Matrices. Calculate the Transpose and Trace of a Matrix

Question 14

a. What does it mean for a matrix to be a **column matrix**? Give an example.

A matrix is called a **column matrix** if it has only one column and multiple rows, meaning its dimensions are $n \times 1$.
Example:

$$C = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

is a 3×1 column matrix.

b. Given the matrix:

$$A = \begin{bmatrix} 6 & -2 \\ 1 & 5 \end{bmatrix}$$

Find $a_{2,1} + a_{2,2}$.

We have:

$$a_{2,1} = 1, \quad a_{2,2} = 5.$$

Adding these values:

$$a_{2,1} + a_{2,2} = 1 + 5 = 6.$$

c. Compute the **trace** of matrix A.

The trace of a square matrix is the sum of its diagonal elements.

$$\text{tr}(A) = 6 + 5 = 11.$$

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Question 15

a. Consider the matrix:

$$B = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

What type of matrix is B?

The matrix B has only one column, so it is called a **column matrix** (or column vector).

b. Compute the **transpose** of the matrix:

$$C = \begin{bmatrix} -2 & 4 \\ 1 & -3 \\ 5 & 0 \end{bmatrix}$$

The transpose of a matrix is obtained by swapping its rows and columns:

$$C^T = \begin{bmatrix} -2 & 1 & 5 \\ 4 & -3 & 0 \end{bmatrix}.$$

c. Find the **trace** of:

$$D = \begin{bmatrix} 9 & -4 & 3 \\ 2 & 7 & -6 \\ 1 & 8 & -2 \end{bmatrix}$$

The trace of a matrix is the sum of its diagonal elements:

$$\text{tr}(D) = 9 + 7 + (-2) = 14.$$

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Question 16

a. If a matrix E is such that $E^T = -E$, what can we conclude about the matrix's structure? Give an example of a 3×3 matrix satisfying this condition.

A matrix satisfying $E^T = -E$ is called a **skew-symmetric matrix**. For a matrix to be skew-symmetric, it must be square, and all diagonal elements must be zero.
Example:

$$E = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$$

is a 3×3 skew-symmetric matrix since $E^T = -E$.

b. Evaluate the **transpose** of:

$$F = \begin{bmatrix} 1 & -4 & 2 & 3 \\ -3 & 5 & 0 & -1 \\ 7 & 2 & -6 & 0 \end{bmatrix}$$

The transpose of F is obtained by swapping its rows and columns:

$$F^T = \begin{bmatrix} 1 & -3 & 7 \\ -4 & 5 & 2 \\ 2 & 0 & -6 \\ 3 & -1 & 0 \end{bmatrix}.$$

c. Explain why the trace of matrix F cannot be found.

The trace of a matrix is only defined for **square matrices**, which have the same number of rows and columns.
Since F is a 3×4 matrix (not square), it does not have a trace.

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Sub-Section [3.5.2]: Perform Matrix Addition, Scalar Multiplication, and Matrix Multiplication

Question 17

a. Given the matrices:

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 5 \\ 6 & 0 \end{bmatrix}$$

Evaluate $A + B$.

Matrix addition is performed element-wise:

$$A + B = \begin{bmatrix} 4 + (-2) & -1 + 5 \\ 2 + 6 & 3 + 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix}.$$

b. Given the matrix:

$$C = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$$

Find $2C$.

Scalar multiplication is applied to each element:

$$2C = 2 \times \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ 2 & 4 \end{bmatrix}.$$

c. Given the matrices:

$$D = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix}$$

Evaluate $D \times E$.

Matrix multiplication is computed as follows:

$$D \times E = \begin{bmatrix} 2(4) + 3(7) & 2(1) + 3(-2) \\ (-1)(4) + 5(7) & (-1)(1) + 5(-2) \end{bmatrix} =$$

$$\begin{bmatrix} 8 + 21 & 2 - 6 \\ -4 + 35 & -1 - 10 \end{bmatrix} = \begin{bmatrix} 29 & -4 \\ 31 & -11 \end{bmatrix}.$$

Question 18

a. Matrices F and G are given as follows:

$$F = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 0 & -4 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -2 & 6 \end{bmatrix}$$

Evaluate $3F - 2G$.

$$3F = \begin{bmatrix} 15 & -6 & 3 \\ 9 & 0 & -12 \end{bmatrix}, \quad 2G = \begin{bmatrix} -2 & 6 & 4 \\ 8 & -4 & 12 \end{bmatrix}$$

Subtracting:

$$3F - 2G = \begin{bmatrix} 15 - (-2) & -6 - 6 & 3 - 4 \\ 9 - 8 & 0 - (-4) & -12 - 12 \end{bmatrix} = \begin{bmatrix} 17 & -12 & -1 \\ 1 & 4 & -24 \end{bmatrix}.$$

b. Consider the matrices:

$$H = \begin{bmatrix} 1 & 4 & -3 \\ 2 & -5 & 7 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$$

Find $H + Q$, if possible.

Matrix addition is only possible if both matrices have the same dimensions. H is a 2×3 matrix, and Q is a 2×2 matrix. Since their dimensions do not match, $H + Q$ is **not possible**.

c. Given:

$$J = \begin{bmatrix} -1 & 2 \\ 4 & 0 \\ 3 & -5 \end{bmatrix}, \quad k = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

Compute $J \times K$.

$$J \times K = \begin{bmatrix} (-1)(3) + (2)(-2) & (-1)(1) + (2)(4) \\ (4)(3) + (0)(-2) & (4)(1) + (0)(4) \\ (3)(3) + (-5)(-2) & (3)(1) + (-5)(4) \end{bmatrix}$$

$$= \begin{bmatrix} -3 - 4 & -1 + 8 \\ 12 + 0 & 4 + 0 \\ 9 + 10 & 3 - 20 \end{bmatrix} = \begin{bmatrix} -7 & 7 \\ 12 & 4 \\ 19 & -17 \end{bmatrix}.$$

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Question 19

a. Consider matrices L and M given by:

$$L = \begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$$

Evaluate $L(L + M)$.

$$L + M = \begin{bmatrix} 3+1 & 2+(-2) \\ -4+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}.$$

$$L(L+M) = \begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3(4)+2(1) & 3(0)+2(4) \\ -4(4)+1(1) & -4(0)+1(4) \end{bmatrix} = \begin{bmatrix} 12+2 & 0+8 \\ -16+1 & 0+4 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ -15 & 4 \end{bmatrix}.$$

b. Consider the matrices:

$$N = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$$

Determine whether $N^2 - P^2 = (N + P)(N - P)$.

First, compute N^2 and P^2:

$$N^2 = N \times N = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2)+(-3)(4) & 2(-3)+(-3)(1) \\ 4(2)+1(4) & 4(-3)+1(1) \end{bmatrix} = \begin{bmatrix} 4-12 & -6-3 \\ 8+4 & -12+1 \end{bmatrix} = \begin{bmatrix} -8 & -9 \\ 12 & -11 \end{bmatrix}.$$

$$P^2 = P \times P = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+(-1)(2) & 1(-1)+(-1)(5) \\ 2(1)+5(2) & 2(-1)+5(5) \end{bmatrix} = \begin{bmatrix} 1-2 & -1-5 \\ 2+10 & -2+25 \end{bmatrix} = \begin{bmatrix} -1 & -6 \\ 12 & 23 \end{bmatrix}.$$

$$N^2 - P^2 = \begin{bmatrix} -8 & -9 \\ 12 & -11 \end{bmatrix} - \begin{bmatrix} -1 & -6 \\ 12 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} -8+1 & -9+6 \\ 12-12 & -11-23 \end{bmatrix} = \begin{bmatrix} -7 & -3 \\ 0 & -34 \end{bmatrix}.$$

Now, compute $N + P$ and $N - P$:

$$N + P = \begin{bmatrix} 2+1 & -3+(-1) \\ 4+2 & 1+5 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 6 & 6 \end{bmatrix}.$$

$$N - P = \begin{bmatrix} 2-1 & -3-(-1) \\ 4-2 & 1-5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}.$$

$$(N+P)(N-P) = \begin{bmatrix} 3 & -4 \\ 6 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1)+(-4)(2) & 3(-2)+(-4)(-4) \\ 6(1)+6(2) & 6(-2)+6(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 3-8 & -6+16 \\ 6+12 & -12-24 \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ 18 & -36 \end{bmatrix}.$$

Since $N^2 - P^2 \neq (N+P)(N-P)$, the matrix identity does not hold.

c. Given the matrices:

$$Q = \begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & -3 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 3 \\ x & 4 \\ 7 & y \end{bmatrix}$$

Find the value of x and y if $QR = \begin{bmatrix} 21 & 6 \\ -3 & 13 \end{bmatrix}$.

Compute the matrix product $Q \times R$:

$$QR = \begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ x & 4 \\ 7 & y \end{bmatrix}$$

$$= \begin{bmatrix} 3(2) + (-2)(x) + 5(7) & 3(3) + (-2)(4) + 5(y) \\ 4(2) + 1(x) + (-3)(7) & 4(3) + 1(4) + (-3)(y) \end{bmatrix}$$

$$= \begin{bmatrix} 6 - 2x + 35 & 9 - 8 + 5y \\ 8 + x - 21 & 12 + 4 - 3y \end{bmatrix}.$$

We are given that:

$$\begin{bmatrix} 41 - 2x & 5y + 1 \\ x - 13 & 16 - 3y \end{bmatrix} = \begin{bmatrix} 21 & 10 \\ -10 & 13 \end{bmatrix}.$$

Equating corresponding elements:
$x = 10$, $y = 1$.

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Sub-Section [3.5.3]: Calculate the Inverse of a Matrix and Determine its Determinant

Question 20

a. Compute $\det(A)$ for:

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$$

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$\det(A) = ad - bc.$$

Substituting the given values:

$$\det(A) = (4)(5) - (-2)(3) = 20 + 6 = 26.$$

b. Is A **invertible**? Justify your answer.

A matrix is invertible if and only if its determinant is nonzero. Since:

$$\det(A) = 26 \neq 0,$$

the matrix A is invertible.

c. If A is invertible, find A^{-1}.

The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Substituting the values:

$$A^{-1} = \frac{1}{26} \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{26} & \frac{2}{26} \\ -\frac{3}{26} & \frac{4}{26} \end{bmatrix}.$$

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Question 21

a. Compute $\det(B)$ for:

$$B = \begin{bmatrix} 6 & -4 \\ 1 & 2 \end{bmatrix}$$

Using the determinant formula:

$$\det(B) = (6)(2) - (-4)(1) = 12 + 4 = 16.$$

b. If $BX = \begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}$, find X.

We solve for X using $X = B^{-1} \times \begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}$.

We find that

$$B^{-1} = \frac{1}{16}\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}.$$

Then evaluating $X = B^{-1} \times \begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}$, we get

$$X = \begin{bmatrix} \frac{10}{16} + \frac{28}{16} & \frac{6}{16} + \frac{4}{16} \\ -\frac{5}{16} + \frac{42}{16} & -\frac{3}{16} + \frac{6}{16} \end{bmatrix} = \begin{bmatrix} \frac{19}{8} & \frac{5}{8} \\ \frac{37}{16} & \frac{3}{16} \end{bmatrix}.$$

c. Determine the possible value of x if the matrix, $\begin{bmatrix} x & 5 \\ -2 & x+3 \end{bmatrix}$ is invertible.

The determinant is given by:

$$\det = x(x+3) - (5)(-2) = x^2 + 3x + 10.$$

Setting determinant equal to zero:

$$x^2 + 3x + 10 = 0.$$

Solving for x, we find the roots using the quadratic formula:

$$x = \frac{-3 \pm \sqrt{9 - 40}}{2} = \frac{-3 \pm \sqrt{-31}}{2}.$$

Since the determinant never equals zero for real x, the matrix is always invertible.

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Question 22

a. Compute $\det(D)$ for:

$$D = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 4 & -1 \\ 2 & 0 & 3 \end{bmatrix}$$

The determinant of a 3×3 matrix may be calculated using:

$$\det(D) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Expanding along the first row:

$$\det(D) = 3(4 \cdot 3 - (-1) \cdot 0) - (-2)(1 \cdot 3 - (-1) \cdot 2) + 5(1 \cdot 0 - 4 \cdot 2).$$

$$= 3(12) + 2(3 + 2) + 5(0 - 8).$$

$$= 36 + 10 - 40 = 6.$$

b. Use the fact that $\det(AB) = \det(A) \cdot \det(B)$, for two invertible matrices, A and B, to prove that $\det(A^{-1}) = \frac{1}{\det(A)}$.

$\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1.$

Thus $\det(A^{-1}) = \dfrac{1}{\det(A)}$

c. Find the determinant of D^{-1} given that, D is invertible.

The determinant of the inverse of a matrix is given by:

$$\det(D^{-1}) = \frac{1}{\det(D)}.$$

Substituting the known determinant:

$$\det(D^{-1}) = \frac{1}{6}.$$

Sub-Section [3.5.4]: Apply Matrix Operations to Solve Systems of Linear Equations

Question 23

a. Write the system of equations:

$$\begin{cases} 3x - 2y = 8 \\ 5x + 4y = -6 \end{cases}$$

in the form $AX = C$.

The system of equations can be written in matrix form as:

$$\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 8 \\ -6 \end{bmatrix}.$$

b. Compute A^{-1} for:

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}$$

The determinant of A is given by:

$$\det(A) = (2)(5) - (1)(-3) = 10 + 3 = 13.$$

Since $\det(A) \neq 0$, the matrix is invertible.
Using the inverse formula:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Substituting values:

$$A^{-1} = \frac{1}{13} \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}.$$

c. If $AX = C$ and $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, find X given that:

$$C = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$$

The solution to $AX = C$ is given by:

$$X = A^{-1}C.$$

Multiplying:

$$X = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

$$= \begin{bmatrix} (2)(-5) + (-1)(7) \\ (3)(-5) + (4)(7) \end{bmatrix} = \begin{bmatrix} -10 - 7 \\ -15 + 28 \end{bmatrix} = \begin{bmatrix} -17 \\ 13 \end{bmatrix}.$$

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Question 24

a. Solve for X in:

$$\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix} X = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Using $X = A^{-1}C$.

$$X = \frac{1}{13}\begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} (2)(4) + (1)(7) \\ (-3)(4) + (5)(7) \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 8 + 7 \\ -12 + 35 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

b. Determine whether the system:

$$\begin{cases} 6x + 2y = 4 \\ 12x + 4y = 10 \end{cases}$$

Has a unique solution, no solution, or infinitely many solutions.

The system in matrix form is:

$$A = \begin{bmatrix} 6 & 2 \\ 12 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 4 \\ 10 \end{bmatrix}.$$

Computing the determinant:

$$\det(A) = (6)(4) - (2)(12) = 24 - 24 = 0.$$

Closer inspection shows that the system will have no solutions because the first equation is equivalent to $12x + 4y = 8$.

c. Determine the value(s) of k for which the system:

$$\begin{cases} 3x + ky = 7 \\ kx + 6y = 5 \end{cases}$$

Has a unique solution, no solution, or infinitely many solutions.

Writing in matrix form:

$$A = \begin{bmatrix} 3 & k \\ k & 6 \end{bmatrix}.$$

Compute determinant:

$$\det(A) = (3)(6) - (k)(k) = 18 - k^2.$$

Setting determinant to zero for dependency:

$$18 - k^2 = 0 \Rightarrow k^2 = 18 \Rightarrow k = \pm\sqrt{18} = 3\sqrt{2}.$$

Test these values of k to see that they both result in distinct paralell lines.
Thus system never has infitely many solutions, no solution for $k = \pm 3\sqrt{2}$ and unique solution for $k \in \mathbb{R} \setminus \{-3\sqrt{2}, 3\sqrt{2}\}$

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Question 25

a. Write the following system in matrix form and solve using the inverse matrix method.

$$\begin{cases} 3x + 4y = 7 \\ 5x - y = 9 \end{cases}$$

The system can be rewritten as:

$$\begin{bmatrix} 3 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 3 & 4 \\ 5 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 7 \\ 9 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{-23} \begin{bmatrix} -1 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{23} & \frac{4}{23} \\ \frac{5}{23} & -\frac{3}{23} \end{bmatrix}.$$

$$X = A^{-1}C = \begin{bmatrix} \frac{1}{23} & \frac{4}{23} \\ \frac{5}{23} & -\frac{3}{23} \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix}.$$

Thus,

$$x = \frac{1}{23}(7) + \frac{4}{23}(9) = \frac{7}{23} + \frac{36}{23} = \frac{43}{23}.$$

$$y = \frac{5}{23}(7) + \left(-\frac{3}{23}(9)\right) = \frac{35}{23} - \frac{27}{23} = \frac{8}{23}.$$

The solution is:

$$x = \frac{43}{23}, \quad y = \frac{8}{23}.$$

b. Write the following system in matrix form and solve using the inverse matrix method.

$$\begin{cases} x + 2y = 5 \\ 4x - 3y = 1 \end{cases}$$

The system can be rewritten as:

$$\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{-11} \begin{bmatrix} -3 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix}.$$

$$X = A^{-1}C = \begin{bmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Thus,

$$x = \frac{3}{11}(5) + \frac{2}{11}(1) = \frac{15}{11} + \frac{2}{11} = \frac{17}{11}.$$

$$y = \frac{4}{11}(5) + \left(-\frac{1}{11}(1)\right) = \frac{20}{11} - \frac{1}{11} = \frac{19}{11}.$$

The solution is:

$$x = \frac{17}{11}, \quad y = \frac{19}{11}.$$

c. Write the following system in matrix form and solve using the inverse matrix method.

$$\begin{cases} 6x + 3y = 4 \\ 2x - 5y = 7 \end{cases}$$

The system can be rewritten as:

$$\begin{bmatrix} 6 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

Thus, we have:

$$A = \begin{bmatrix} 6 & 3 \\ 2 & -5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{-36} \begin{bmatrix} -5 & -3 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} \frac{5}{36} & \frac{3}{36} \\ \frac{2}{36} & -\frac{6}{36} \end{bmatrix}.$$

$$X = A^{-1}C = \begin{bmatrix} \frac{5}{36} & \frac{3}{36} \\ \frac{2}{36} & -\frac{6}{36} \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

Thus,

$$x = \frac{5}{36}(4) + \frac{3}{36}(7) = \frac{20}{36} + \frac{21}{36} = \frac{41}{36}.$$

$$y = \frac{2}{36}(4) + \left(-\frac{6}{36}(7)\right) = \frac{8}{36} - \frac{42}{36} = -\frac{34}{36} = -\frac{17}{18}.$$

The solution is:

$$x = \frac{41}{36}, \quad y = -\frac{17}{18}.$$

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