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VCE Specialist Mathematics ½
Advanced Trigonometric Functions [3.4]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [3.4.1]** – Trigonometric Identities and Solving Exact Values of Reciprocal Functions
- ☐ **SM12 [3.4.2]** – Graph Reciprocal Trigonometric Functions
- ☐ **SM12 [3.4.3]** – Apply compound and Double Angle Formula to Solve Exact Values
- ☐ **SM12 [3.4.4]** – Find Domain, Range and Rule of the Inverse Trigonometric Function
- ☐ **SM12 [3.4.5]** – Graphing the Inverse Trigonometric Functions

Section A: Reciprocal Trigonometric Functions

Sub-Section: Introduction of Reciprocal Trigonometric Functions

What are Reciprocal Trigonometric Functions?

Reciprocal Trigonometric Functions

- The reciprocal of **sine** is **cosecant:**

$$\mathbf{cosec}(x) = \frac{1}{\sin(x)}$$

- The reciprocal of **cosine** is **secant:**

$$\sec(x) = \frac{1}{\cos(x)}$$

- The reciprocal of **tangent** is **cotangent:**

$$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$$

Space for Personal Notes

Question 1

Evaluate the following.

a. $\sec\left(-\frac{\pi}{3}\right)$

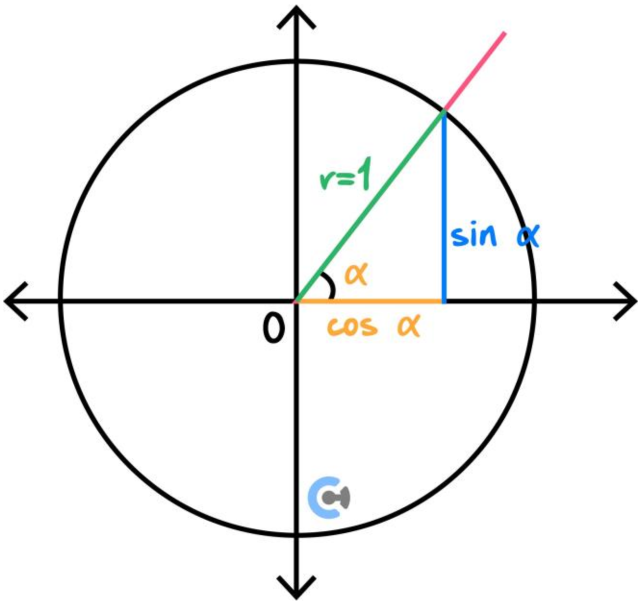
b. $\text{cosec}\left(\frac{2\pi}{3}\right)$

c. $\cot\left(-\frac{5\pi}{6}\right)$

TIP: Look at their third alphabet!

Sub-Section: Trigonometric Identities

REMINDER



- Trigonometric Identity is given by,

$$\sin^2(\theta) + \cos^2(\theta) = \_\_\_\_\_\_\_\_$$

What Happens when we Divide Both Sides by $\sin^2(\theta)$ and $\cos^2(\theta)$?

Exploration: Other Trigonometric Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

- **Divide each term by $\cos^2(\theta)$.**

$$\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_$$

- We can **divide each term by $\sin^2(\theta)$.**

$$\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_$$

Trigonometric Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \text{cosec}^2(\theta)$$

Question 2

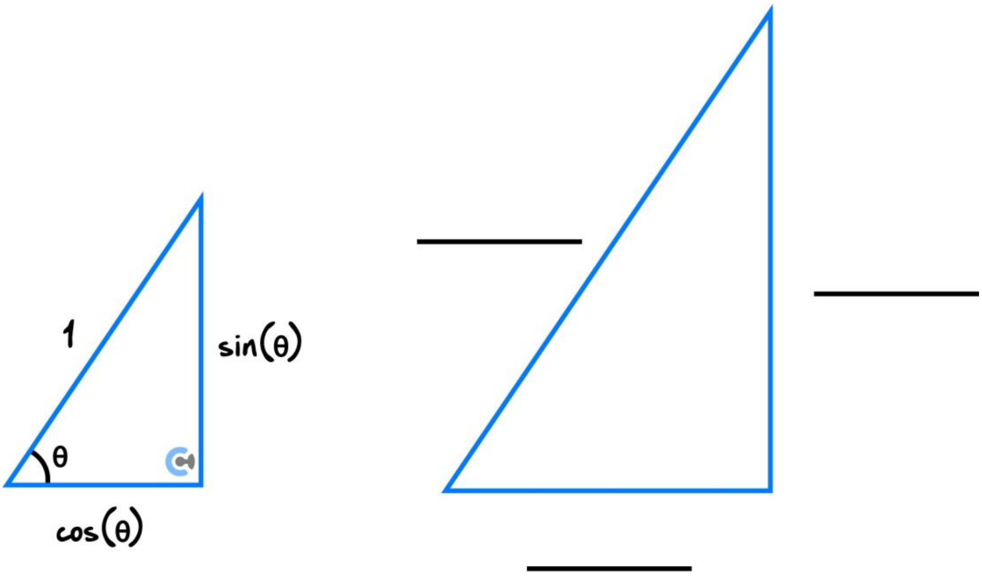
Given that $\sec(x) = -4$ and $x \in \left[\frac{\pi}{2}, \pi\right]$, find $\text{cosec}(x)$ and $\tan(x)$. Show your working.

Sub-Section: Further Understanding of Reciprocal Trigonometric Functions

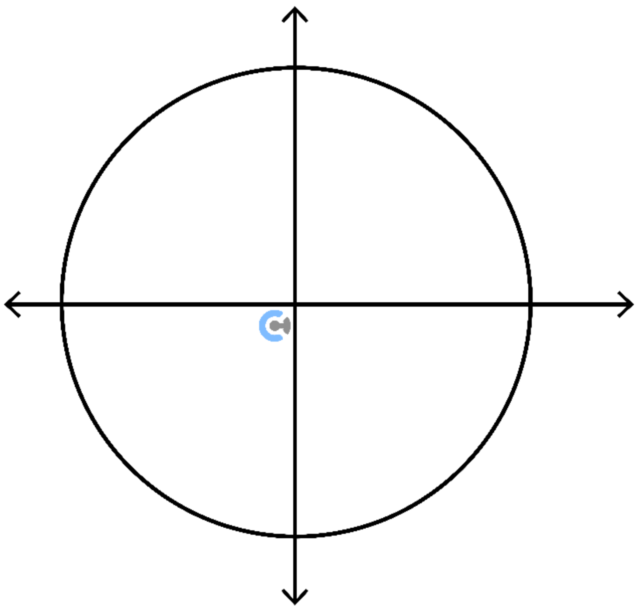
Let's use the Trigonometric Identities to visualise the Reciprocal Trig Values!

Exploration: Visualisation of $1 + \tan^2(\theta) = \sec^2(\theta)$

- Visualise $1 + \tan^2(\theta) = \sec^2(\theta)$ on the right angle triangle below!



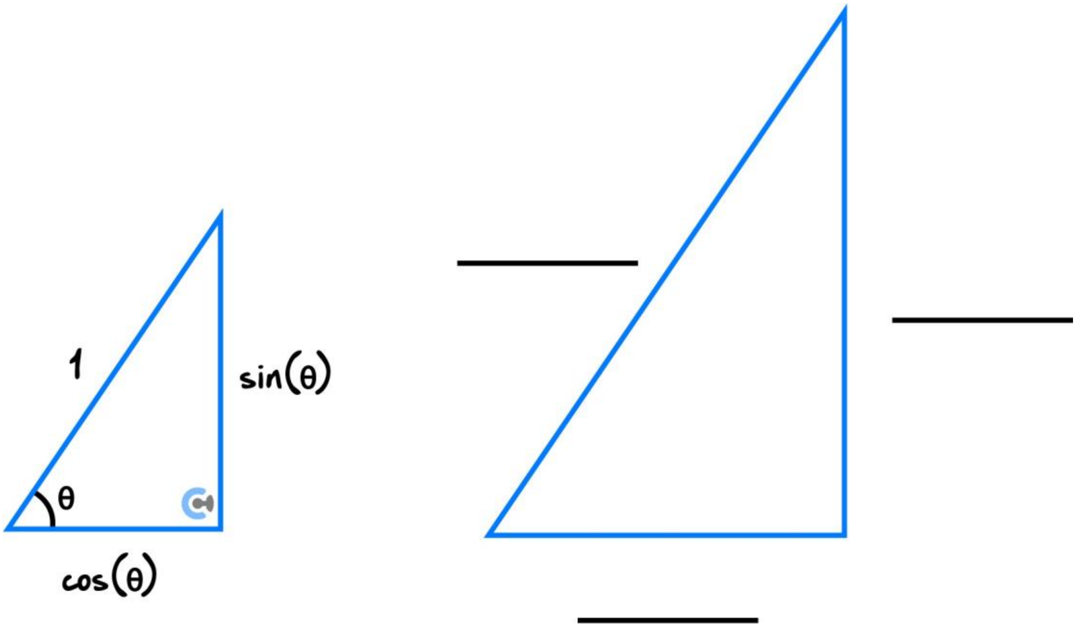
- Hence, visualise $1 + \tan^2(\theta) = \sec^2(\theta)$ on the unit circle below!



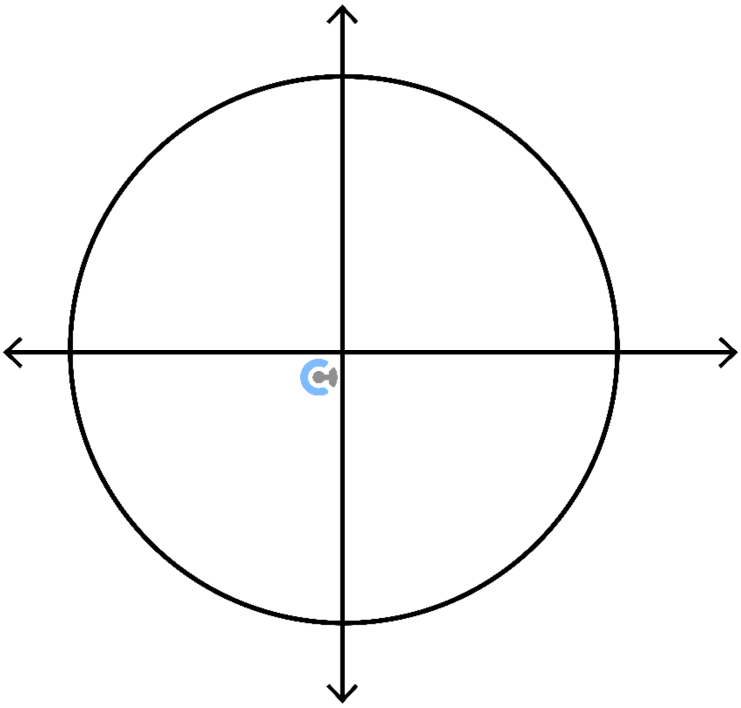
Try the Next Exploration Yourself!

Exploration: Visualisation of $1 + \cot^2(\theta) = \text{cosec}^2(\theta)$

- Visualise $1 + \cot^2(\theta) = \text{cosec}^2(\theta)$ on the right angle triangle below!



- Hence, visualise $1 + \cot^2(\theta) = \text{cosec}^2(\theta)$ on the unit circle below!



Section B: Graphs of Reciprocal Trigonometric Functions

Sub-Section: Graphs of Reciprocal Functions

Discussion: What would the graph of $\frac{1}{f(x)}$ have when the $f(x) = 0$?

Discussion: What would the graph of $\frac{1}{f(x)}$ have, when the $f(x)$ is increasing?

Discussion: What would the graph of $\frac{1}{f(x)}$ have, when the $f(x)$ is decreasing?

Properties of Reciprocal Graphs

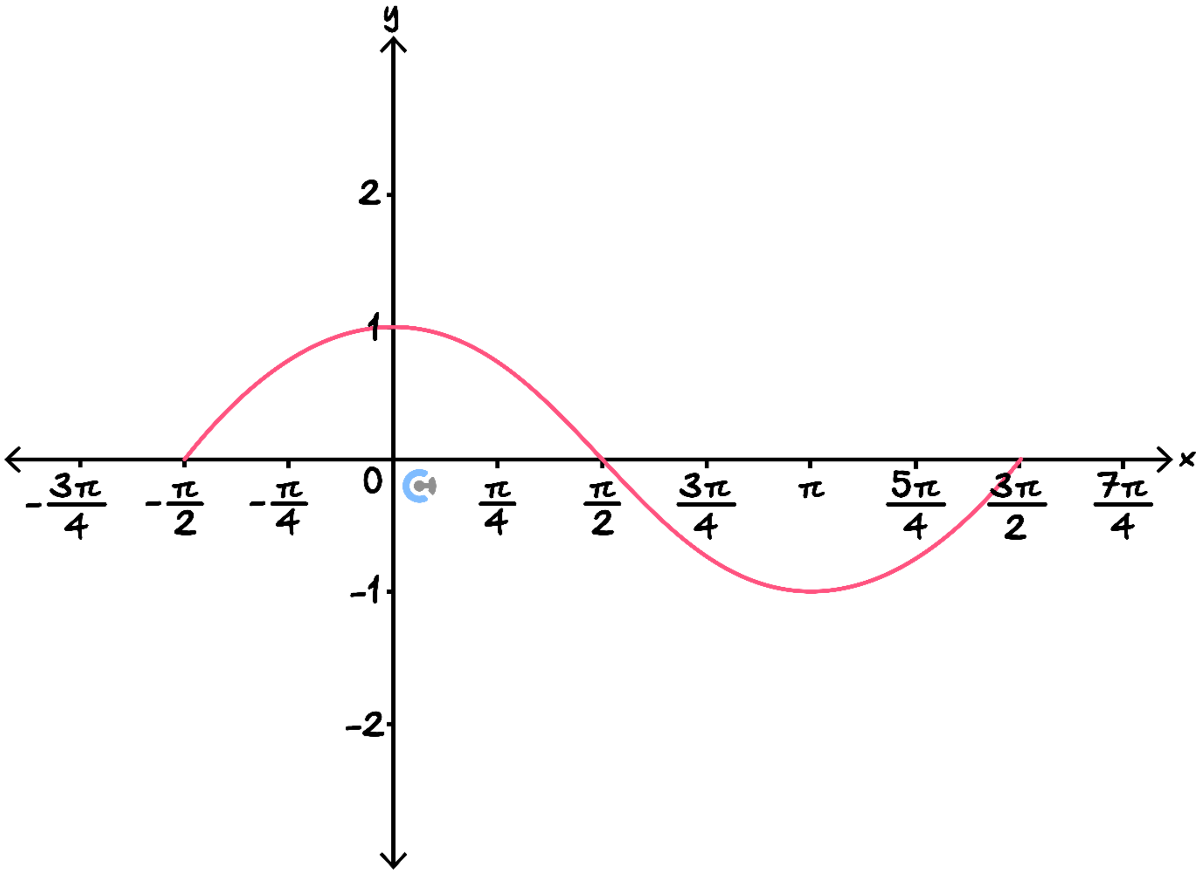
Feature on $y = f(x)$	Feature on $y = \frac{1}{f(x)}$
x-intercept	Vertical asymptote
Positive y-values	Positive y-values
Negative y-values	Negative y-values
Increasing	Decreasing
Decreasing	Increasing
The graphs intersect only when $f(x) = 1$ or $f(x) = -1$.	

Sub-Section: Graphs of Reciprocal Trigonometric Functions

Let's Try Sketching the Reciprocal Trigonometric Functions!

Question 3 Walkthrough.

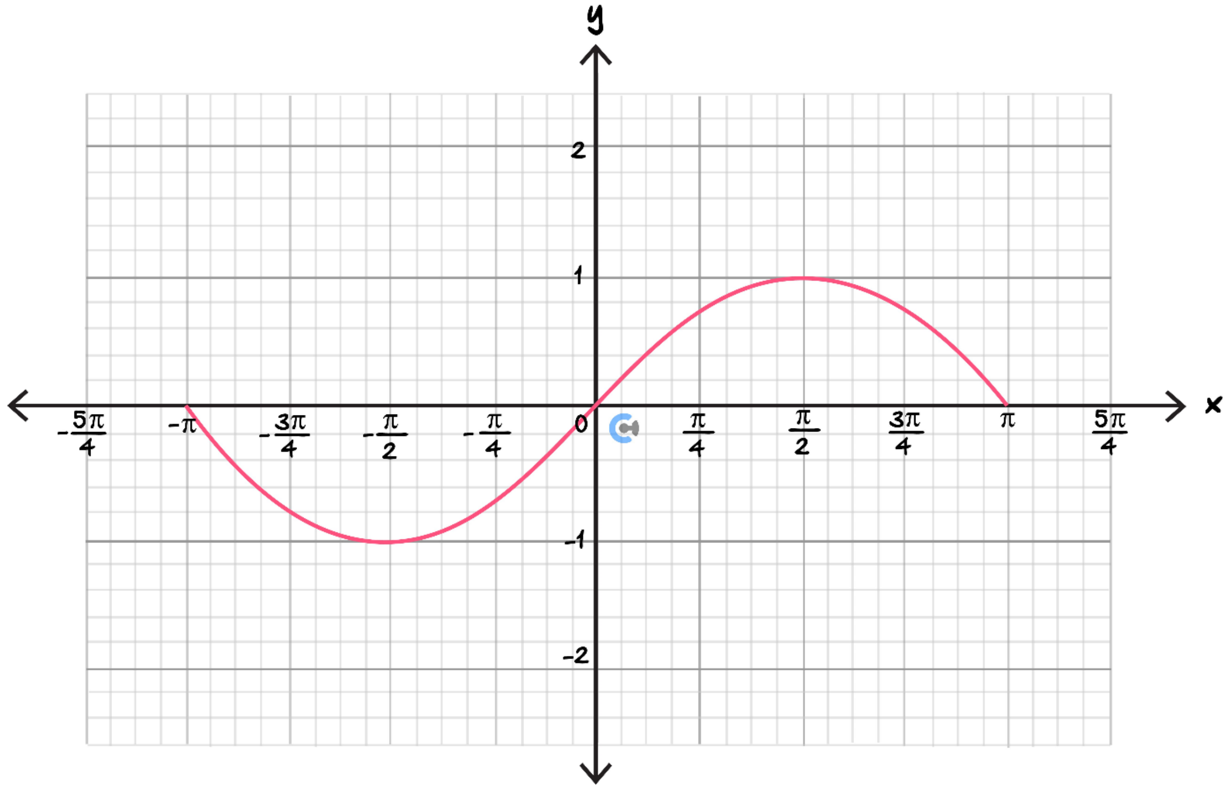
a. On the same axes below, sketch $\sec(x)$ for $x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Label all your asymptotes and turning points.



b. State the range of $\sec(x)$, and the value(s) of x where $\sec(x)$ is not defined in the given interval.

Question 4

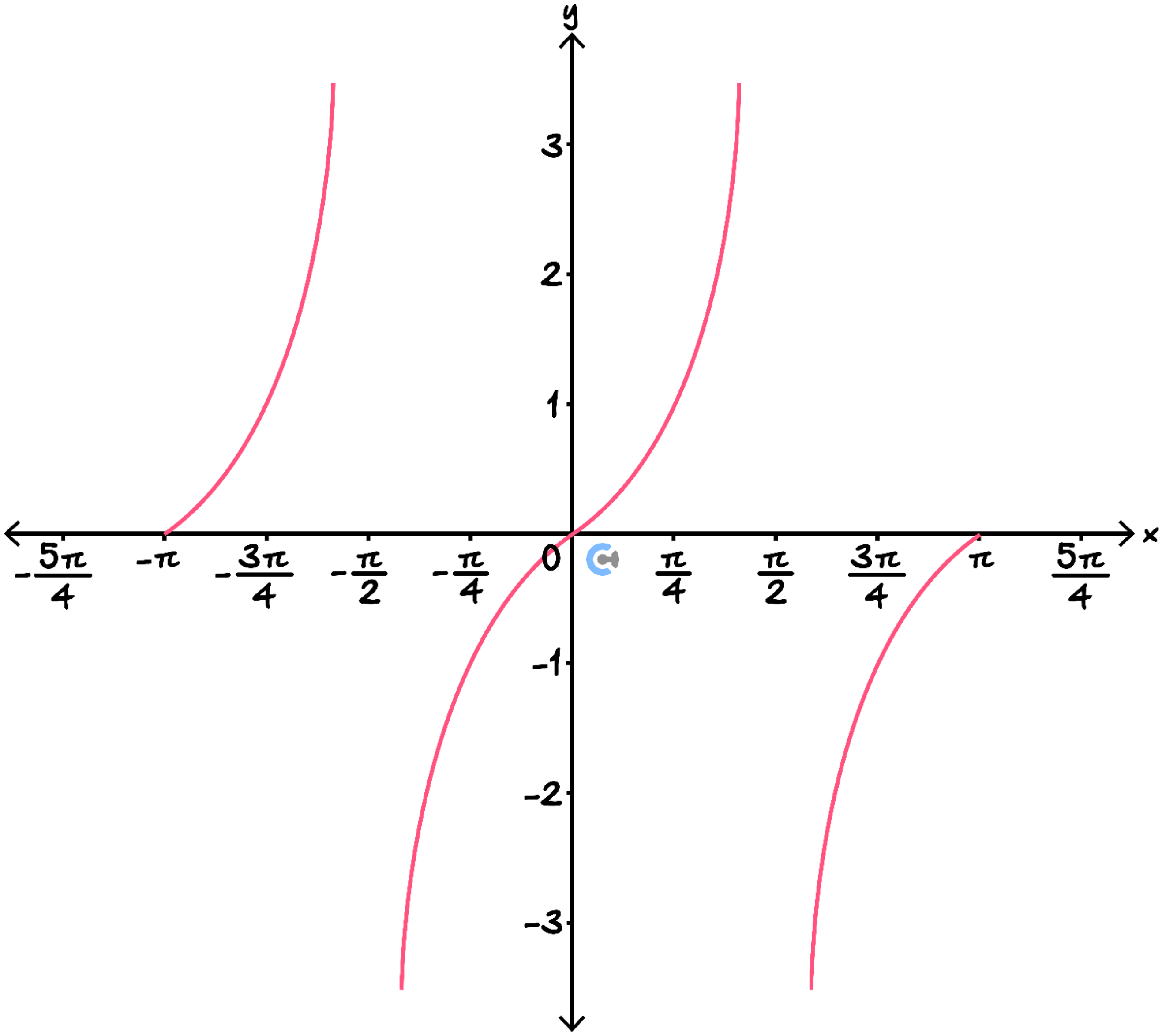
a. On the same axes below, sketch $\text{cosec}(x)$ for $x \in [-\pi, \pi]$.



b. State the range of $\text{cosec}(x)$, and the value(s) of x where $\text{cosec}(x)$ is not defined in the given interval.

Question 5 Walkthrough.

a. On the same axes below, sketch $\cot(x)$ for $x \in [-\pi, \pi]$.



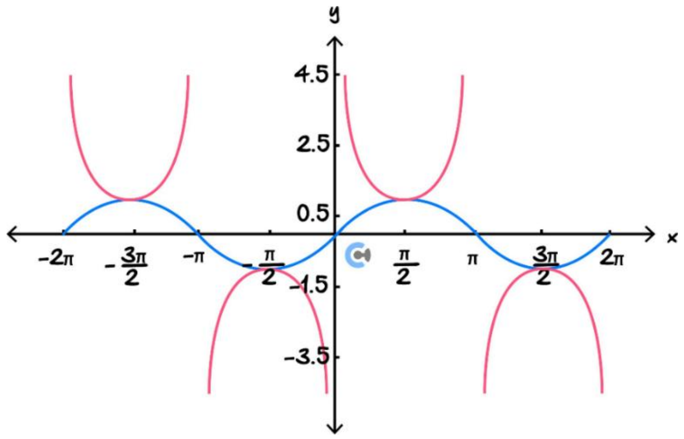
b. State the range of $\cot(x)$, and the value(s) of x where $\cot(x)$ is not defined in the given interval.

Discussion: Now, what does $\frac{1}{\tan(x)}$ graph look like?

In Summary!

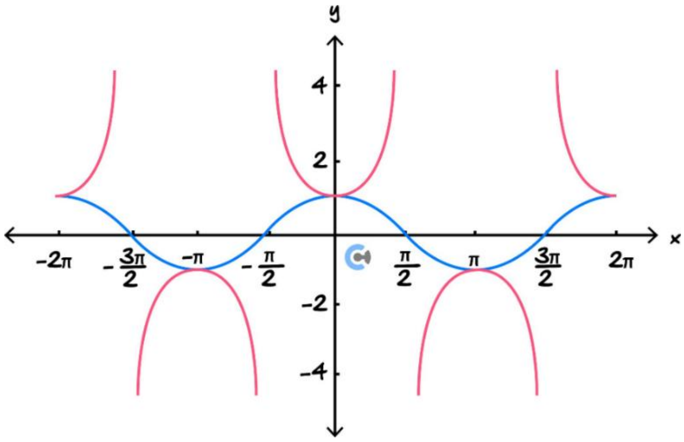
Graphing Reciprocal Trigonometric Functions

$$y = \text{cosec}(x)$$



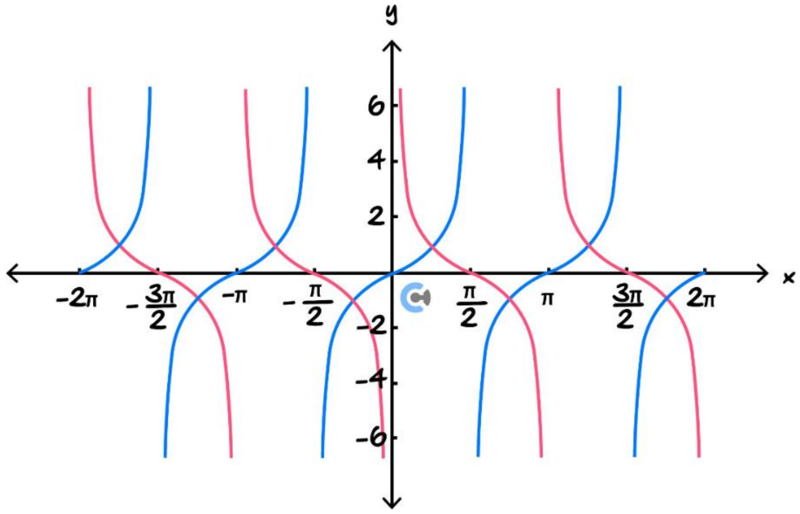
- Maximal Domain: $R\backslash\{x: \sin(x) = 0\}$.
- Range: $(-\infty, -1] \cup [1, \infty)$.

$$y = \sec(x)$$



- Maximal Domain: $R\backslash\{x: \cos(x) = 0\}$.
- Range: $(-\infty, -1] \cup [1, \infty)$.

$$y = \cot(x)$$



- Maximal Domain: $R\backslash\{x: \tan(x) = 0\}$.
- Range: R.

<u>Discussion:</u> How often do the asymptotes occur for cosec and sec?

<u>Discussion:</u> How often do the asymptotes occur for cot?

Sub-Section: Graphs of Complicated Reciprocal Trigonometric Functions

Okay, now, about do we Sketch Harder Ones with Transformations?

Steps for Sketching Reciprocal Trig Graphs

- Find **an asymptote.**

$$equate\ \boldsymbol{Angle} = \boldsymbol{0}\ for\ \text{cosec and cot}\ graphs$$

$$equate\ \boldsymbol{Angle} = \boldsymbol{\frac{\pi}{2}}\ for\ \text{sec}\ graphs$$

- Find and mark **all other asymptotes** in the domain.

$$\boldsymbol{Add/Subtract}\ \boldsymbol{\frac{\pi}{n}}\ from\ first\ asymptotes$$

- Plot a point in between the two asymptotes.

$$Midpoint = \boldsymbol{Turning\ Point}\ for\ \text{cosec and sec}\ graphs$$

$$Midpoint = \boldsymbol{Inflection\ Point}\ for\ \text{cot}\ graphs$$

- Solve for axes intercept (if applicable).
- Repeat the shape over the entire domain.
  - For cosec and sec graphs, the "U" shapes **alternate** between asymptotes, while cot graphs look the same between asymptotes.

NOTE: Don't forget to label endpoints, and/or axes-intercept(s), turning point(s) and point(s) as required!

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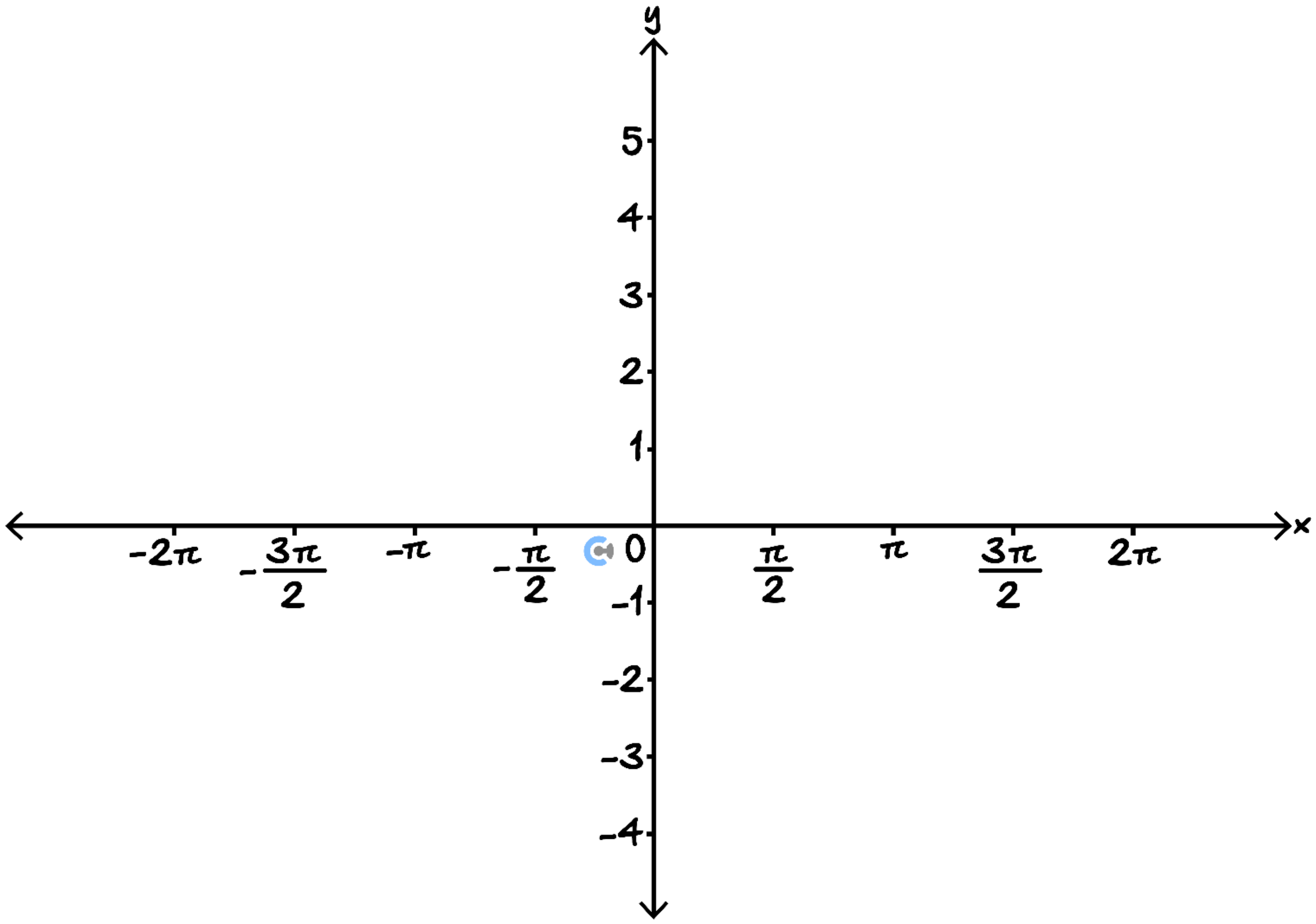
Question 6 Walkthrough.

Sketch the graph of $y = -\frac{1}{2}\text{cosec}(x) + 1$ for $-2\pi \le x \le 2\pi$, labelling all stationary points, axes-intercepts and asymptotes.



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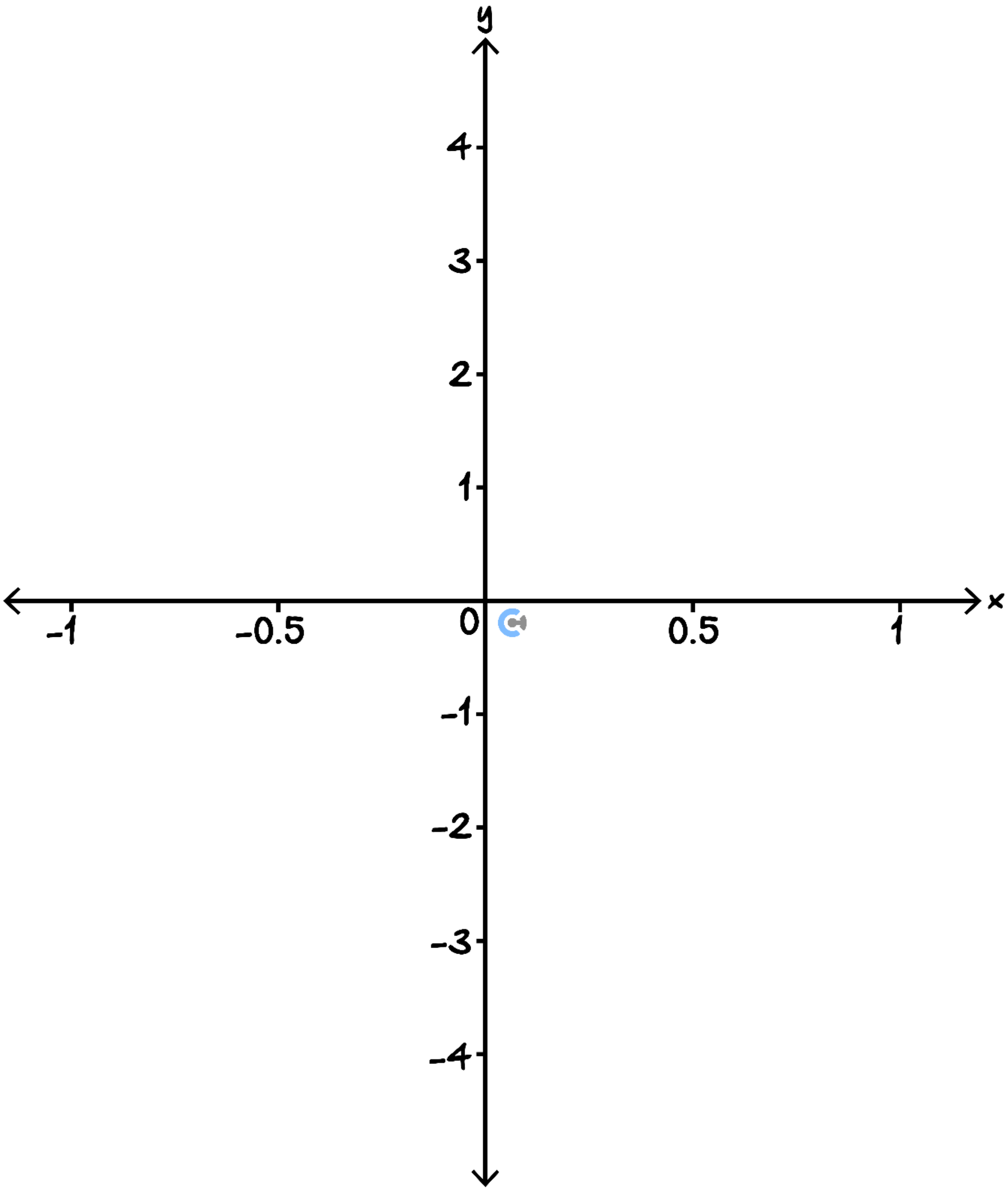
Your Turn!

Question 7

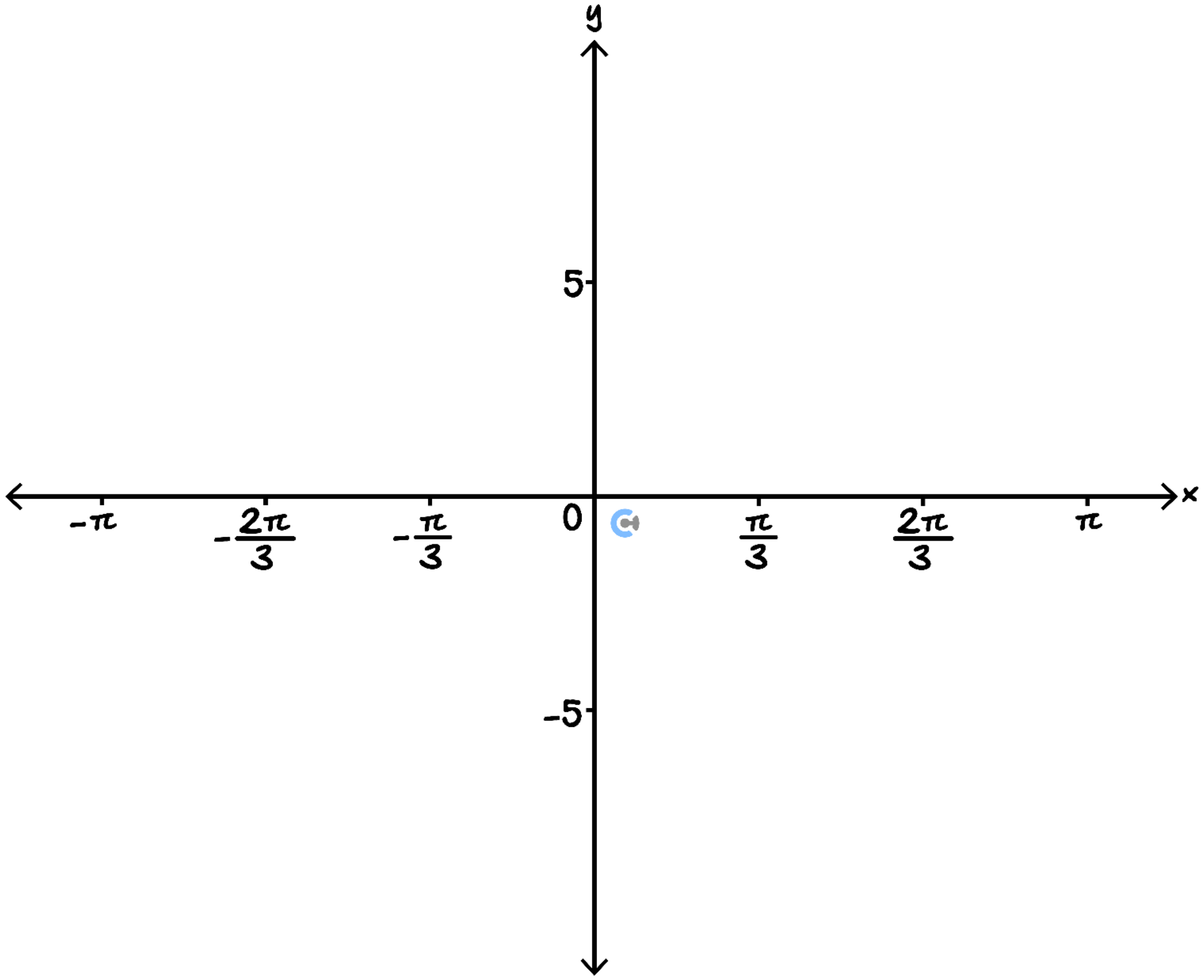
Sketch the graph of $y = 2\sec(\pi(2x - 1))$ for $-1 \leq x \leq 1$, labelling all stationary points, axes-intercepts and asymptotes.



Now **Cot!** *Remember they have an Inflection Instead of Turning Points!*

Question 8

Sketch the graph of $y = 1 - \cot\left(x + \frac{\pi}{2}\right)$ for $-\pi \leq x \leq \pi$, labelling all stationary points, axes-intercepts and asymptotes.



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Section C: Compound and Double Angle Formula

Sub-Section: Compound Angle Formula

Let's look at the compound angle formula!

Compound Angle Formula

- $\sin$ compound angle formulae.

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

- $\cos$ compound angle formulae.

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

- $\tan$ compound angle formulae.

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$

$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

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Question 9 Walkthrough.

Using compound angle formula, evaluate $\sin\left(\frac{-\pi}{12}\right)$.

Question 10

Using compound angle formula, evaluate $\cos\left(\frac{5\pi}{12}\right)$.

Sub-Section: Double Angle Formula

What do We Get if x and y Were the Same for the Compound Angle Formula?

Double Angle Formulae

- $\sin$ double angle formula.

$$\sin(2x) = 2\sin(x)\cos(x)$$

- $\cos$ double angle formula.

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 2\cos^2(x) - 1$$

$$= 1 - 2\sin^2(x)$$

- $\tan$ double angle formula.

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

Question 11 Walkthrough.

Find $\sin(2t)$, where $\cos(t) = -\frac{1}{8}$.

Question 12

Find $\cos(2t)$, where $\sin(t) = -\frac{1}{8}$.

Calculator Commands: Expanding Trigonometric Identities

Mathematica	TI-Nspire	Casio Classpad
"$TrigExpand$"	"$texpand$"	"$texpand$"

Question 13 Tech-Active.

Expand $\sin(2x + y)$ in terms of x and y.

Section D: Inverse Trigonometric Functions

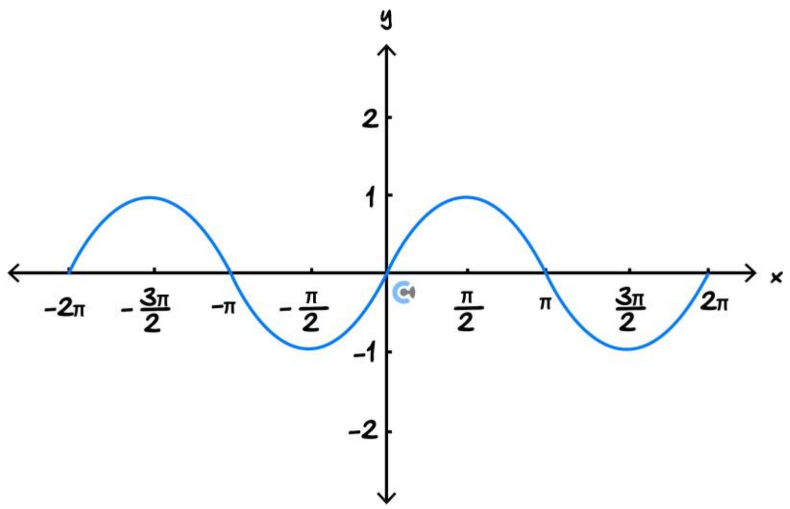
Sub-Section: Inversing Trigonometric Functions

Discussion

- What does the original function need to be for the inverse function to exist?

Question 14 Walkthrough.

Consider the function $\sin(x)$ sketched on the axes below.



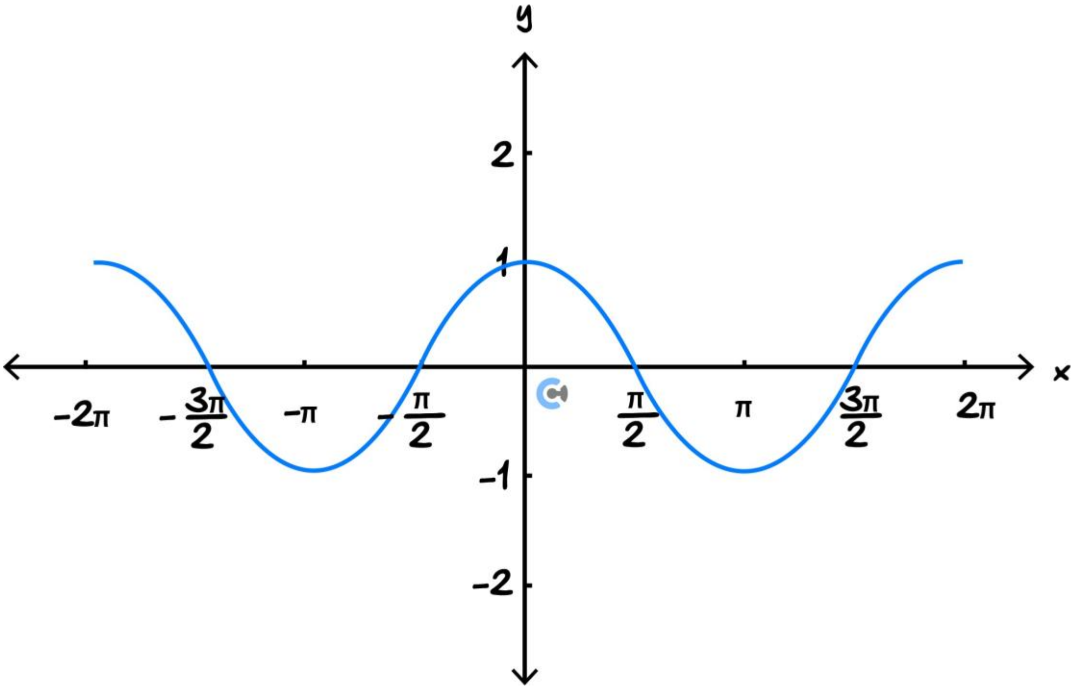
a. Shade the part of the graph such that the $\sin(x)$ is $1:1$.

b. State the domain and range of $\sin(x)$ such that the $\sin^{-1}(x)$ exists.

c. Hence, state the domain and range of $\sin^{-1}(x)$.

Question 15 Walkthrough.

Consider the function $\cos(x)$ sketched on the axes below.



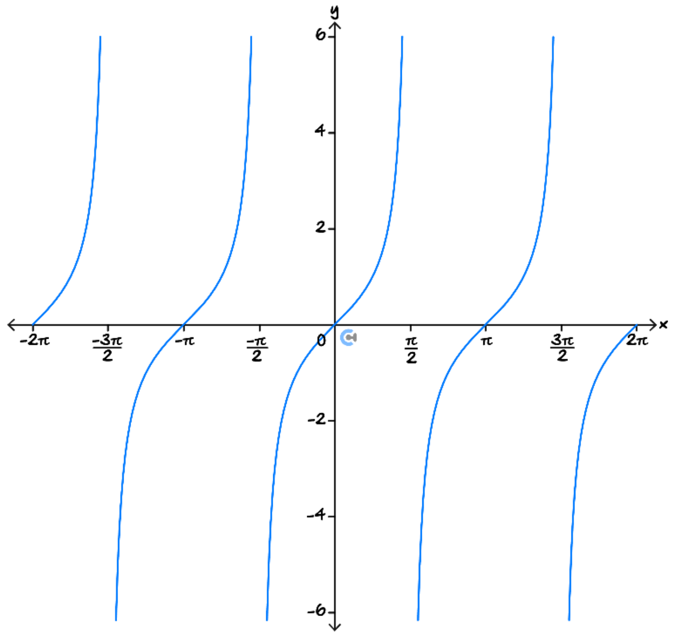
a. Shade the part of the graph such that the $\cos(x)$ is $1:1$.

b. State the domain and range of $\cos(x)$, such that the $\cos^{-1}(x)$ exists.

c. Hence, state the domain and range of $\cos^{-1}(x)$.

Question 16 Walkthrough.

Consider the function $\tan(x)$ sketched on the axes below.



a. Shade the part of the graph such that the $\tan(x)$ is $1:1$.

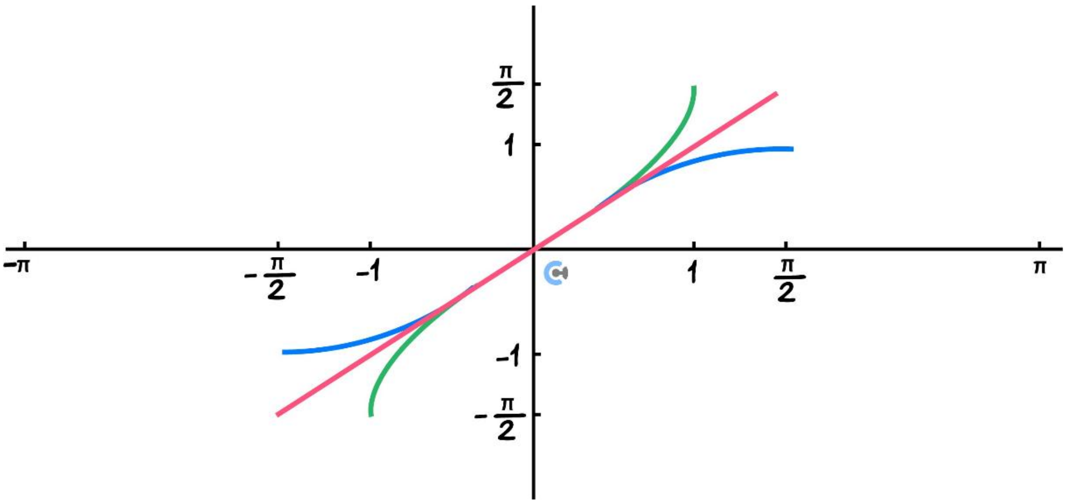
b. State the domain and range of $\tan(x)$ such that the $\tan^{-1}(x)$ exists.

c. Hence, state the domain and range of $\tan^{-1}(x)$.

In Summary!

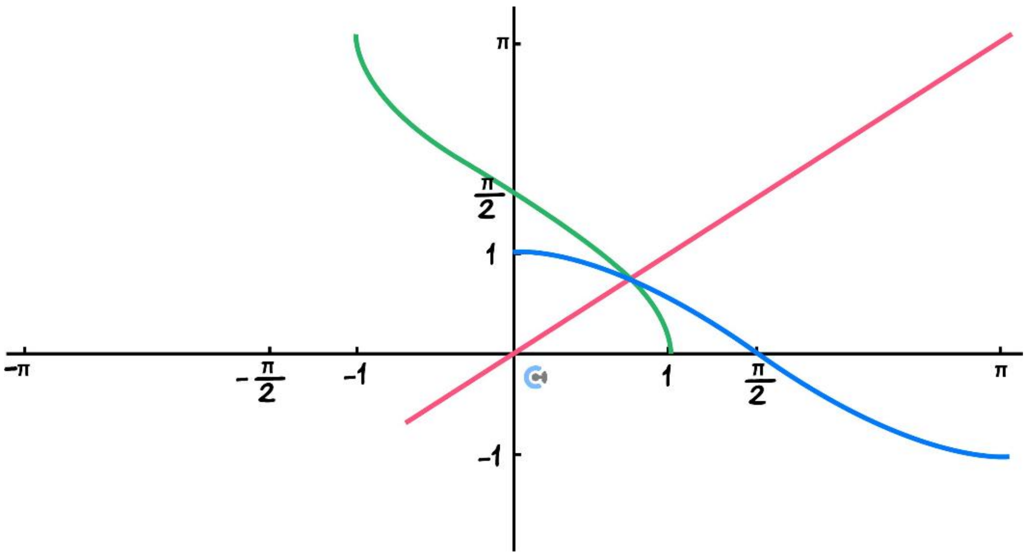
Inverse Trig Functions

- $\sin^{-1}(x)$



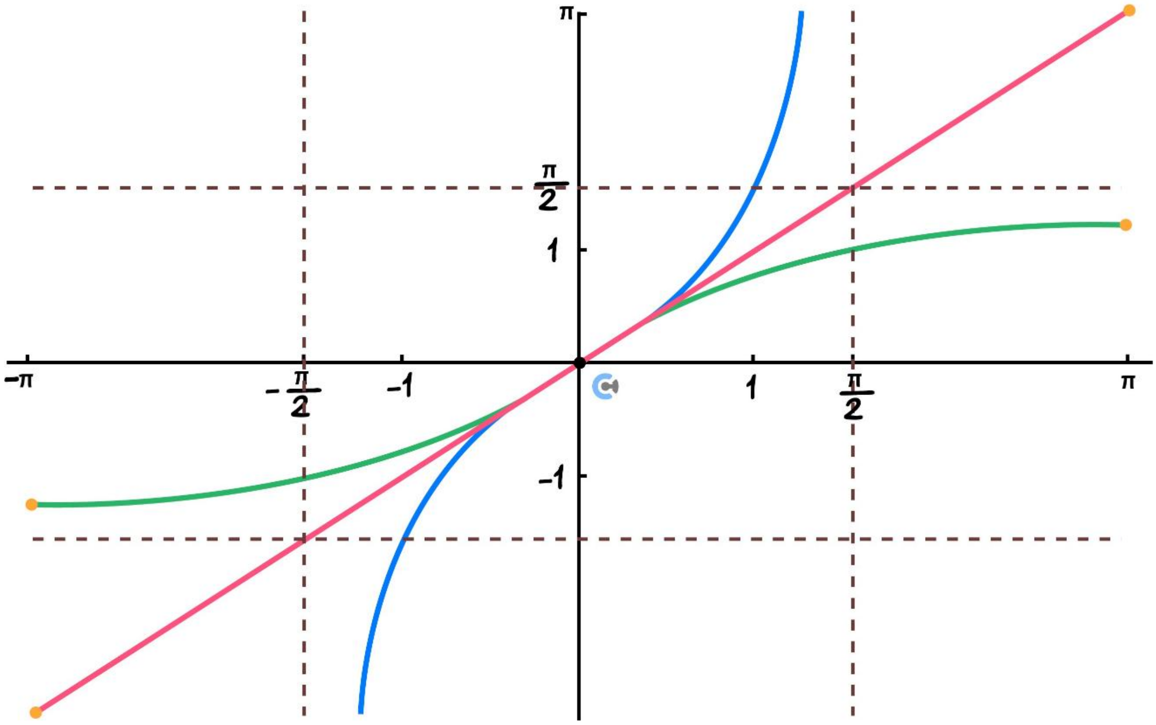
- The domain of the $\arcsin$ function = Range of $\sin = [-1,1]$.
- The range = Domain of restricted $\sin = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- $\cos^{-1}(x)$



- The domain of the $\arccos$ function = Range of $\cos = [-1,1]$.
- The range = Domain of restricted $\cos = [0, \pi]$.

- $\tan^{-1}(x)$



- The domain of the $\arctan$ function = Range of $\tan = R$.
- The range = Domain of restricted $\tan = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

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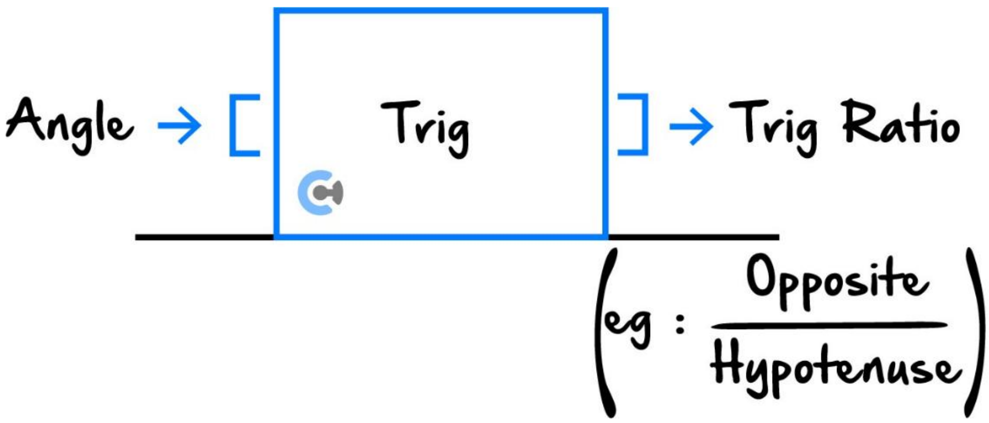
Sub-Section: Understanding Inverse Trigonometric Functions

What do Inverse Trigonometric Functions Do?

Exploration: Understanding Inverse Trig Functions

- We can consider the normal trigonometric function to be the following:



- Hence, inverse trigonometric functions can be visualised to perform the following:

- In summary, inverse trig functions have:
  - x-value: ________________ $=$ y-value original trig functions.
  - y-value: ________________ $=$ x-value of original trig functions.

Space for Personal Notes

Question 17

Evaluate the following, or explain why they are undefined.

a. $\arcsin\left(-\frac{1}{2}\right)$

b. $\arccos\left(\frac{3}{\sqrt{3}}\right)$

c. $\arctan\left(\frac{1}{\sqrt{3}}\right)$

NOTE: Inverse functions are **angles.**

NOTE: Consider the range of the inverse trig functions!

Sub-Section: Graphs of Inverse Trigonometric Functions

How do we Sketch Inverse Trigonometric Functions with Transformations?

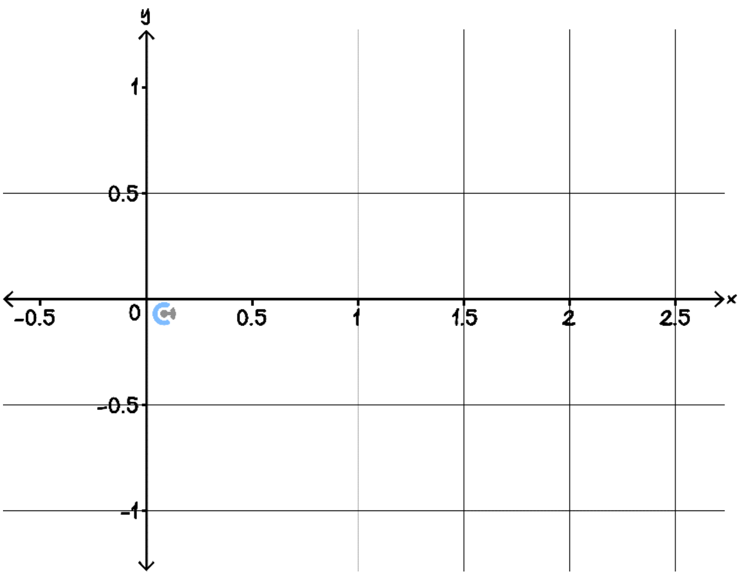
Steps for Graphing General Arcsin and Arccos

- Find the ____________________ of the function.
  - Restrict **inside** to be within ______________.
- Find and plot the ____________________ of the graph by substituting ends of the domain.
- Find and plot the ______________ of the ends. (It is an inflection point.)
- Find and plot the ____________________ if required.
- Using the previously plotted points as a guide, sketch a ______________.

Question 18 Walkthrough.

Following the steps above, sketch:

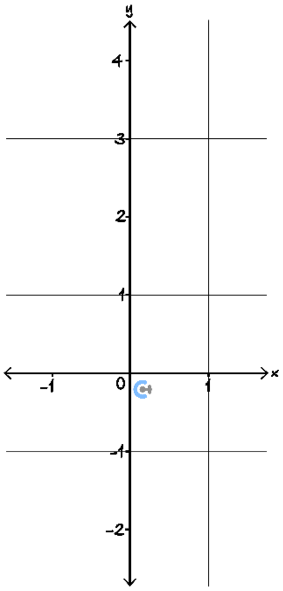
$$y = \frac{1}{2}\arccos(1 - x) - \frac{\pi}{4}$$



Question 19

Following the steps above, sketch:

$$y = \frac{\pi}{4} - 2\arcsin x$$



Now arctan *Functions!*

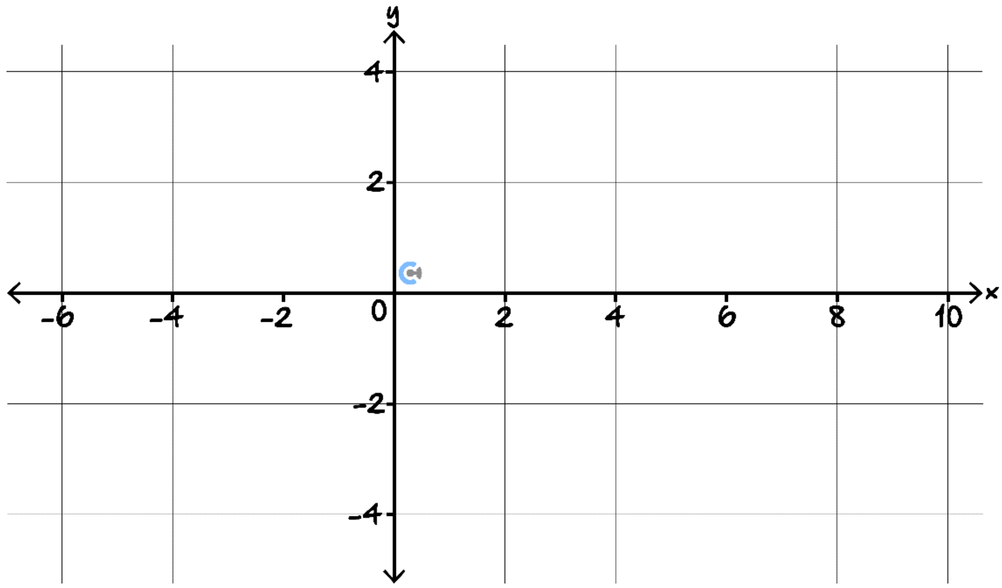
Steps for Graphing General Graphs of arctan

- Find the ______________________________ of the graph and plot them.
  - You can find the asymptotes by finding the **range** of the arctan function.
  - E.g., the range of $\arctan(x) + \pi$ is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, so the asymptotes are $y = \frac{\pi}{2}$ and $y = \frac{3\pi}{2}$.
- Inflection point is given by (h, k).
  - The x-value can be found by ______________________________.
  - The y-value can be found by ______________________________.
- Find and plot the ______________________________ if required.
- Using the previously plotted points and asymptotes as a guide, sketch the function.

Question 20 Walkthrough.

Following the steps above, sketch:

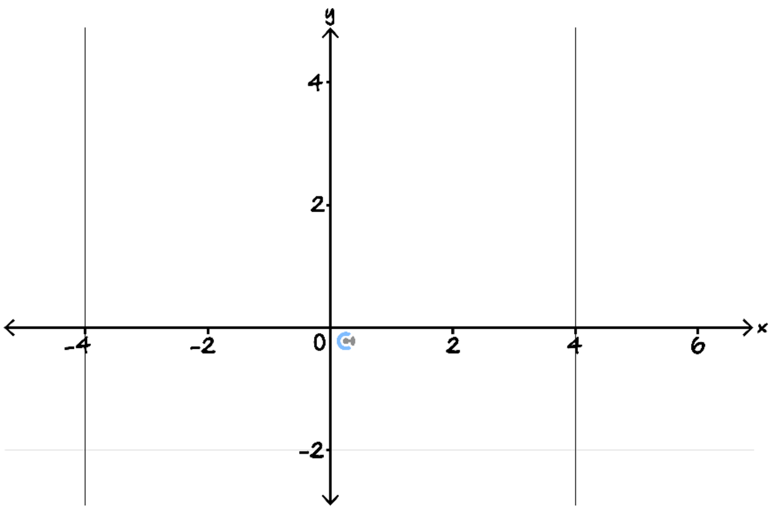
$$y = \arctan\left(x - \sqrt{3}\right) - \frac{\pi}{3}$$



NOTE: $\tan^{-1}(2) > \tan^{-1}(1) = \frac{\pi}{4}$.

Question 21 (3 marks)

Consider the function $y = \arctan(2(x-1)) + \frac{\pi}{2}$. Sketch its graph.



Contour Check

☐ Learning Objective: [3.4.1] – Trigonometric Identities and Solving Exact Values of Reciprocal Functions

Key Takeaways

$$\sin^2(\theta) + \cos^2(\theta) = \_\_\_\_\_\_$$

$$1 + \tan^2(\theta) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$1 + \cot^2(\theta) = \_\_\_\_\_\_\_\_\_\_\_\_$$

☐ Learning Objective: [3.4.2] – Graph Reciprocal Trigonometric Functions

Key Takeaways

- ☐ If the inside of a reciprocal trig function is nx, it has asymptotes every __________ units.
- ☐ cosec and $\sec$ have the same [shape] / [flip] after every asymptote.
- ☐ $\cot$ has the [same shape] / [flips] after every asymptote.

Learning Objective: [3.4.3] - Apply compound and Double Angle Formula to Solve Exact Values

Key Takeaways

- To find $\cos\left(\frac{\pi}{8}\right)$, it is more appropriate to use the [double] / [compound] angle formula.
- To find $\sin\left(\frac{7\pi}{12}\right)$, it is more appropriate to use the [double] / [compound] angle formula.

- **Sin double angle formula:**

$$\sin(2x) = 2\_\_\_\_\_\_\_\_\_\_\_\_$$

- **Cos double angle formula:**

$$\cos(2x) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_$$

- **Tan double angle formula:**

$$\tan(2x) = \_\_\_\_\_\_\_\_\_\_\_\_$$

☐ Learning Objective: [3.4.4] – Find Domain, Range and Rule of the Inverse Trigonometric Function

Key Takeaways

- ☐ $\cos^{-1}(x)$ and $\sin^{-1}(x)$ have the domain ______________.
- ☐ $\tan^{-1}(x)$ has the domain ______________________________.
- ☐ $\tan^{-1}(x)$ has an inflection point at ______________.
- ☐ Inverse trig functions output an ________________.
- ☐ To get the inverse of a trig function, we restrict the function's domain so that it is _____________.

☐ Learning Objective: [3.4.5] – Graphing the Inverse Trigonometric Functions

Key Takeaways

- ☐ $\mathbf{tan^{-1}(x)}$ has [horizontal] / [vertical] asymptotes at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- ☐ An untransformed $\sin^{-1}$ graph has a point of inflection at ____________.

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