CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Specialist Mathematics ½
Trigonometry II [3.2]
Workbook

Outline:

Learning Objectives:

- **SM12 [3.2.1]** – Find Trig Ratios of Supplementary Relationships
- **SM12 [3.2.2]** – Find Particular and General Solutions
- **SM12 [3.2.3]** – Graph Sine, Cosine and Tangent functions

Section A: Introduction to Circular Functions

Sub-Section: Radians and Degrees

Radians and Degrees

$$1^c = \left(\frac{180}{\pi}\right)^o$$

$$1^o = \left(\frac{\pi}{180}\right)^c$$

$$180^o = \pi^c$$

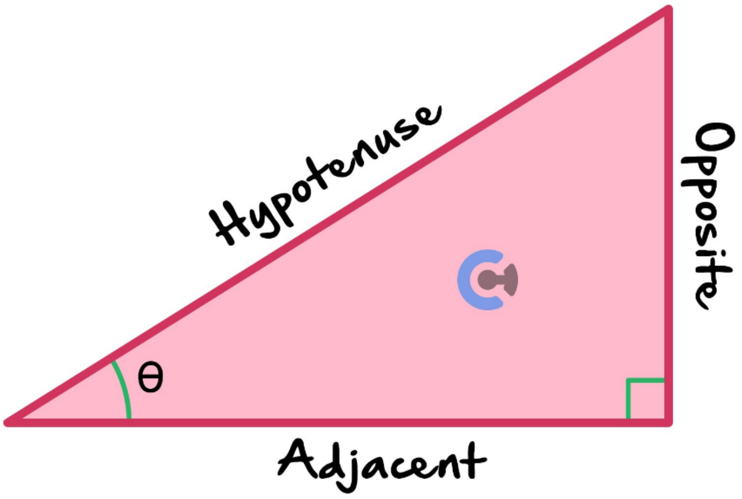
Question 1

a. Find $\left(\frac{\pi}{4}\right)^c$ in degrees.

b. Find 12^o in radians.

Sub-Section: Unit Circle

Active Recall



$\sin =$ ______________________

$\cos =$ ______________________

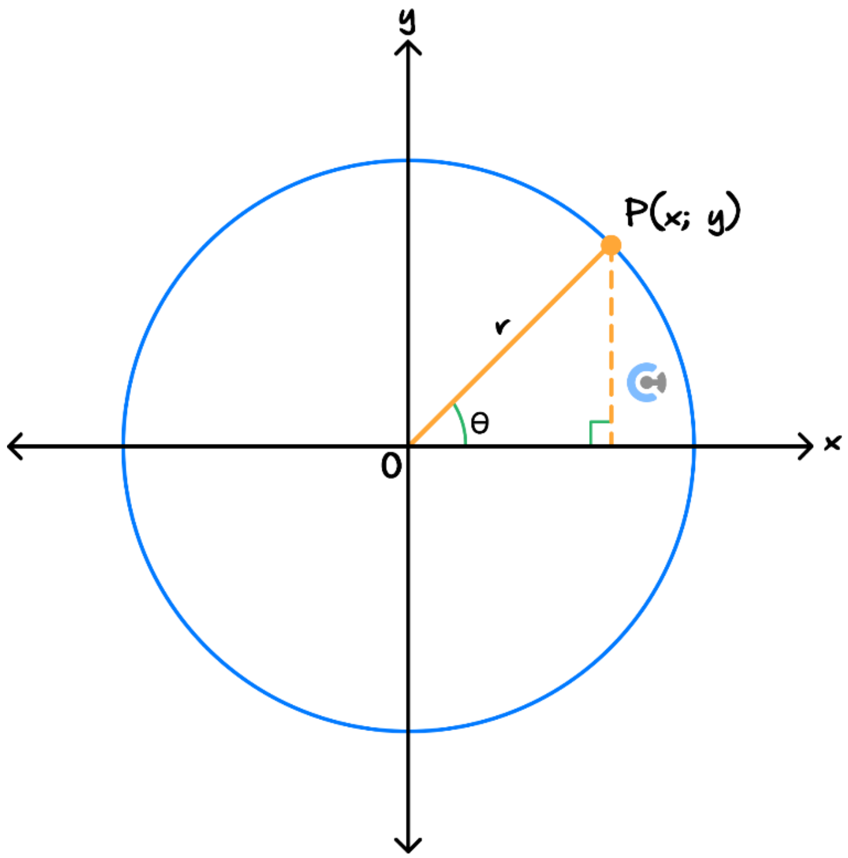
$\tan =$ ______________________

Space for Personal Notes

What is a unit circle, and how do we use it?

Exploration: Unit Circle

- The unit circle is simply a circle of radius _____.
- Angles are measured from the ____.
- It can be divided into **four quadrants**:



- We can use the elementary definition of the trigonometric functions.
- Select the option below!

$$\sin(\theta) = [X\ Value, Y\ Value\ , Gradient]$$

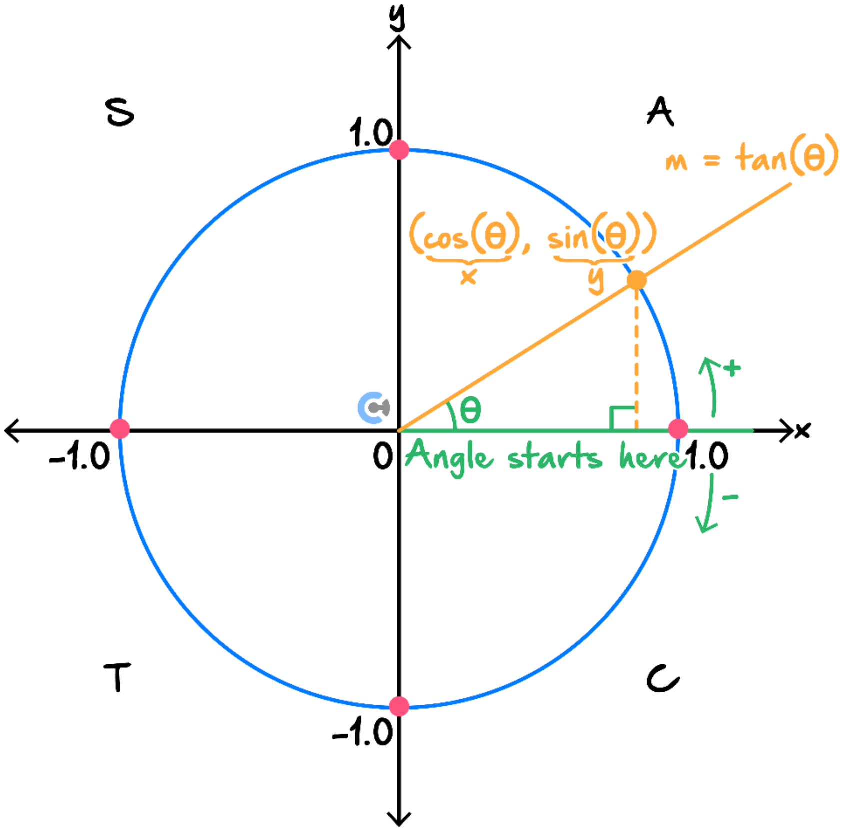
$$\cos(\theta) = [X\ Value, Y\ Value, Gradient]$$

$$\tan(\theta) = [X\ Value, Y\ Value, Gradient]$$

Definition

<u>Unit Circle</u>

- The unit circle is simply a circle of radius 1.



$$\sin(\theta) = y$$

$$\cos(\theta) = x$$

$$\tan(\theta) = \text{gradient}$$

<u>Discussion:</u> For which quadrant is $\cos$, $\sin$ and tangent positive?

Sub-Section: Period

Discussion: For what angle does $\cos$, $\sin$ and tangent repeats itself?

Period of a Trigonometric Function

$$\text{period of } \sin(nx) \text{ and } \cos(nx) \text{ functions} = \frac{2\pi}{|n|}$$

$$\text{period of } \tan(nx) \text{ functions} = \frac{\pi}{|n|}$$

$$\text{where } n = \text{coefficient of } x.$$

Question 2

Find the period of each of the following trigonometric functions:

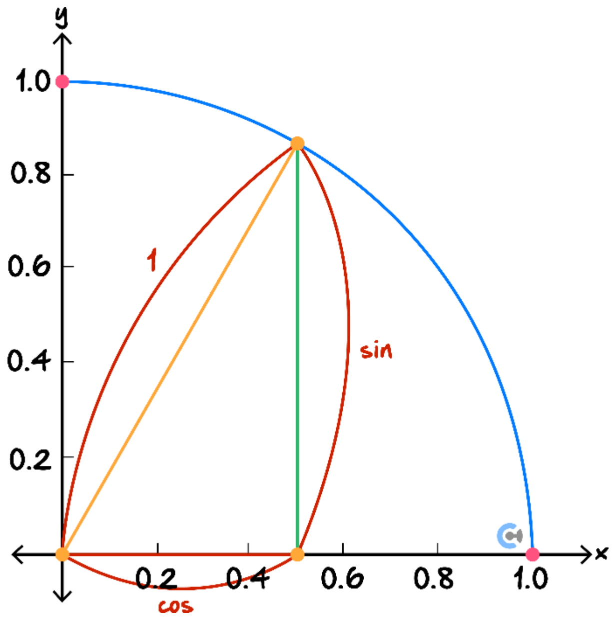
a. $p(x) = \tan(2x)$

b. $q(x) = \cos\left(\frac{5}{2}x + \frac{\pi}{3}\right)$

Sub-Section: Pythagorean Identities

Discussion: What does $\sin^2(\theta) + \cos^2(\theta)$ equal to?

Pythagorean Identities



$$\sin^2(\theta) + \cos^2(\theta) = 1$$

- Can be used for finding one trigonometry function by using the other.

Space for Personal Notes

How can we use it?

Question 3 Walkthrough.

Find the value of $\sin(x)$ given that $\cos(x) = \frac{1}{4}$ and x is the first quadrant.

NOTE: Always show the rejection by the quadrant.

Question 4

Find the value of $\cos(x)$ given that $\sin(x) = \frac{1}{3}$ and x is the second quadrant.

Sub-Section: Exact Values

Exact values are super important to remember!

The Exact Values Table

x	$0\ (0^o)$	$\frac{\pi}{6}\ (30^o)$	$\frac{\pi}{4}\ (45^o)$	$\frac{\pi}{3}\ (60^o)$	$\frac{\pi}{2}\ (90^o)$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

TIP: Use the fact that $\sin$ is the y value, $\cos$ is the x value, and the tangent is the gradient to remember the values well!

Discussion: With your classmate next to you, test each other on the exact value table!

Discussion: The exact value table only has first-quadrant angles! How do we evaluate other quadrants?

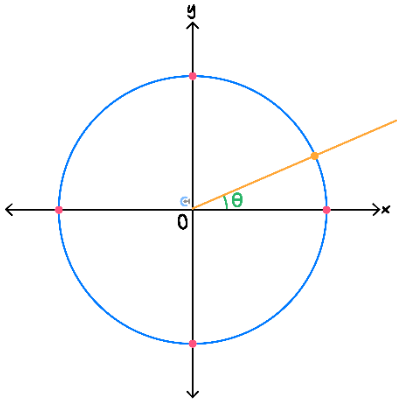
Section B: Symmetry

Sub-Section: Supplementary Relationships

What does reflection in the y-axis look like?

Exploration: Reflection in y-axis

- Consider the unit circle.



- Reflect the angle around the y-axis on the unit circle above.
- What is the angle in terms of θ?

Question 5

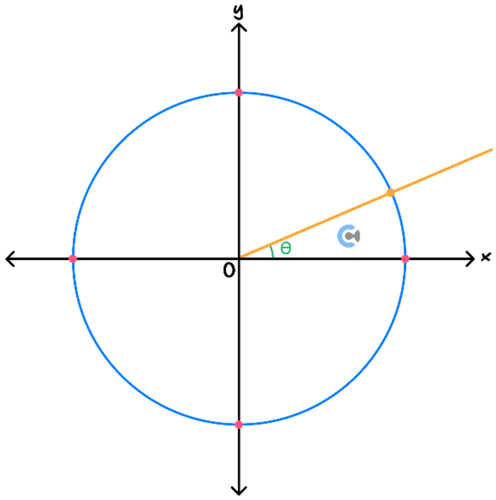
Consider the angle $\frac{\pi}{6}$.

Find the angle after the reflection in the y-axis.

What does a reflection in the x-axis look like?

Exploration: Reflection in x-axis

- Consider the unit circle.



- Reflect the angle around the x-axis on the unit circle above.
- What is the angle in terms of θ?

Question 6

Consider the angle $\frac{\pi}{6}$.

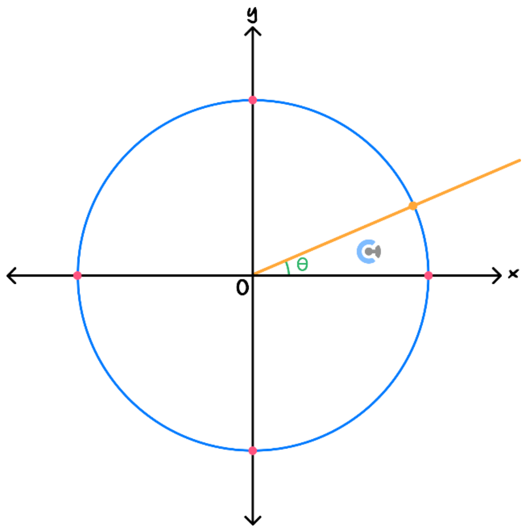
Find the angle after the reflection in the x-axis.

NOTE: Simply make the angle negative!

What does reflection in both axes look like?

Exploration: Reflection on Both Axes

- Consider the unit circle.



- Reflect the angle around both axes on the unit circle above.
- What is the angle in terms of θ?

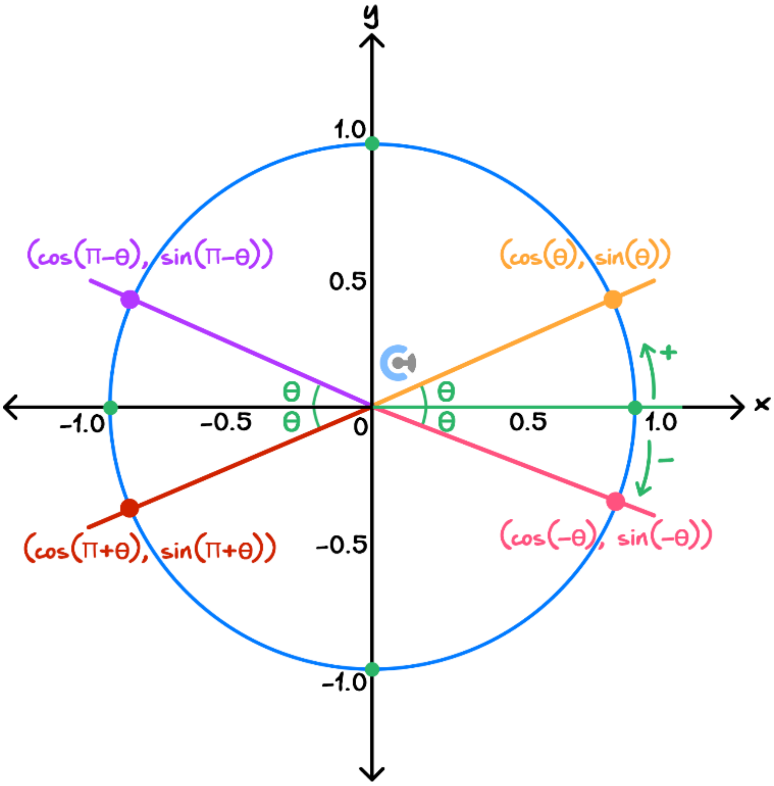
Question 7

Consider the angle $\frac{\pi}{6}$.

Find the angle after the reflection in both axes.

Let's summarise!

Supplementary Relationships



- Simply look at the quadrant to find the correct sign.
  - Second Quadrant $(\pi - \theta)$:

$$\cos(\pi - \theta) = -\cos(\theta)$$

$$\sin(\pi - \theta) = +\sin(\theta)$$

$$\tan(\pi - \theta) = -\tan(\theta)$$

  - Third Quadrant $(\pi + \theta)$:

$$\cos(\pi + \theta) = -\cos(\theta)$$

$$\sin(\pi + \theta) = -\sin(\theta)$$

$$\tan(\pi + \theta) = +\tan(\theta)$$

Fourth Quadrant $(-\theta)$:

$$\cos(-\theta) = +\cos(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

Try the following question!

Question 8

If $\sin(\theta) = -0.6$ where θ is a third quadrant angle, evaluate the following.

a. $\sin(\pi + \theta)$

b. $\cos(\pi + \theta)$

c. $\tan(\pi - \theta)$

NOTE: The aim of the question is to convert the angle to theta!

Section C: Particular and General Solutions

Sub-Section: Particular Solutions

Active Recall: Period of Trigonometric Function

$$\text{period of } \sin(nx) \text{ and } \cos(nx) \text{ functions} = \_\_\_\_\_\_\_\_\_\_$$

$$\text{period of } \tan(nx) \text{ functions} = \_\_\_\_\_\_\_\_\_\_$$

$$\text{where } n = \text{coefficient of } x.$$

Discussion: How often would the solution to $\sin(x) = \frac{1}{2}$ repeat?

Particular Solutions

- Solving trigonometric equations **for finite solutions.**
- **Steps:**
  - Make the trigonometric function the subject.
  - Find the necessary angle for one period.
  - Solve for x by equating the necessary angles to the inside of the trigonometric functions.
  - Add and subtract the period to find all other solutions in the domain.

Question 9 Walkthrough.

Solve the following equations for x over the domains specified.

$$2\sin(2x+\pi)-\sqrt{3}=0 \text{ for } x\in[0,2\pi]$$

Question 10

Solve the following equations for x over the domains specified.

a. $\sin\left(x-\frac{\pi}{2}\right)=-\frac{1}{2}$ for $x\in[-\pi,\pi]$

b. $2\cos\left(2x+\frac{\pi}{6}\right)+1=0$ for $x\in[0,2\pi]$

Question 11 Walkthrough.

Solve the following equations for x over the domains specified.

$$\tan\left(x + \frac{\pi}{3}\right) - \sqrt{3} = 0 \text{ for } x \in [0, 2\pi]$$

<u>Discussion:</u> Why do we need to find one angle only for tangents?

Question 12

Solve the following equations for x over the domains specified.

$$\sqrt{3}\tan\left(x+\frac{\pi}{4}\right)+1=0 \text{ for } x \in (0, 3\pi)$$

Sub-Section: General Solutions

Discussion: How many solutions would there be for $x \in R$?

General Solutions

- Finding ______________ solutions to a trigonometric equation.
- **Steps:**
  - Make the trigonometric function the subject.
  - Find the necessary angle for one period.
  - Solve for x by equating the necessary angles to the inside of the trigonometric functions.
  - Add $\text{Period} \cdot n$ where $n \in Z$.

Space for Personal Notes

Question 13 Walkthrough.

Find the general solutions to the following equations:

$$2\sin\left(2x+\frac{\pi}{2}\right)-1=0$$

NOTE: The steps are exactly the same as a particular solution except for adding the period. We simply add period $\times\, n$ instead.

ALSO NOTE: We must state that $n \in Z$.

Discussion: What does the n have to be a whole number?

Question 14

Find the general solutions to the following equation:

$$2\sin\left(-2x + \frac{\pi}{4}\right) = \sqrt{2}$$

Question 15 Walkthrough.

Find the general solutions to the following equation:

$$\tan\left(3x - \frac{\pi}{4}\right) + \frac{1}{\sqrt{3}} = 0$$

NOTE: For tangents, we always get one general solution!

Question 16

Find the general solutions to the following equation:

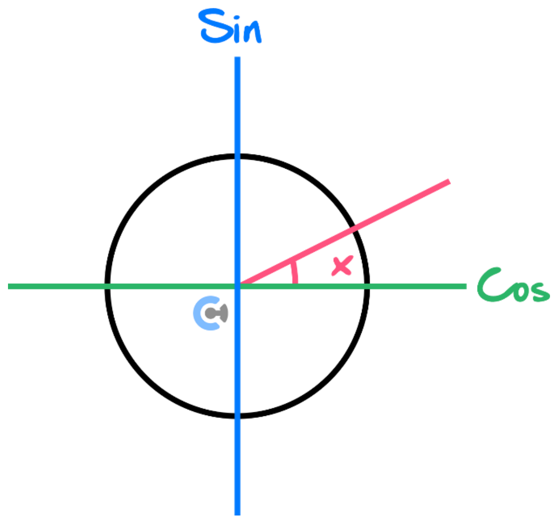
$$2\sqrt{3} + 2\tan\left(2\left(x + \frac{\pi}{6}\right)\right) = 0$$

Section D: Graphs of Sine and Cosine

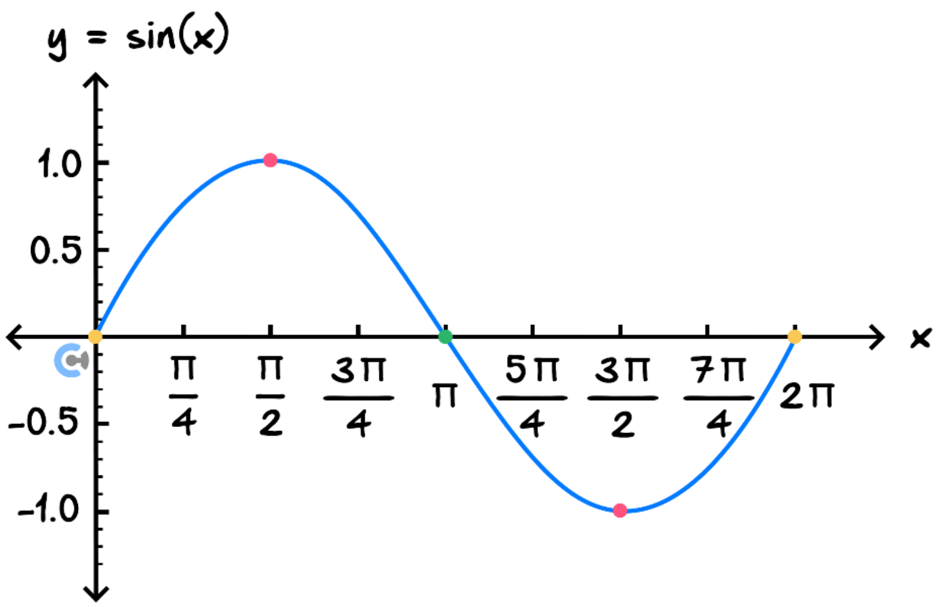
Sub-Section: Basics of Sine and Cosine Functions

What does a Sine and Cosine graph look like?

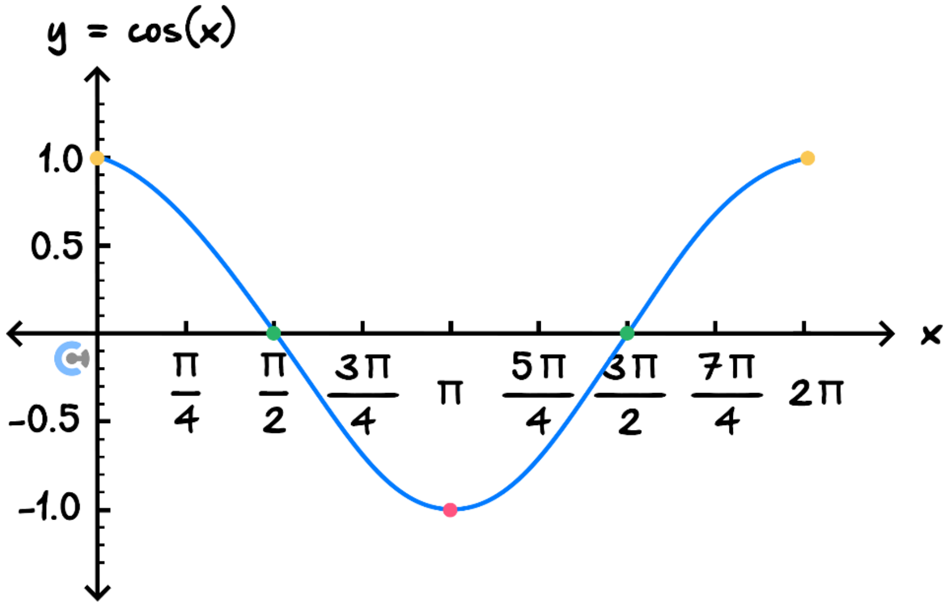
Exploration: Graph of Sine and Cosine



- Label below $Q1, Q2, Q3, Q4$ for the section of the graph that corresponds to respective quadrants.
- **Sin**



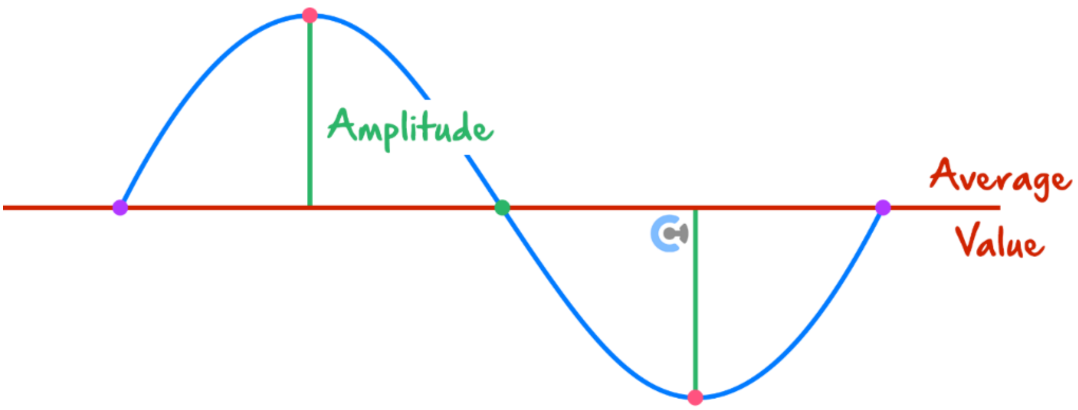
- **Cos**



Definition

Amplitude, Period and Average Value

For $y = A\sin/\cos\,(nx + b) + k$



Consider the sign of our graph

$$\text{Amplitude} = |A|$$

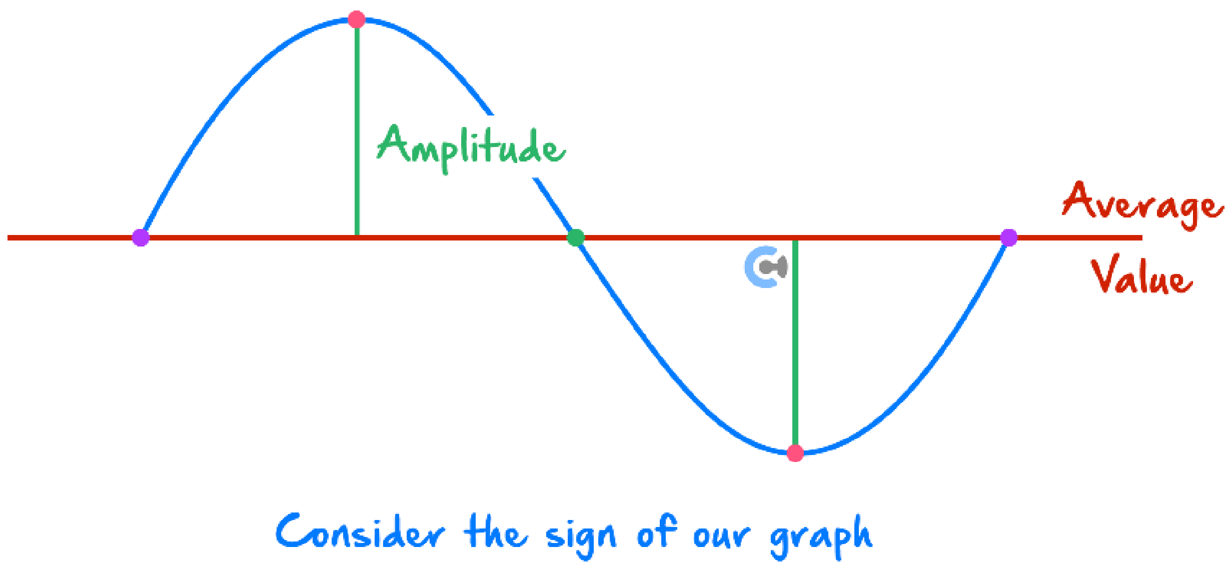
$$\text{Period} = \frac{2\pi}{n}$$

$$\text{Average Value} = k$$

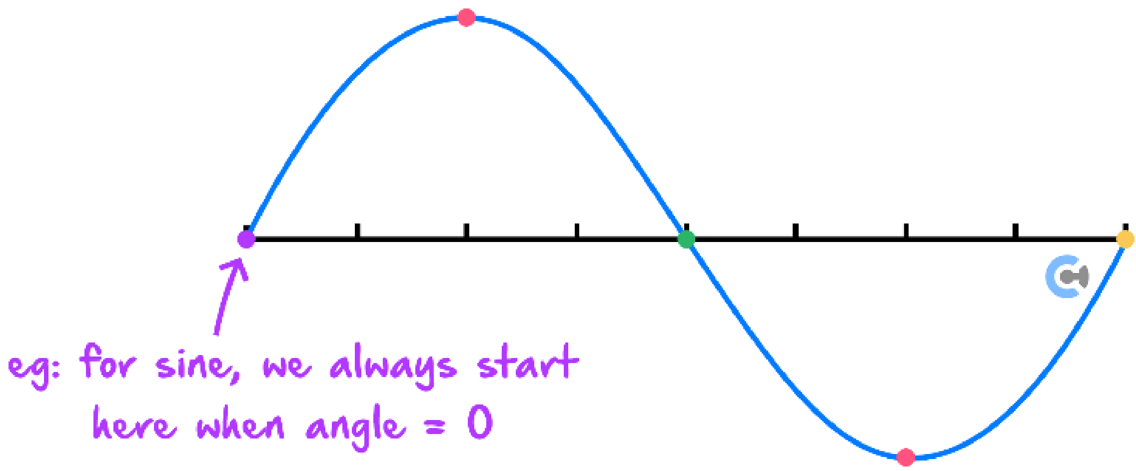
Sub-Section: Graphing Sine and Cosine Functions

Steps for Sketching Transformations of sin and cos Functions

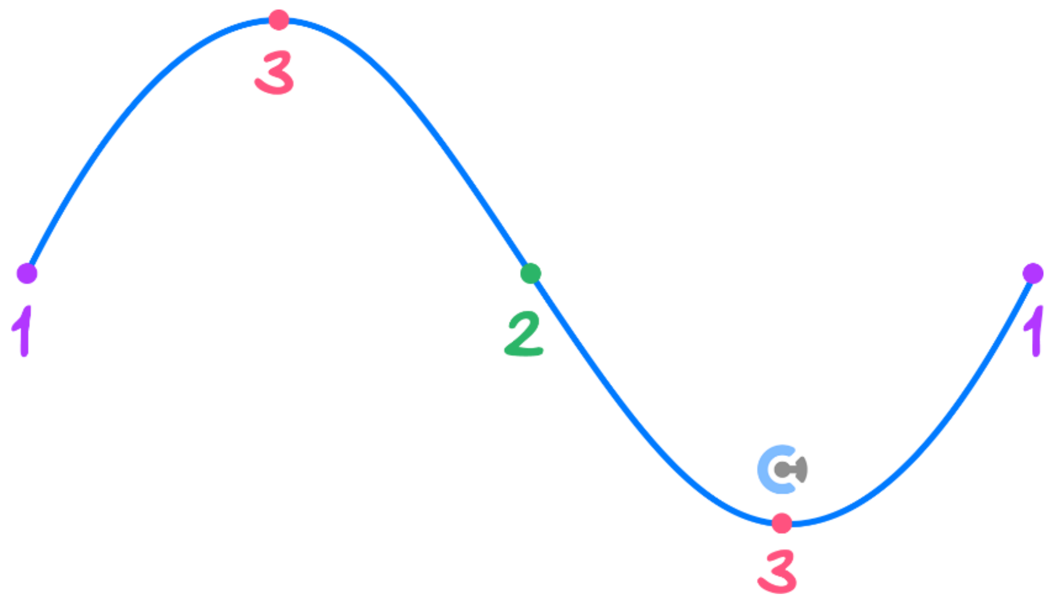
- Identify:
  - Amplitude
  - Period
  - Mean Value
  - Positive/Negative Shape
- And create a "mini version" of the graph you are about to draw.



- Start plotting the function from when the angle $= 0$.
  - For instance, for $\sin\left(2x - \frac{\pi}{3}\right)$, start from $x = \frac{\pi}{6}$.
  - Why?



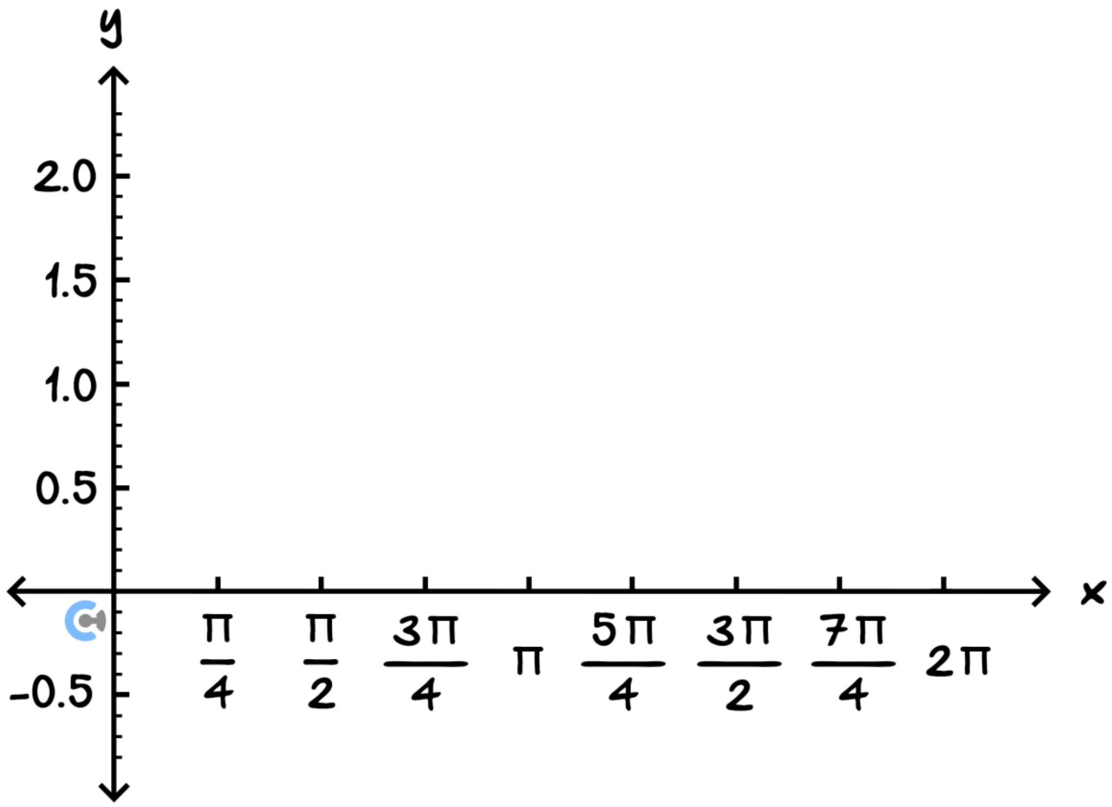
- Draw the start and end of the periods, and plot the halves (turning points).



- Find any x-intercepts.
- Join all the points!

Question 17 Walkthrough.

Sketch the graph of $f(x) = -\sin(2x) + 1$ for $x \in [0, 2\pi]$ on the axes below, labelling all intercepts and endpoints with their coordinates.

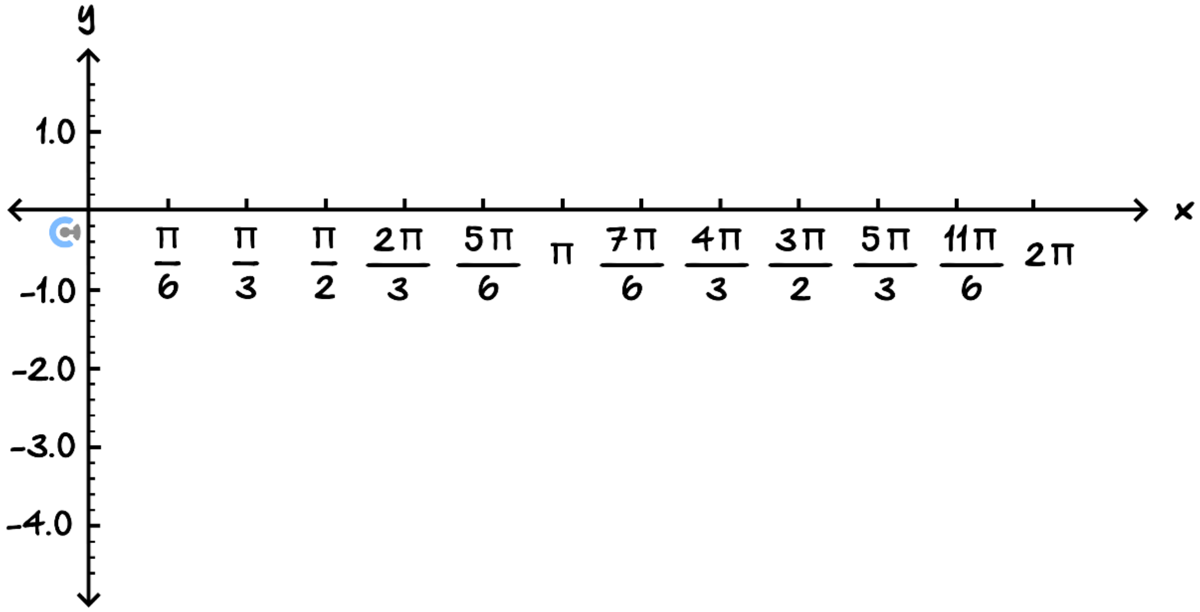


Space for Personal Notes

Question 18

Sketch the following on the axes below, labelling all intercepts, endpoints, and turning points with their coordinates.

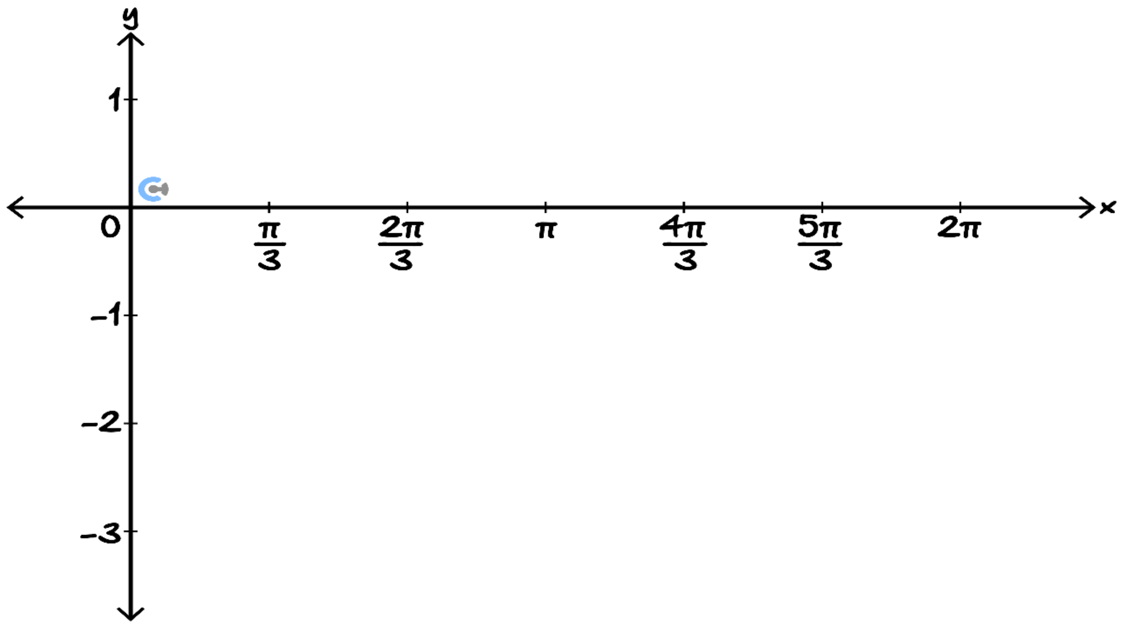
$$y = 2\sin\left(2\left(x - \frac{\pi}{3}\right)\right) - \sqrt{3} \text{ for } x \in [0, 2\pi]$$



Question 19

Sketch the following on the axes below, labelling all intercepts, endpoints, and turning points with their coordinates.

$$y = 2\cos\left(2x + \frac{\pi}{3}\right) - 1 \text{ for } x \in [0, 2\pi]$$

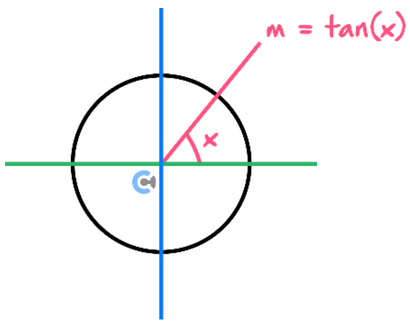


Section E: Graphs of Tangent

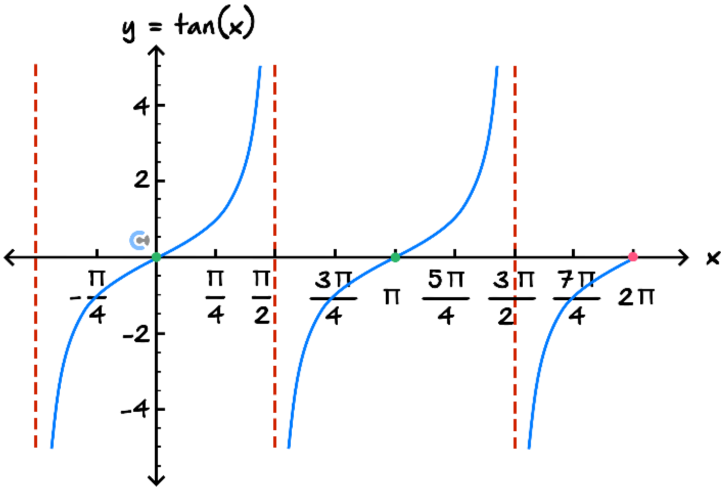
Sub-Section: Basics of Tangent Graphs

What does the tangent graph look like?

Exploration: Graph of Tangents



- Label below $Q1, Q2, Q3, Q4$ for the section of the graph which corresponds to respective quadrants.
- $\textbf{Tan}(x)$



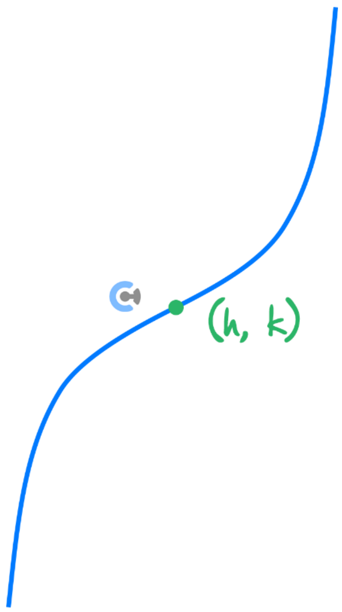
Discussion: Why do we have a vertical asymptote for a tangent?

Sub-Section: Graphing Tangent Functions

Steps for Sketching $\tan$ Functions

- Identify
  - The period $= \frac{\pi}{n}$.
- Find the vertical asymptotes by solving for the angle $= \frac{\pi}{2}$.
- Find other vertical asymptotes within the domain by adding the period to answer from the previous step.
  - For instance, for $\tan\left(2x - \frac{\pi}{3}\right)$, solve $2x - \frac{\pi}{3} = \frac{\pi}{2}$ for x.
- Plot the inflection point (h, k) (Midpoint of the two vertical asymptotes).
  - x value of inflection point $= x$ value, which makes an angle $= 0$.
  - y value of inflection point $=$ vertical translation of the function.

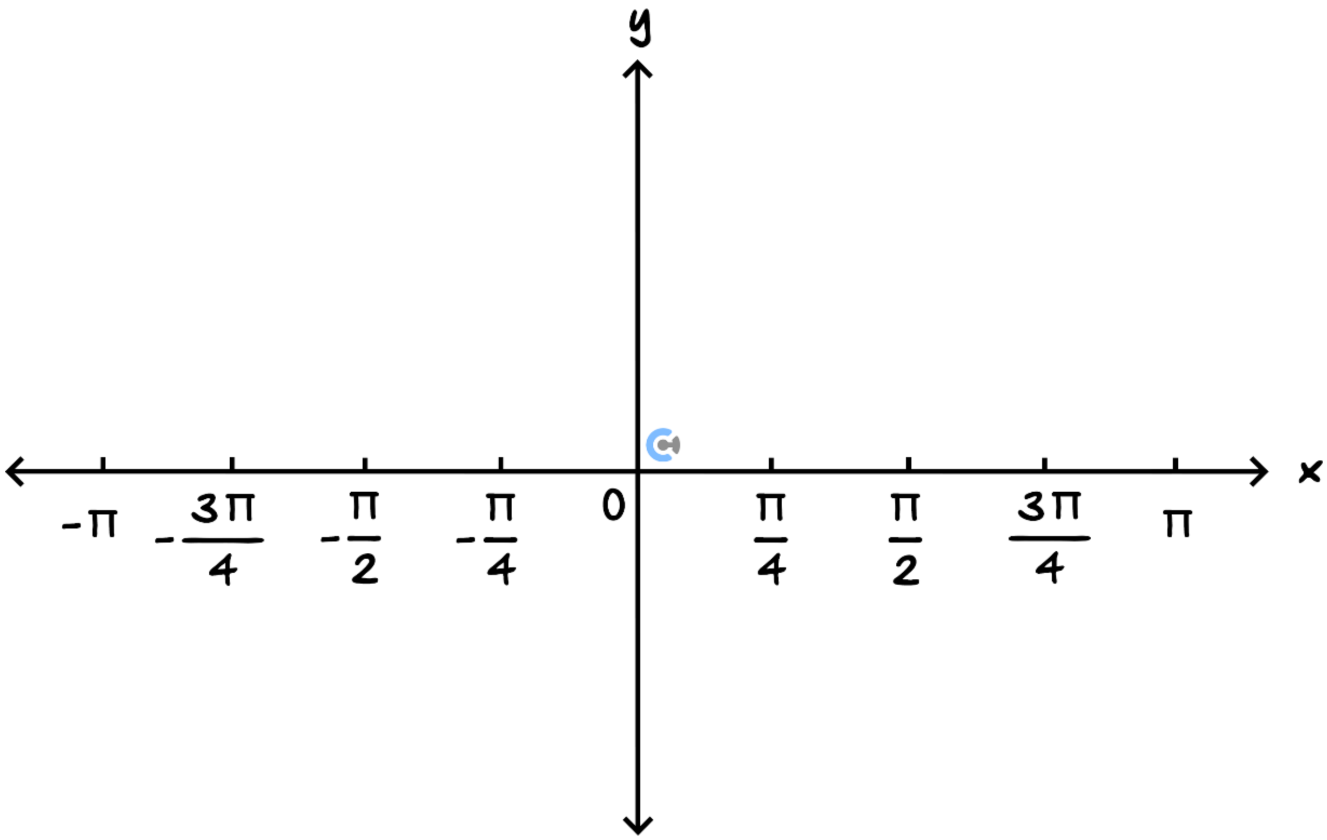
eg: tan(x−h)+k



- Find any x-intercepts.
- Sketch a "cubic-like" shape.

Question 20 Walkthrough.

Sketch the graph of $y = 3\tan(2x)$ for $x \in [-\pi, \pi]$.



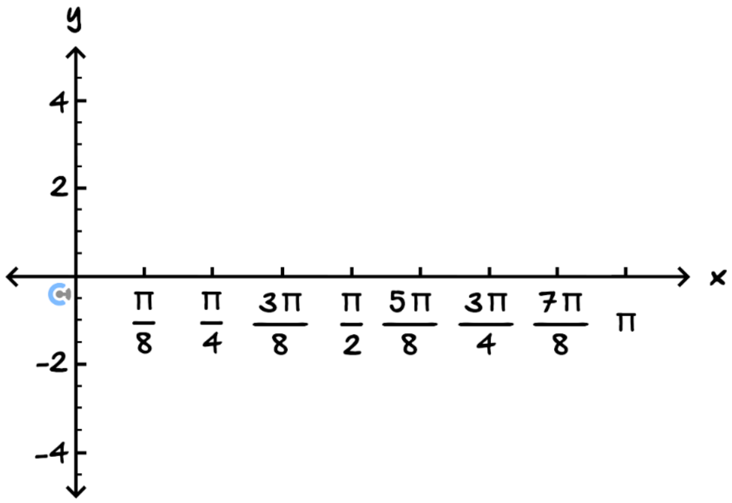
Space for Personal Notes

Your turn!

Question 21

Sketch the following on the axes below, labelling all intercepts, points of inflection, and endpoints with their coordinates, and all asymptotes with their equations.

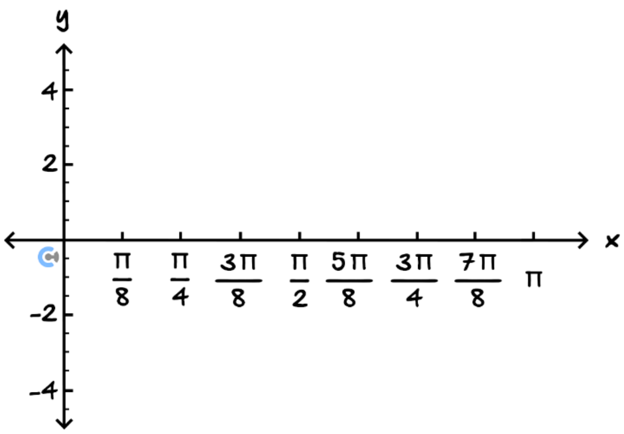
$$y = \tan\left(2x + \frac{\pi}{2}\right) + 1 \text{ for } x \in (0, \pi)$$



Question 22

Sketch the following on the axes below, labelling all intercepts, points of inflection, and endpoints with their coordinates and all asymptotes with their equations.

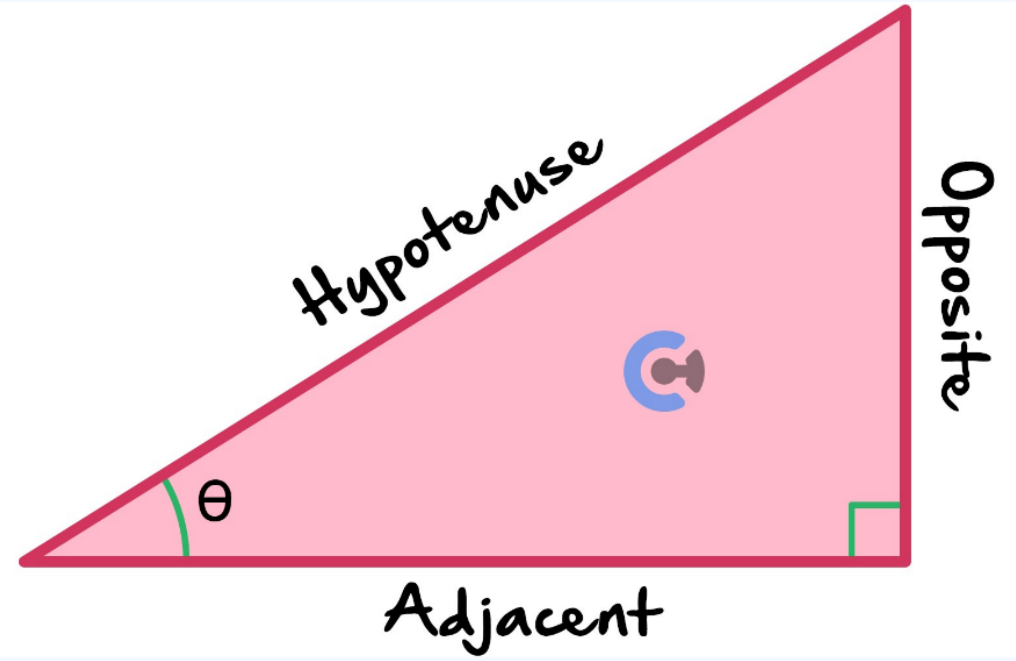
$$f: [0, \pi] \to \mathbb{R}, f(x) = -3\tan(\pi + 4x) + \sqrt{3}$$



Contour Checklist

☐ **Learning Objective**: [3.2.1] - Find Trig Ratios of Supplementary Relationships

Key Takeaways

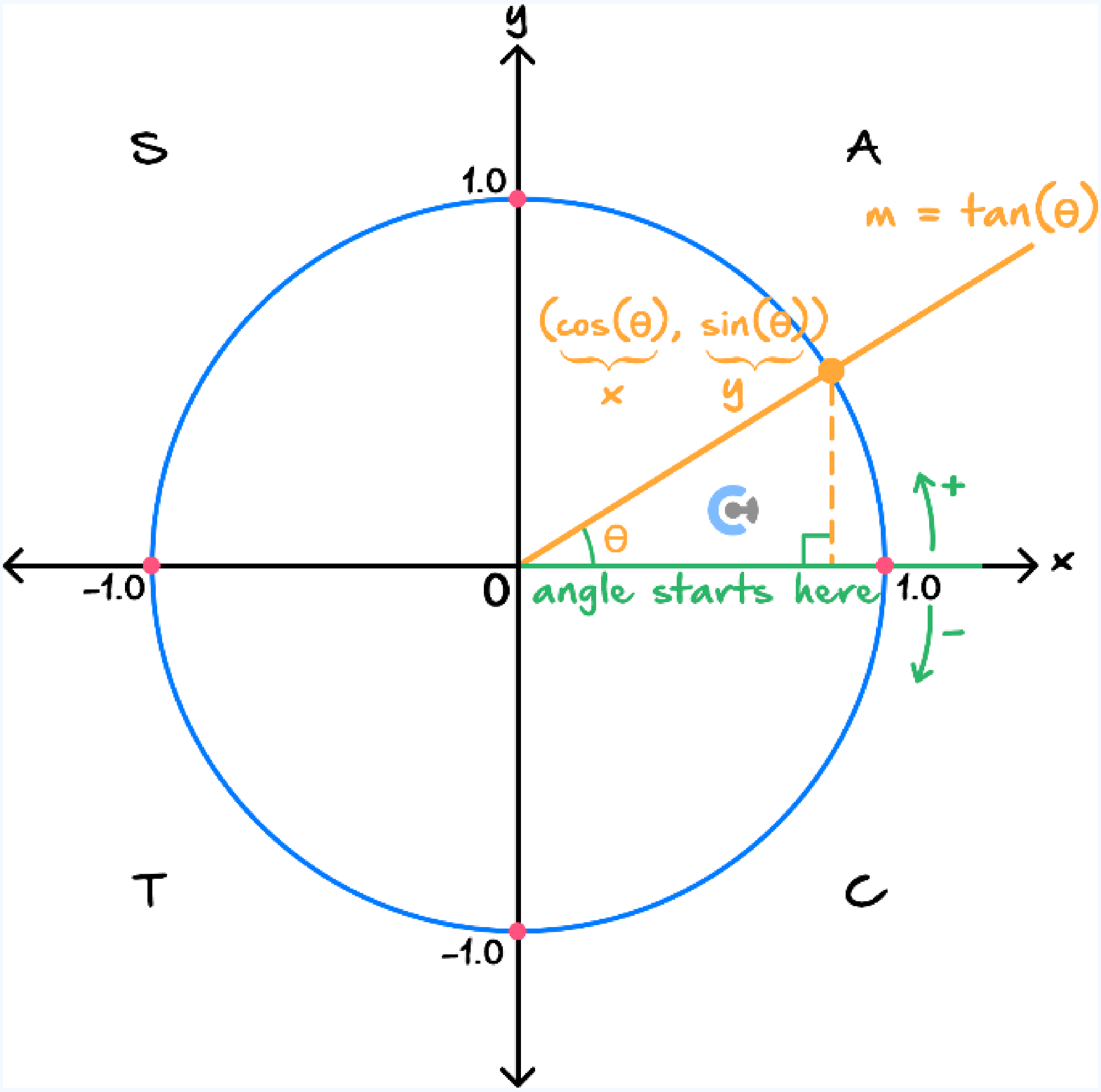


$\sin =$ ______________________________

$\cos =$ ______________________________

$\tan =$ ______________________________

- **Unit Circle**
  - The unit circle is simply a circle of radius 1.



$$\sin(\theta) = \_\_\_\_\_\_\_\_$$

$$\cos(\theta) = \_\_\_\_\_\_\_\_\_\_$$

$$\tan(\theta) = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Period of a Trigonometric Function

$$\text{period of } \sin(nx) \text{ and } \cos(nx) \text{ functions} = \_\_\_\_\_$$

$$\text{period of } \tan(nx) \text{ functions} = \_\_\_\_\_$$

$$\text{where } n = \text{coefficient of } x.$$

- Pythagorean identity:

$$\sin^2(\theta) + \cos^2(\theta) = \_\_\_\_\_\_\_\_$$

- Supplementary relationships:
  - Second Quadrant $(\pi - \theta)$

$$\cos(\pi - \theta) = \_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\sin(\pi - \theta) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\tan(\pi - \theta) = \_\_\_\_\_\_\_\_\_\_\_\_$$

  - Third Quadrant $(\pi + \theta)$

$$\cos(\pi + \theta) = \_\_\_\_\_\_\_\_\_\_$$

$$\sin(\pi + \theta) = \_\_\_\_\_\_\_\_\_\_$$

$$\tan(\pi + \theta) = \_\_\_\_\_\_\_\_\_$$

  - Fourth Quadrant $(-\theta)$

$$\cos(-\theta) = \_\_\_\_\_\_\_\_\_\_$$

$$\sin(-\theta) = \_\_\_\_\_\_\_\_\_\_\_$$

$$\tan(-\theta) = \_\_\_\_\_\_\_\_\_\_\_\_$$

Learning Objective: [3.2.2] - Find Particular and General Solutions

Key Takeaways

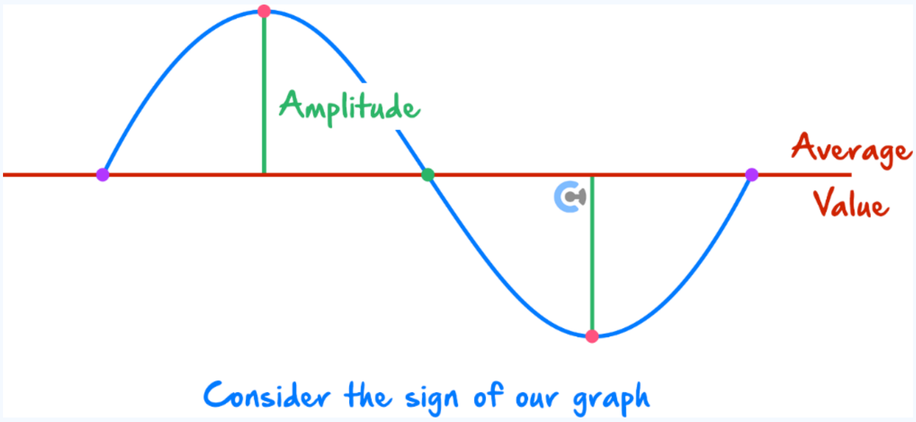
- **Particular Solutions**
  - Solving trigonometric equations **for finite solutions.**
  - **Steps:**
    - Make the trigonometric function the subject.
    - Find the necessary ______________ for one period.
    - Solve for x by equating the necessary angles to the inside of the trigonometric functions.
    - Add and subtract the ____________ to find all other solutions in the __________________.
- **General Solutions**
  - Finding ____________ solutions to a trigonometric equation.
  - **Steps:**
    - Make the trigonometric function the subject.
    - Find the necessary ____________ for one period.
    - Solve for x by equating the necessary angles to the inside of the trigonometric functions.
    - Add _________________ where $n \in Z$.

Learning Objective: [3.2.3] - Graph Sine, Cosine and Tangent functions

Key Takeaways

- Amplitude, Period and Average Value

For $y = A\sin/\cos\,(nx + b) + k$



Amplitude = ________

Period = ________

Average Value = ________

- Tan function:

Period = _____

- Find the asymptotes by solving for angle ____________.
- Find the other asymptotes by adding the __________________ period to the previous answer.
- For the point of inflection:
  - x value of inflection point = x value, which makes an angle = __________.
  - y value of inflection point = vertical translation of the function.

CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300 | **Email:** hello@contoureducation.com.au

VCE Specialist Mathematics ½

Free 1-on-1 Consults

What Are 1-on-1 Consults?

- **Who Runs Them?** Experienced Contour tutors (45 + raw scores and 99 + ATARs).
- **Who Can Join?** Fully enrolled Contour students.
- **When Are They?** 30-minute 1-on-1 help sessions, after-school weekdays, and all-day weekends.
- **What To Do?** Join on time, ask questions, re-learn concepts, or extend yourself!
- **Price?** Completely free!
- **One Active Booking Per Subject:** Must attend your current consultation before scheduling the next. :)

SAVE THE LINK, AND MAKE THE MOST OF THIS (FREE) SERVICE!

Booking Link

bit.ly/contour-specialist-consult-2025