3.1

Website: contoureducation.com.au | Phone: 1800 888 300
Email: hello@contoureducation.com.au

VCE Specialist Mathematics ½
Trigonometry I [3.1]
Workbook

Outline:

Learning Objectives:

- ☐ SM12 [3.1.1] - Find lengths, angles, and area of triangles using sine and cosine rule
- ☐ SM12 [3.1.2] - Find arc lengths, chord lengths, sector, and segment areas
- ☐ SM12 [3.1.3] - Apply angle of elevation/depression and bearing to solve geometric problems (Only 2D)

Section A: Introduction to Trigonometry

Sub-Section: Introduction to Trigonometry

Why is trigonometry useful?

Context: Trigonometry in Real Life

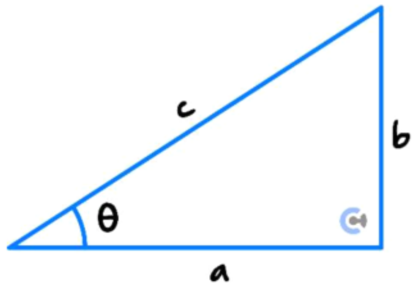
- Let's say Pranit is leaning on the wall.

- He knows he is $175\ cm$ tall, and wants to calculate how far his feet are from the wall.
- To calculate this, what information is important? ANS: Angles and his height.
- Trigonometry is a topic which links the angle with the length.

Space for Personal Notes

Sub-Section: Trigonometric Ratios

Trigonometric Ratios



$$\sin(\theta) = \frac{b}{c} = \frac{opposite}{hypotenuse}$$

$$\cos(\theta) = \frac{a}{c} = \frac{adjacent}{hypotenuse}$$

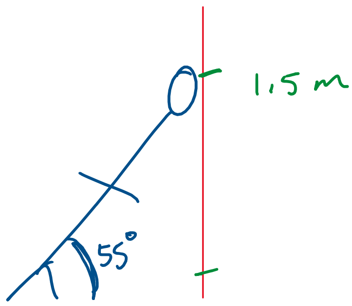
$$\tan(\theta) = \frac{b}{a} = \frac{opposite}{adjacent}$$

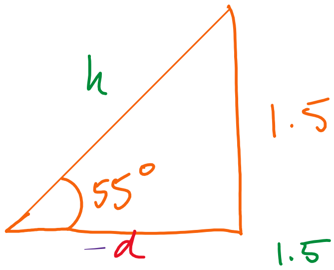
Space for Personal Notes

Question 1 Walkthrough.

Sam is leaning against a vertical wall makes an angle of 55° with the ground. His head touches the wall at 1.5 m above the ground. Calculate:

a. Sam's height, correct to two decimal places.





$$\sin(55^\circ) = \frac{1.5}{h}$$

$$h = \frac{1.5}{\sin(55^\circ)} = 1.83\text{m}$$

b. The distance between his feet and the wall, correct to two decimal places.

$$\tan(55^\circ) = \frac{1.5}{d}$$

$$d = \frac{1.5}{\tan(55^\circ)} = 1.05\text{ m}$$

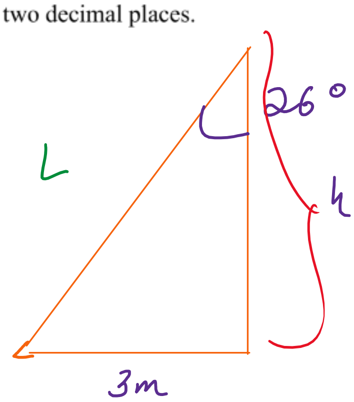
Space for Personal Notes

Your turn!

Question 2 Tech-Active.

A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

a. The length of the ladder, correct to two decimal places.



$$\sin(26^\circ) = \frac{3}{L}$$

$$L = \frac{3}{\sin(26^\circ)}$$

$$= 6.84\,m$$

b. The height it reaches above the ground, correct to two decimal places.

$$\tan(26^\circ) = \frac{3}{h}$$

$$h = \frac{3}{\tan 26^\circ} = 6.15\,m$$

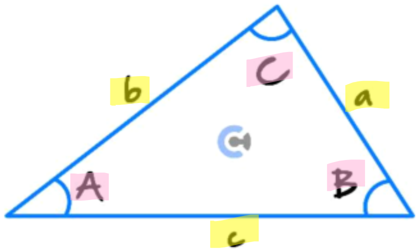
Space for Personal Notes

Section B: Triangle Rules

Sub-Section: Sine Rule

The Sine Rule

- The sine rule states that for a triangle ABC:



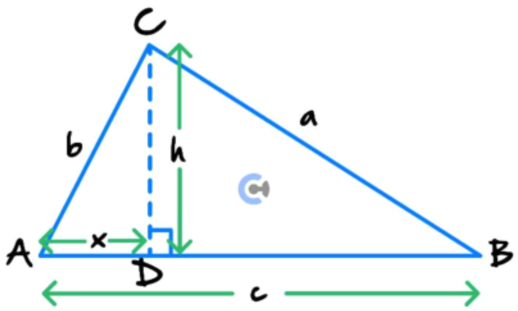
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Space for Personal Notes

How does this work?

<u>Exploration</u>: Proof of the Sine Rule

- We will give a proof of acute-right angled triangles. The proof for obtuse-angled triangles is similar.



- In triangle ACD:

$$\sin(A) = \frac{h}{b}$$

$$h = b\sin(A)$$

- In triangle BCD:

$$\sin B = \frac{h}{a}$$

- Hence, if you were to substitute $h = b\sin(A)$:

$$\sin B = \frac{b\sin(A)}{a}$$

- If you rearrange,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

- **That proves the sine rule!**

Space for Personal Notes

When is the sine rule used?

Application of Sine Rule

$\frac{a}{\sin A} = \frac{b}{\sin B}$

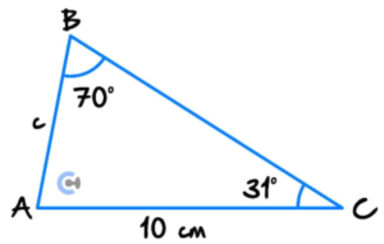
- We can use it to solve for length or angles within the triangle.
- CASE 1: One side and two angles are given.
  - In CASE 1, the triangle is uniquely defined up to congruence.
- CASE 2: Two sides and a non-included angle are given (the angle is not 'between' the two sides).
  - In CASE 2, there may be two possible triangles.

Let's take a look at the first case!

Question 3 Walkthrough.

Case 1: One side and two angles given.

Find the length AB using the sine rule, in cm correct to two decimal places.



$$\frac{c}{\sin(31^\circ)} = \frac{10}{\sin(70^\circ)}$$

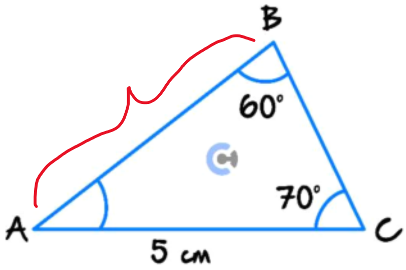
$$c = \frac{10\sin(31^\circ)}{\sin(70^\circ)}$$

$$c = 5.48\ cm$$

Question 4

Case 1: One side and two angles given.

Find the length AB using the sine rule, in cm correct to two decimal places.



$$\frac{AB}{\sin(70^\circ)} = \frac{5}{\sin(60^\circ)}$$

$$AB = \frac{5\sin 70^\circ}{\sin 60^\circ} = 5.43$$

Space for Personal Notes

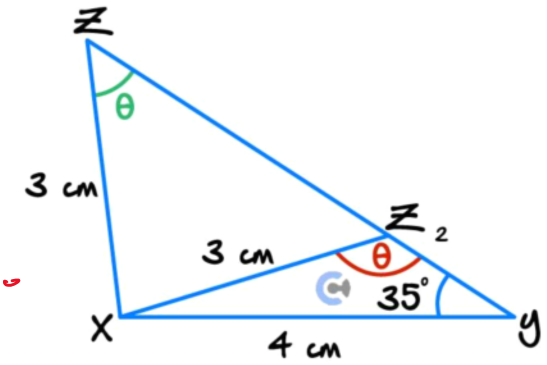
Let's look at the second case!

Question 5 Walkthrough.

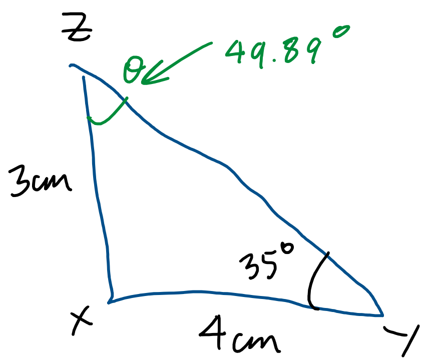
Case 2: Two sides and non-included angle given.

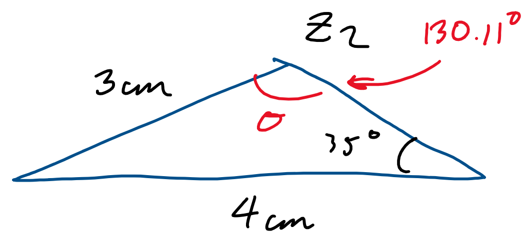
Consider a triangle XYZ. Find the magnitude of angle Z in the triangle, given that $Y = 35°$, $XZ = 3\ cm$, and $XY = 4\ cm$. Give your answer in degrees, correct to two decimal places.

$$\frac{4}{\sin\theta} = \frac{3}{\sin 35°}$$



$$\frac{4}{\sin\theta} = \frac{3}{\sin 35}$$





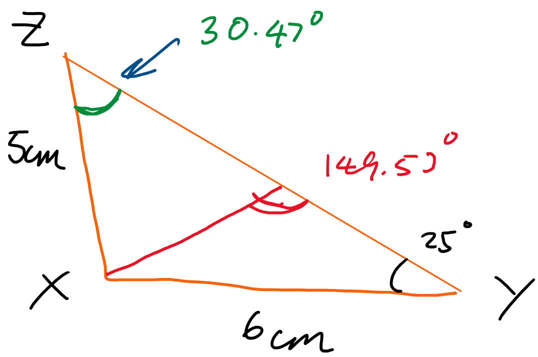
NOTE: $\text{Sin}(180 - \theta) = \text{Sin}(\theta°)$.

Space for Personal Notes

Question 6

Case 2: Two sides and non-included angle given.

Consider a triangle XYZ. Find the magnitude of angle Z in the triangle, given that $Y = 25°$, $XZ = 5\ cm$, and $XY = 6\ cm$. Give your answer in degrees, correct to two decimal places.



$$\frac{6}{\sin(z)} = \frac{5}{\sin(25°)}$$

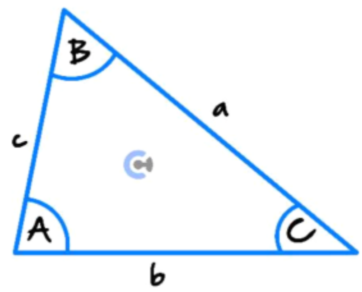
$$\angle Z = 30.47° \text{ or } 149.53°$$

Space for Personal Notes

Sub-Section: Cosine Rule

The Cosine Rule

- The cosine rule states that for a triangle ABC:



$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

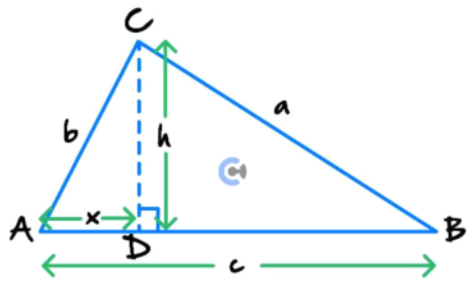
$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

Space for Personal Notes

How does this work?

Exploration: Proof of Cosine Rule

- In triangle ACD:



- In triangle ACD:

$$\cos(A) = \frac{x}{b}$$

$$x = b\cos(A)$$

- Using Pythagoras' theorem in Triangles ACD and BCD:

$$b^2 = x^2 + h^2$$

$$a^2 = (c-x)^2 + h^2$$

- Expanding the second equation gives us:

$$a^2 = c^2 - 2cx + x^2 + h^2$$

- Substituting $b^2 = x^2 + h^2$ gives us:

$$a^2 = c^2 - 2cx + b^2$$

- Substituting $x = b\cos(A)$ gives us:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

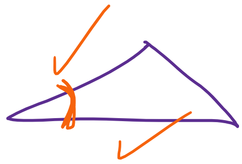
- That proves the cosine rule!

When is the cosine rule used?

Application of Cosine Rule

$a^2 = b^2 + c^2 - 2bc\cos(A)$

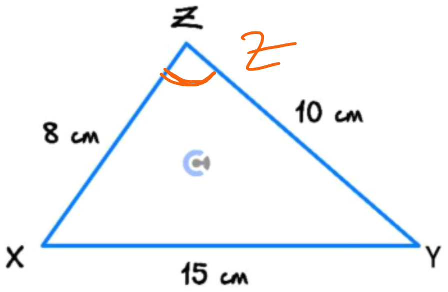
- We can use it to solve for length or angles within the triangle.
  - Same as sine rule in terms of the aim.
    - CASE 1: Three sides are given.
    - CASE 2: Two sides and the included angle are given (the angle **IS** between the two sides).
- In each case, the triangles are uniquely defined up to congruence. ← 1 possibility

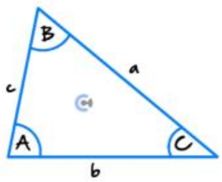


Question 7 Walkthrough.

Case 1: Three sides are given.

Find the angle $Z°$. Give your answer correct to two decimal places.





$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

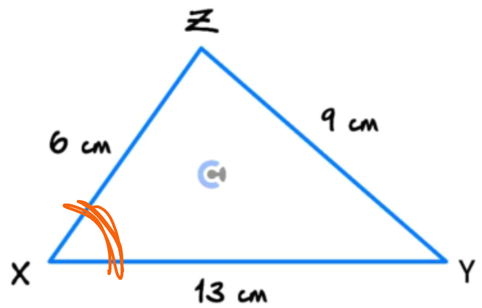
$$15^2 = 8^2 + 10^2 - 2(8)(10)\cos(Z)$$

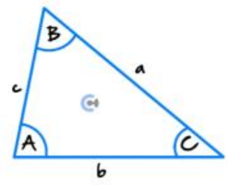
$$Z = 112.41°$$

Question 8

Case 1: Three sides are given.

Find the angle $X°$. Give your answer correct to two decimal places.





$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$9^2 = 13^2 + 6^2 - 2(6)(13)\cos(X)$$

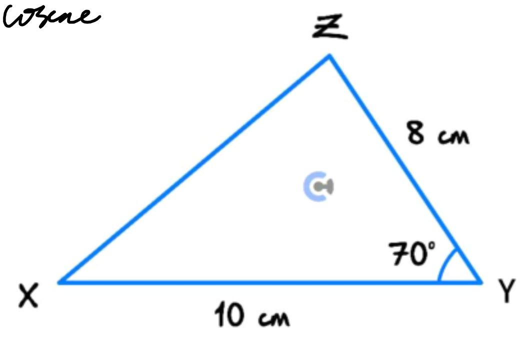
$$X = 37.36^{\circ}$$

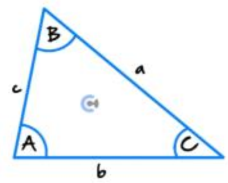
Space for Personal Notes

Question 9 Walkthrough.

Case 2: Two sides and the included angle are given.

Find the length of XZ using the ~~sine~~ cosine rule, in cm correct to two decimal places.





$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$XZ = \sqrt{10^2 + 8^2 - 2(10)(8)\cos(70^\circ)}$$

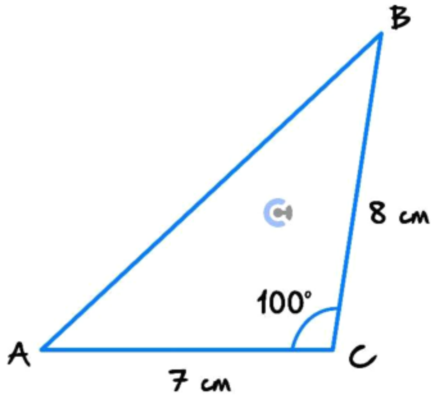
$$XZ = 10.45 \text{ cm}$$

Space for Personal Notes

Question 10

Case 2: Two sides and the included angle are given.

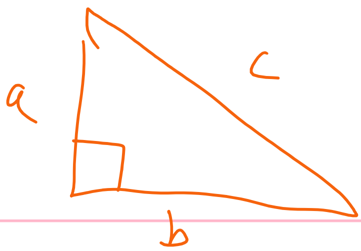
Find the length of AB using the sine rule, in cm correct to two decimal places.



$$AB = \sqrt{7^2+8^2-2(7)(8)\cos(100^\circ)}$$

$$= 11.51 \text{ cm}$$

<u>Discussion:</u> What would happen to cosine rule if the angle was 90 degrees?



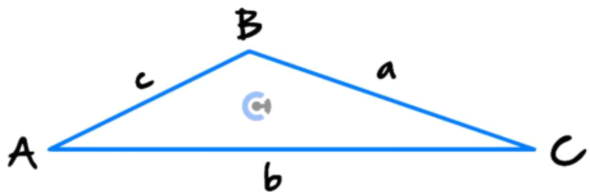
$$c^2 = a^2+b^2-\cancel{2ab\cos(90^\circ)}$$

$$\boxed{c^2 = a^2+b^2}$$

Sub-Section: Area of a Triangle

Area of a Triangle

- In terms of two given sides, and the included angle:



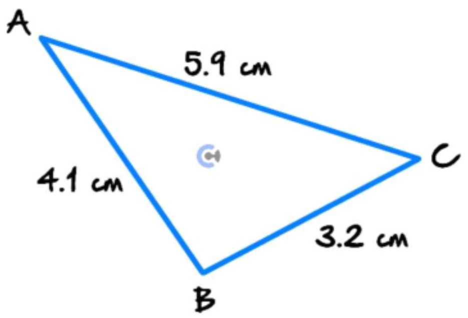
$$A = \frac{1}{2} bc \sin(A)$$

NOTE: Angle must be an angle between the two sides b and c.

Space for Personal Notes

Question 11 Walkthrough. Tech-Active.

Find the area of the triangle, correct to three decimal places. Include a unit in your answer.



$$5.9^2 = 4.1^2 + 3.2^2 - 2(4.1)(3.2)\cos(B)$$

$$B = 107.201^\circ$$

$$A = \frac{1}{2}(4.1)(3.2)\sin(107.201^\circ) = 6.267\text{ cm}^2$$

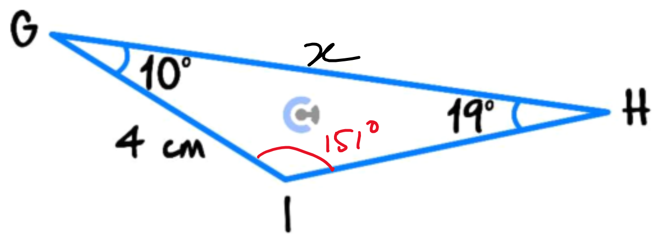
NOTE: We use cosine rule to solve for one angle first!

Space for Personal Notes

Your turn!

Question 12 Tech-Active.

Find the area of the triangle, correct to three decimal places. Include a unit in your answer.



$$\frac{4}{\sin(19^\circ)} = \frac{x}{\sin 151^\circ} \longrightarrow x = 5.9564$$

$$A = \frac{1}{2} \times 4 \times 5.9564 \sin(10^\circ)$$

$$= 2.069\,cm^2$$

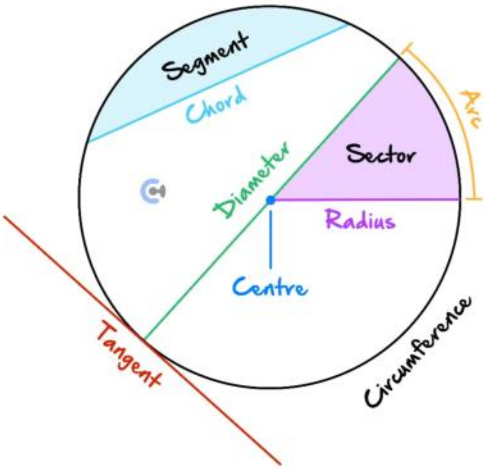
Space for Personal Notes

Section C: Circle Mensuration

Sub-Section: Definitions

What is circle mensuration?

Mensuration



- Part of geometry concerned with finding lengths, areas, and volumes of shapes and objects.
- Circle mensuration is about finding the lengths and areas of different features on circles.

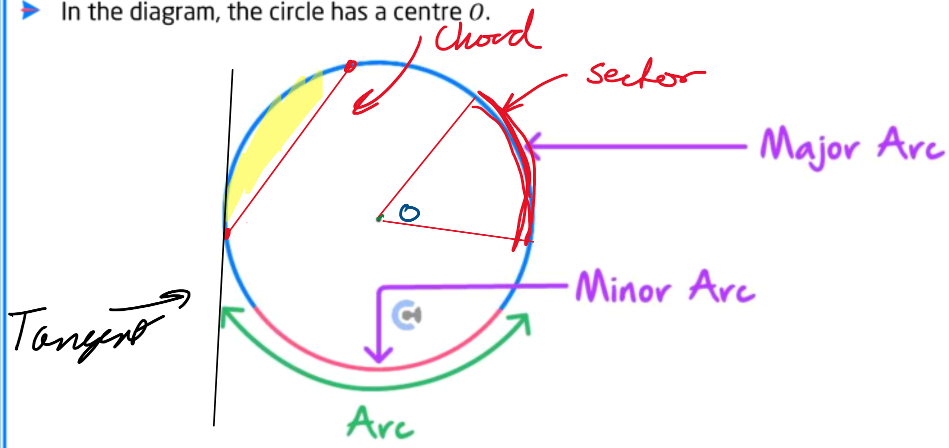
NOTE: This topic is important for SM34 complex planes/subsets!

Space for Personal Notes

Definition

Key Terminology

- In the diagram, the circle has a centre O.



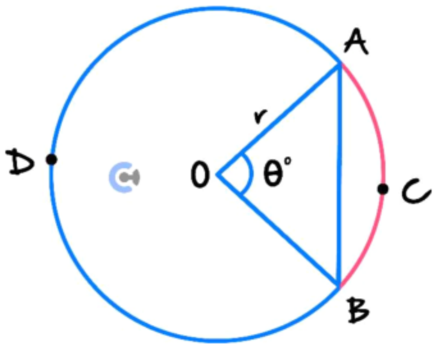
- **Chord** = Line segment with endpoints on the circle.
  - Chord passing through the centre is called the diameter.
- **Arc** = Any curved part of the circle.
  - The shorter arc is called the minor **arc** and the longer is the major **arc**.
- **Segment** = Every chord divides the interior of a circle into two segments.
  - The smaller segment is called the minor **segment** and the larger is the major **segment**.
- **Sector** = Pizza slice. Two radii and an arc define a sector.
- **Tangent** = Line outside a circle that touches the circle exactly once (and does not pass through it).

Space for Personal Notes

Sub-Section: Arc Length

How do we calculate the arc length?

Arc Length



- The arc ACB and the corresponding chord AB are said to subtend the angle $\angle AOB$ at the centre of the circle.

$$l = 2\pi r \times \%$$

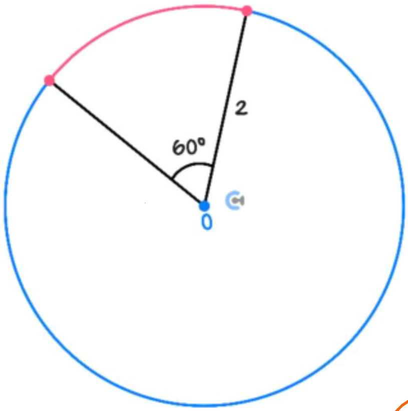
$$\text{Where, } \% = \frac{\theta^c}{2\pi} = \frac{\theta^\circ}{360}$$

- We simply find the % of circumference.
  - % is defined by the angle θ divided by the entire rotation.

Space for Personal Notes

Question 13 Walkthrough.

Find the arc length highlighted in red below.



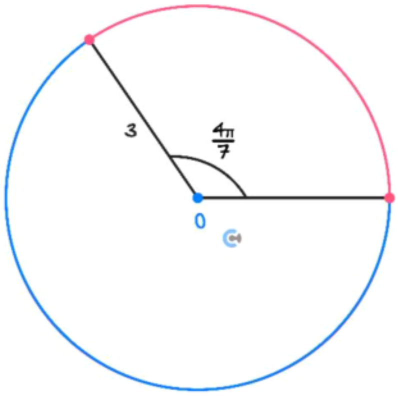
$$\text{Arc length} = 4\pi \times \frac{60}{360} \quad \frac{1}{6}$$

$$= \frac{2\pi}{3}$$

Space for Personal Notes

Question 14

Find the arc length highlighted in red below.



Arc length $= 6\pi \times \dfrac{\frac{4\pi}{7}}{2\pi}$

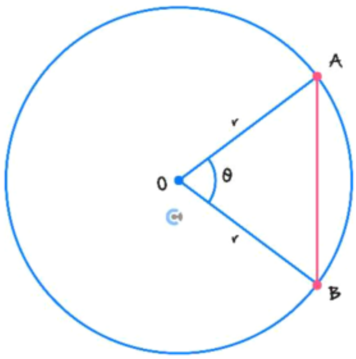
$= \dfrac{12\pi}{7}$

Space for Personal Notes

Sub-Section: Chord Lengths

How do we calculate the chord length?

Chord Length



$$AB = 2r\sin\left(\frac{\theta}{2}\right)$$

Space for Personal Notes

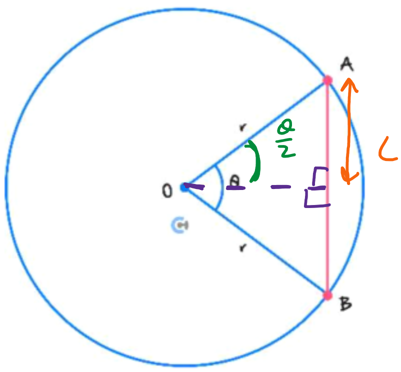
How does this work?

Exploration: Derivation of Chord Length Formula

- Consider the chord length below.

$\sin\left(\frac{\theta}{2}\right) = \frac{l}{r}$

$l = r\sin\left(\frac{\theta}{2}\right)$



Chord length $= 2l$

- Simply cut the length in half in the diagram above.
- How can we solve for the chord length?

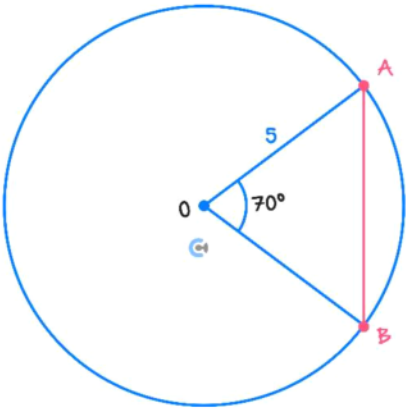
$$\textit{Chord Length} = 2l$$

$$= 2r\sin\left(\frac{\theta}{2}\right)$$

Space for Personal Notes

Question 15 Walkthrough. Tech-Active.

Find the chord length AB highlighted in red below. Give your answer correct to two decimal places.



$$\text{Length} = 2r\sin\left(\tfrac{\theta}{2}\right)$$

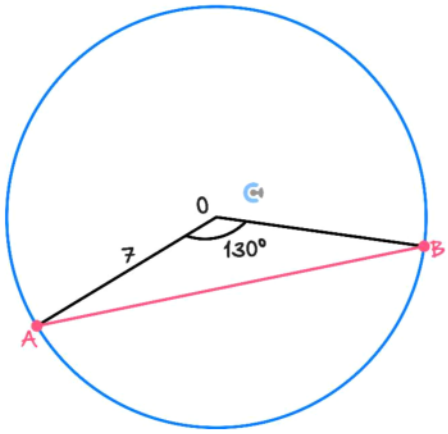
$$= 2 \times 5 \times \sin(35^\circ)$$

$$= 5.74$$

Space for Personal Notes

Question 16 Tech-Active.

Find the chord length AB highlighted in red below. Give your answer correct to two decimal places.



$$\text{Length} = 2r\sin\left(\frac{\theta}{2}\right)$$

$$= 2 \times 7 \times \sin\left(\frac{130}{2}\right)$$

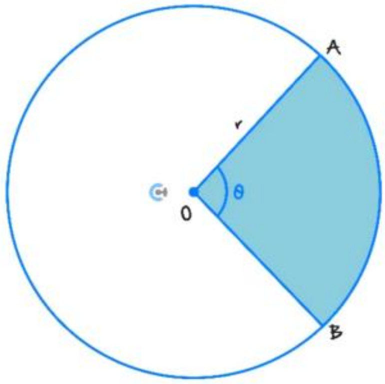
$$= 12.69$$

Space for Personal Notes

Sub-Section: Sector Area

How do we calculate the sector area?

Area of Sector



$$l = \pi r^2 \times \%$$

$$\text{Where, } \% = \frac{\theta^c}{2\pi} = \frac{\theta^\circ}{360}$$

- We simply find the % of the circle area.
  - % is defined by the angle θ divided by the entire rotation.

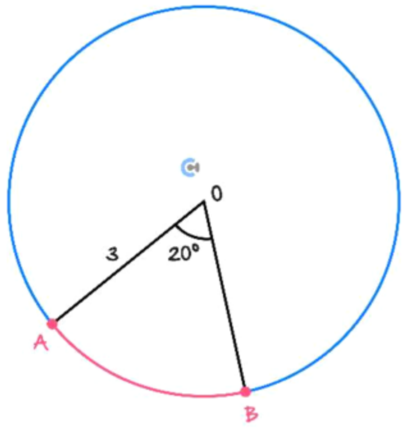
$$\textbf{Using degrees: Area of sector} = \frac{\pi r^2 \theta^\circ}{360}$$

$$\textbf{Using radians: Area of sector} = \frac{1}{2} r^2 \theta^c$$

Space for Personal Notes

Question 17 Walkthrough.

Find the area of the sector AOB. ~~Give your answer correct to two decimal places~~.



Area of Sector $= 9\pi \times \left(\frac{20}{360}\right)$

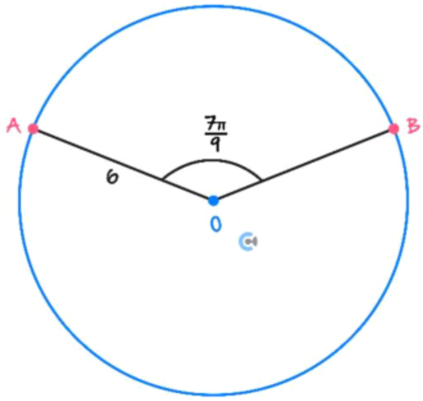
$= 9\pi \times \frac{1}{18}$

$= \frac{\pi}{2}$ units^2

Space for Personal Notes

Question 18

Find the area of the sector AOB.



$$\text{Area} = 36\pi \times \frac{\frac{7\pi}{9}}{2\pi}$$

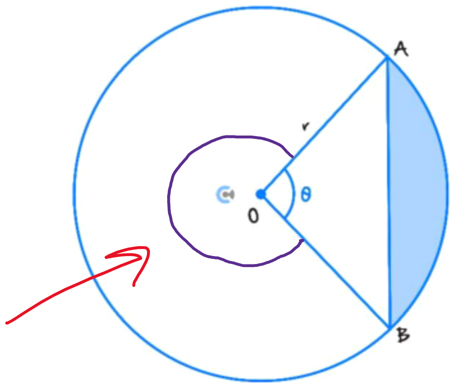
$$= 14\pi$$

Space for Personal Notes

Sub-Section: Segment Area

How do we calculate the segment area?

Area of Segment



- The **area of the segment** is the area of the sector OAB minus the area of the triangle OAB.
- Using the area of a triangle formula, the area of triangle OAB is $\frac{1}{2}r^2\sin(\theta)$.

$$A = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin(\theta) = \frac{1}{2}r^2(\theta - \sin(\theta))\ (\text{radians})$$

$$A = \left(\frac{\theta}{360}\right) \times (\pi r^2) - \frac{1}{2}r^2\sin(\theta)\ \ (\text{degrees})$$

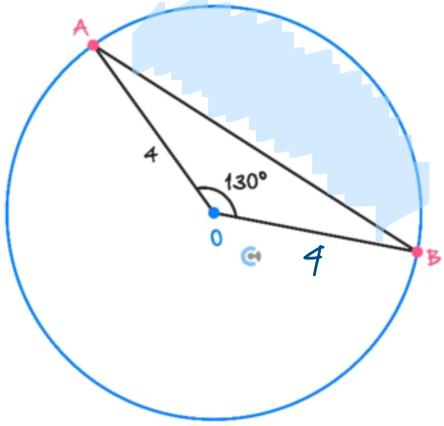
Discussion: How does it work for major segment?

$\sin(>180) < 0$

Adding triangle

Question 19 Walkthrough. Tech-Active.

Find the area of the minor segment given by the line segment AB. Give your answer correct to two decimal places.



Area =  − 

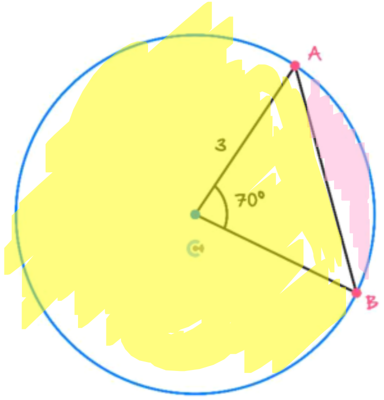
$$= 16\pi \times \left(\frac{130}{360}\right) - \frac{1}{2} \times 4 \times 4 \times \sin(130^\circ)$$

$$= 12.02$$

Space for Personal Notes

Question 20 Tech-Active.

Consider the diagram below.



a. Find the area of the minor segment given by the line segment AB. Give your answer correct to two decimal places.

Area $=$ $-$

$= 9\pi\left(\frac{70}{360}\right) - \frac{1}{2} \times 3 \times 3 \times \sin(70^\circ)$

$= 1.27$

b. Find the area of the major segment given by the line segment AB. Give your answer correct to two decimal places.

Area $= 9\pi - 1.26917$

$= 27.01$

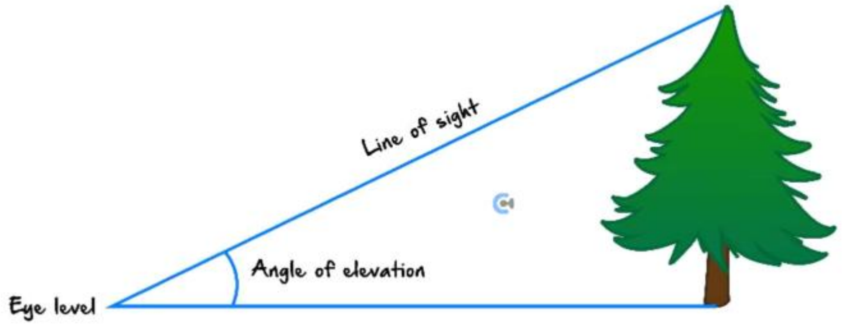
NOTE: Simply change the angle to $360 - \theta$.

Section D: Angles

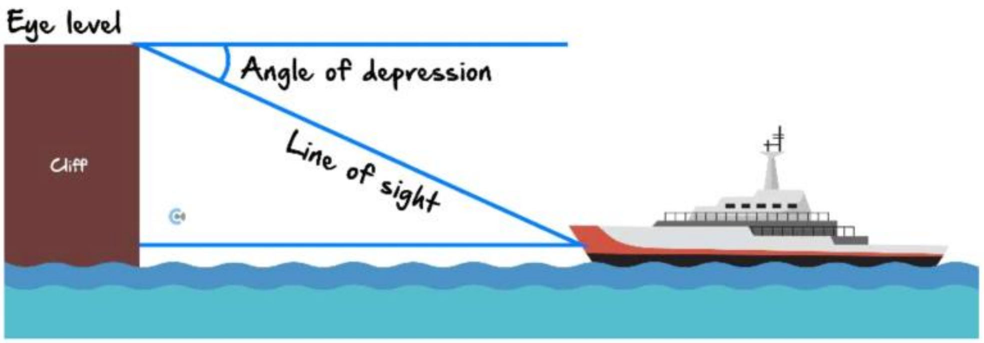
Sub-Section: Angle of Elevation and Depression

Angle of Elevation, Angle of Depression

- **Angle of Elevation**
  - The angle of elevation is the angle between the **horizontal** and a **direction** ___above___ the horizontal.



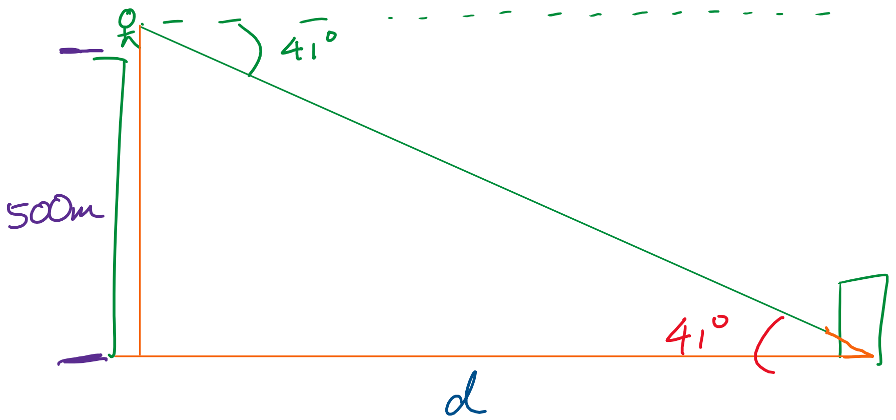
- **Angle of Depression**
  - The angle of elevation is the angle between the **horizontal** and a **direction** ___below___ the horizontal.



Space for Personal Notes

Question 21 Walkthrough.

A hiker standing on top of a mountain observes that the angle of depression to the base of a building is 41°. If the height of the hiker above the base of the building is 500 m, find the horizontal distance from the hiker to the building in metres, correct to two decimal places.



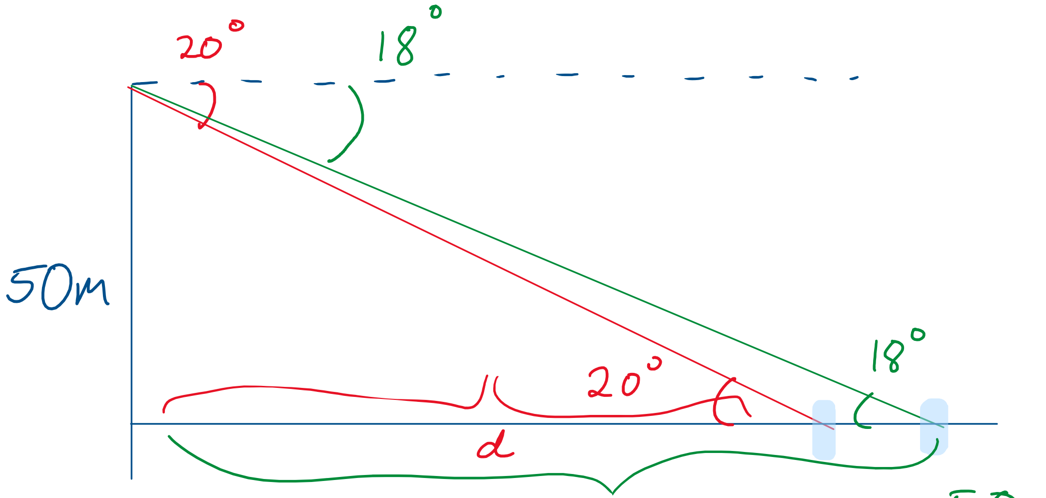
$$\tan 41^\circ = \frac{500}{d}$$

$$d = \frac{500}{\tan 41^\circ} = 575.18m$$

Space for Personal Notes

Question 22

A person standing on top of a cliff 50 m high is in line with two buoys, whose angles of depression are 18° and 20°. Calculate the distance between the buoys in metres, correct to two decimal places.



$\tan(20°) = \frac{50}{d}$

$d = 137.373$

$\tan(18°) = \frac{50}{d}$

$d = 153.884$

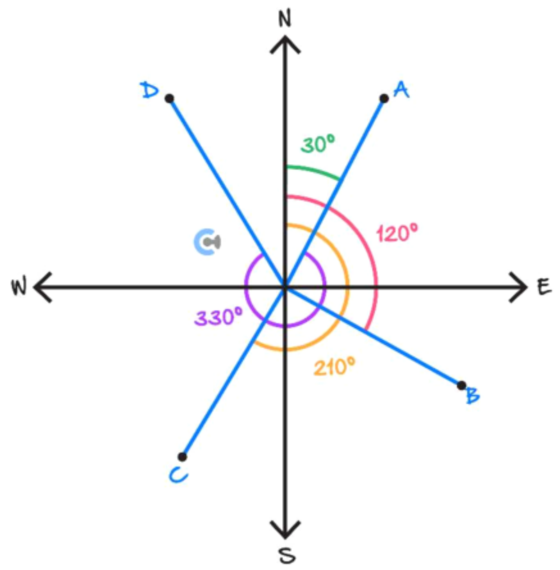
$d - d = 16.51m$

Space for Personal Notes

Sub-Section: Bearing

Bearing

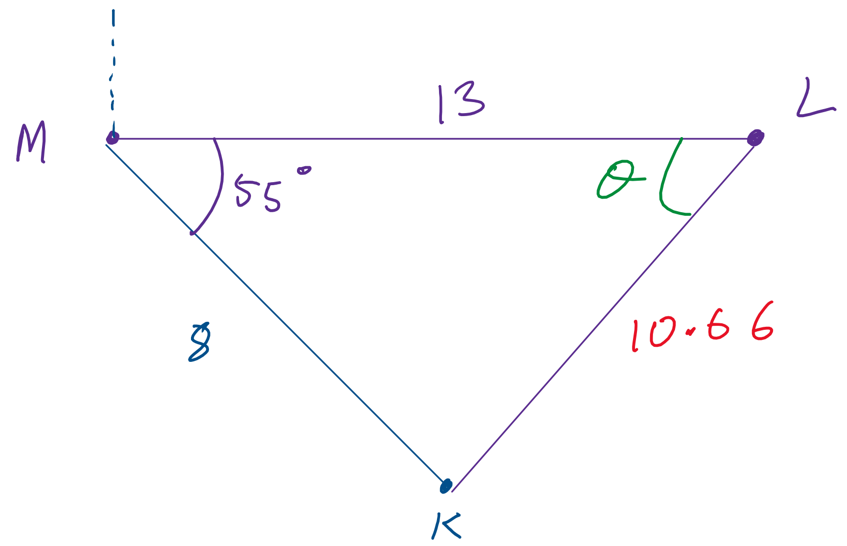
- The **bearing** is the angle measured from North **in the** clockwise **direction.**



Space for Personal Notes

Question 23 Walkthrough.

A yacht starts from a dock at a point L and sails 13 km due west to M. It then sails 8 km on a bearing of 145° to K. Find the magnitude of the angle MLK in degrees, correct to three decimal places.



$$LK^2 = 13^2 + 8^2 - 2(13)(8)\cos(55^\circ)$$

$$LK = 10.66$$

$$\frac{10.66}{\sin(55^\circ)} = \frac{8}{\sin(\theta)} \rightarrow \theta = 37.922^\circ$$

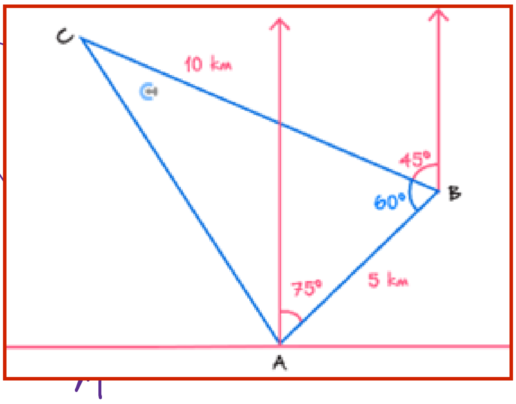
$$\angle MLK = 37.922^\circ$$

Space for Personal Notes

Question 24

A yacht sails from point A to point B on a bearing of $75°$ for $5\ km$, then from point B to point C on a bearing of $315°$ for $10\ km$. Find:

a. The distance between point A and point C.

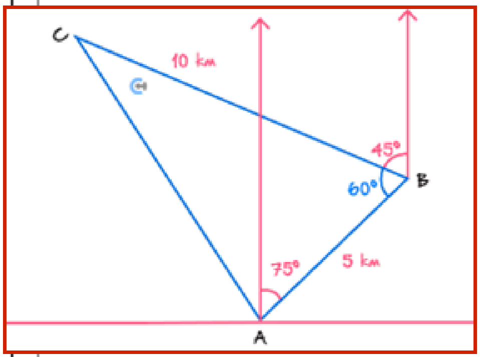


$$AC = \sqrt{5^2 + 10^2 - 2(5)(10)\cos(60°)}$$

$$= 5\sqrt{3}\ km$$

b. The distance that the yacht is from point A when it is closest to point A on the BC leg.

$$\sin(60°) = \frac{d}{5}$$



$$d = 5\sin 60°$$

$$= \frac{5\sqrt{3}}{2}$$

HINT: The closest distance between the line AB and point C is the connecting line (DC), where D lies on AB, is perpendicular to AB.

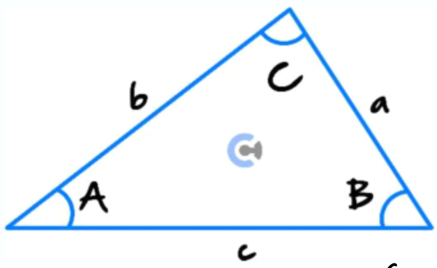
Space for Personal Notes

Contour Checklist

☐ Learning Objective: [3.1.1] – Find lengths, angles, and area of triangles using sine and cosine rule

Key Takeaways

☐ The sine rule states that for a triangle ABC:



$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(c)}$$

Application of Sine Rule

☐ We can use it to solve for length or angles within the triangle.

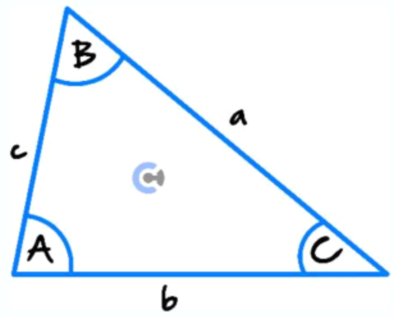
☐ CASE 1: One side and two angles are given.

- ○ In CASE 1, the triangle is uniquely defined up to congruence.

☐ CASE 2: Two sides and a non-included angle are given (the angle is not 'between' the two sides).

- ○ In CASE 2, there may be 2 possible triangles.

- The cosine rule states that for a triangle ABC:



$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

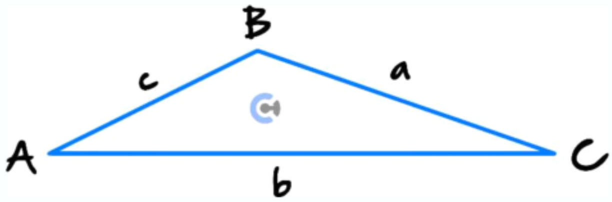
$$\cos(A) = \frac{b^2+c^2-a^2}{2bc}$$

Application of Cosine Rule

- We can use it to solve for length or angles within the triangle.
  - CASE 1: Three sides are given.
  - CASE 2: Two sides and the included angle are given (the angle **IS** between the two sides).
- In each case, the triangles are uniquely defined up to congruence

Area of a Triangle

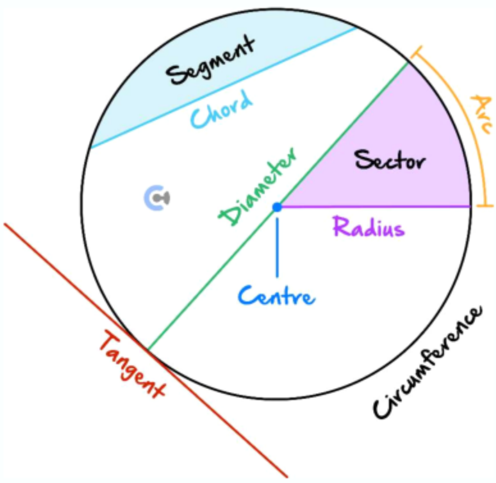
- In terms of two given sides, and the included angle:



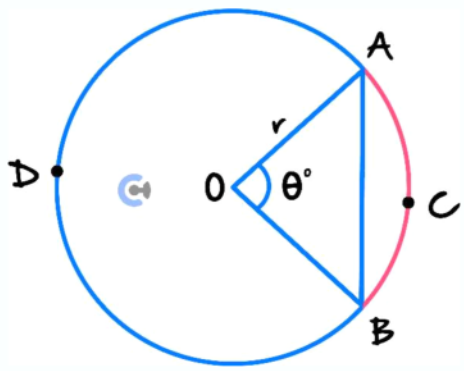
$$A = \frac{1}{2} bc \sin A$$

Learning Objective: [3.1.2] – Find arc lengths, chord lengths, sector, and segment areas

Key Takeaways



Arc Length

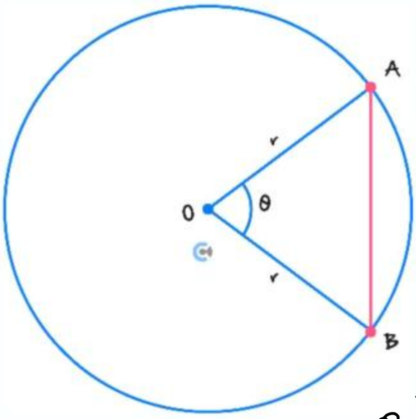


- The arc ACB and the corresponding chord AB are said to subtend the angle $\angle AOB$ at the centre of the circle.

$$l = 2\pi r \times \%$$

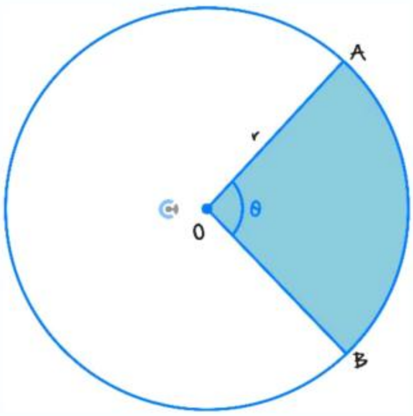
$$\text{Where, } \% = \frac{\theta^c}{2\pi} = \frac{\theta^\circ}{360}$$

Chord Length



$AB = 2r\sin\left(\frac{\theta}{2}\right)$

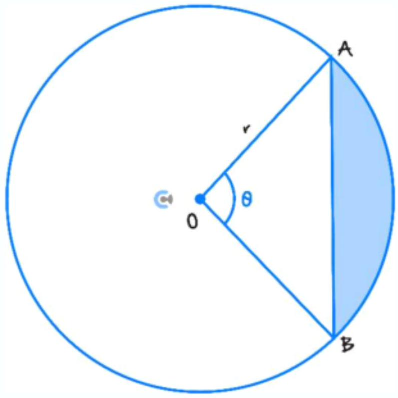
Area of Sector



Using degrees: Area of sector = $\frac{\pi r^2 \theta^\circ}{360}$

Using radians: Area of sector = $\frac{1}{2} r^2 \theta^c$

Area of Segment



- The **area of the segment** is the area of the sector OAB minus the area of the triangle OAB.
- Using the area of a triangle formula, the area of triangle OAB is $\frac{1}{2}r^2\sin(\theta)$.

$$A = \frac{1}{2}r^2(\theta - \sin\theta) \quad \textbf{(radians)}$$

$$A = \left(\frac{\theta}{360}\right)\pi r^2 - \frac{1}{2}r^2\sin\theta \quad \textbf{(degrees)}$$

Learning Objective: [3.1.3] – Apply angle of elevation/depression and bearing to solve geometric problems (Only 2D)

Key Takeaways

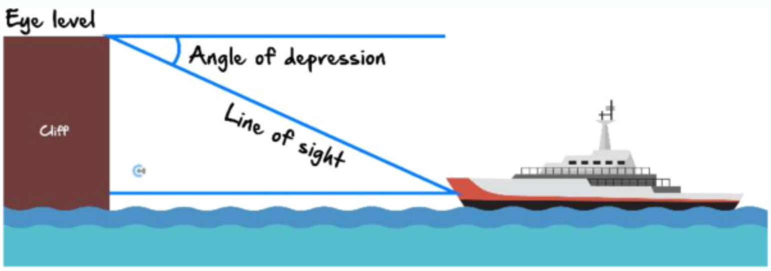
Angle of Elevation, Angle of Depression

Angle of Elevation

- The angle of elevation is the angle between the **horizontal** and a **direction** above the horizontal.

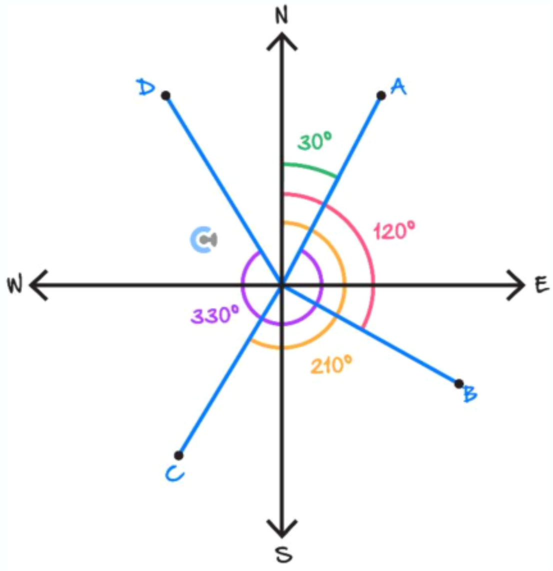
Angle of Depression

- The angle of elevation is the angle between the **horizontal** and a **direction** below the horizontal.



Bearing

- The **bearing** is the angle measured from North in the clockwise direction.



CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300 | **Email:** hello@contoureducation.com.au

VCE Specialist Mathematics ½

Free 1-on-1 Consults

What Are 1-on-1 Consults?

- **Who Runs Them?** Experienced Contour tutors (45+ raw scores and 99+ ATARs).
- **Who Can Join?** Fully enrolled Contour students.
- **When Are They?** 30-minute 1-on-1 help sessions, after school weekdays, and all day weekends.
- **What To Do?** Join on time, ask questions, re-learn concepts, or extend yourself!
- **Price?** Completely free!
- **One Active Booking Per Subject:** Must attend your current consultation before scheduling the next :)

SAVE THE LINK, AND MAKE THE MOST OF THIS (FREE) SERVICE!

Booking Link

bit.ly/contour-specialist-consult-2025