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Email: hello@contoureducation.com.au

VCE Specialist Mathematics ½
AOS 3 Revision [3.0]
Contour Check Solutions

Contour Check

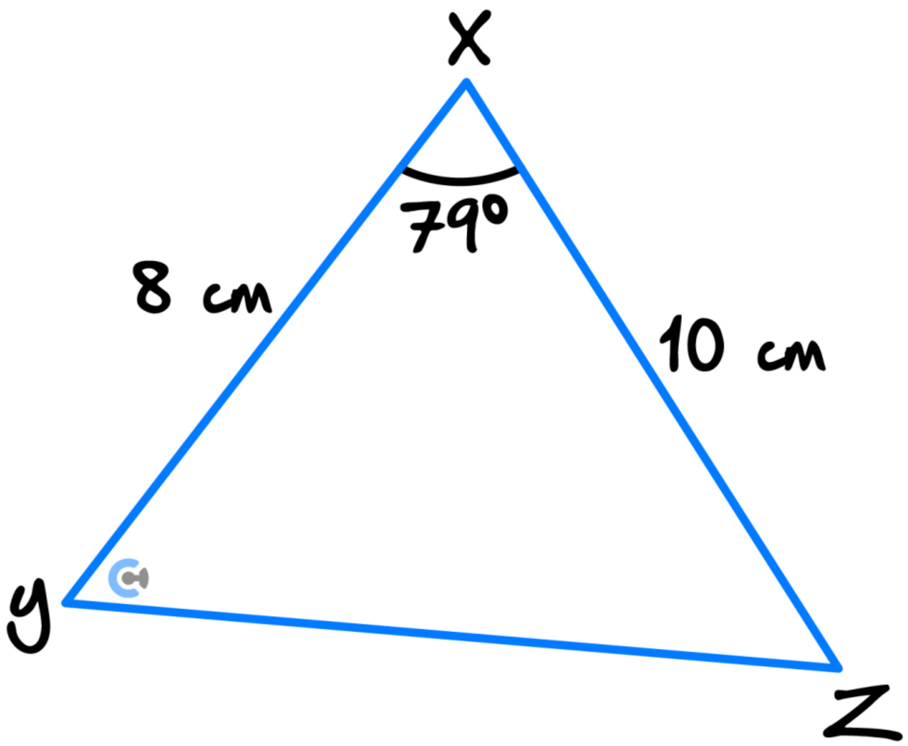
Section A: [3.1] - Trigonometry I (Checkpoints)

Sub-Section: [3.1.1] - Find Lengths, Angles and Area of Triangles Using Sine and Cosine Rule

Question 1

You may use a CAS for the following questions. Give your answers correct to two decimal places.

a. Find the length of YZ in the following triangle.

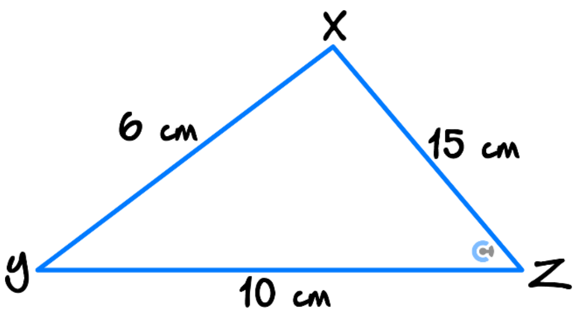


Using cosine rule:

$$(YZ)^2 = 8^2 + 10^2 - 2(8)(10)\cos(79°)$$

$$YZ = \sqrt{64 + 100 - 160\cos(79°)} \approx 11.55$$

b. Find the angle YXZ in the following triangle.

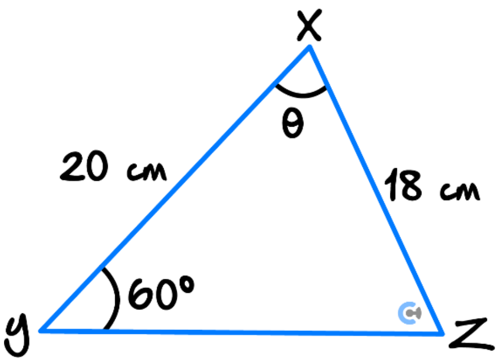


Let $\angle YXZ$ be x.
Using cosine rule:

$$\cos(x) = \frac{15^2 + 6^2 - 10^2}{2(15)(6)} = \frac{161}{180}$$

$$x = \cos^{-1}\left(\frac{161}{180}\right) \approx 26.56°$$

c. Find the angle θ in the following triangle given that $\angle XZY$ is acute.



Let $\angle XZY = \varphi$.
Using sine rule:

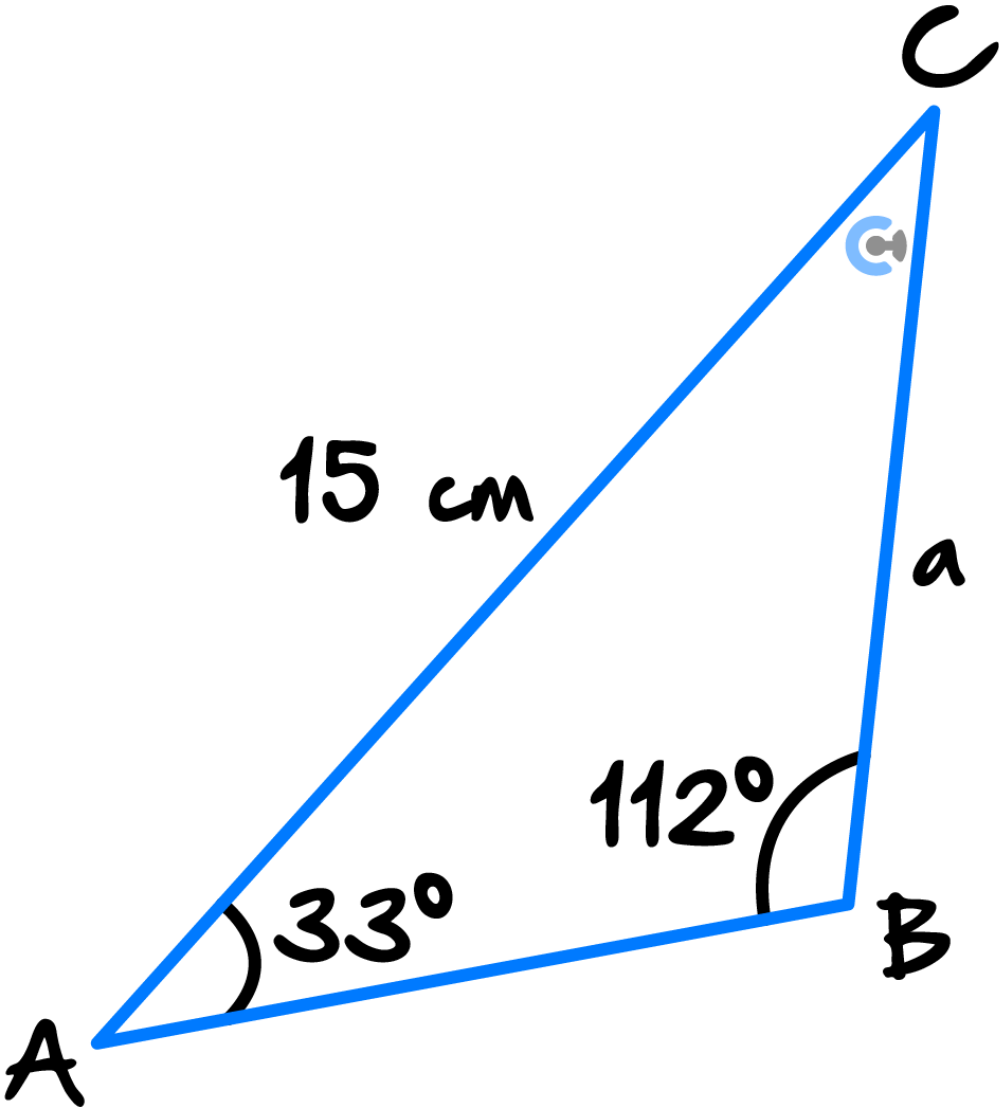
$$\frac{\sin\varphi}{20} = \frac{\sin(60°)}{18}$$

$$\varphi = \sin^{-1}\left(\frac{20\sin(60°)}{18}\right) \approx 74.21°$$

Now, by angle sum property of a triangle,
$\theta = 180° - 60° - 74.21° = 45.79°$.

Question 2 Tech-Active.

Find the area of the following triangle. Give your answer correct to two decimal places.



Using sine rule:

$$\frac{a}{\sin(33°)} = \frac{15}{\sin(112°)}$$

$$a = \frac{15\sin(33°)}{\sin(112°)} = 53.2238\ cm$$

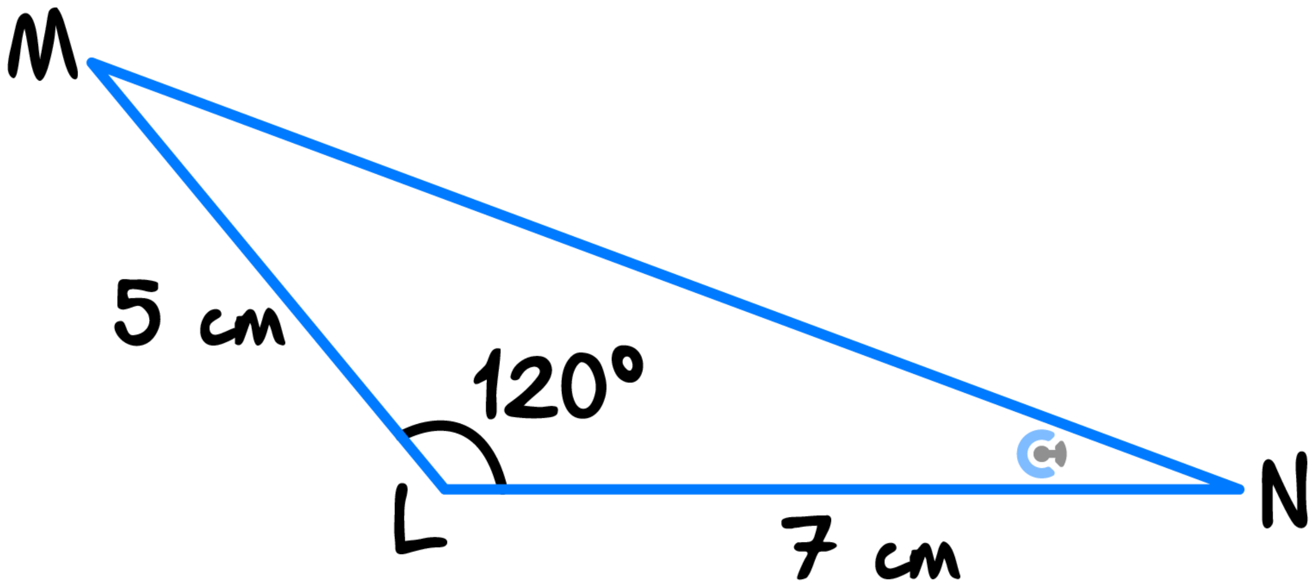
By angle sum property of triangle:

$$\angle ACB = 180° - 33° - 112° = 35°$$

Now, Area $= \frac{1}{2} \times 15 \times 53.2238 \times \cos(35°) \approx 326.99\ cm^2$

Question 3 Tech-Active.

Find all side lengths and angles for the following triangle. Give your answers correct to two decimal places.



By cosine rule:

$$MN^2 = 5^2 + 7^2 - 2(5)(7)\cos(120°) = 109$$

$$MN = \sqrt{109} = 10.44\ cm$$

By sine rule:

$$\frac{\sin(M)}{7} = \frac{\sin(120°)}{10.44}$$

$$M = \sin^{-1}\left(\frac{7\sin(120°)}{10.44}\right) \approx 35.50°$$

By angle sum property of triangle:

$$N = 180° - 120° - 35.50° = 24.50°$$

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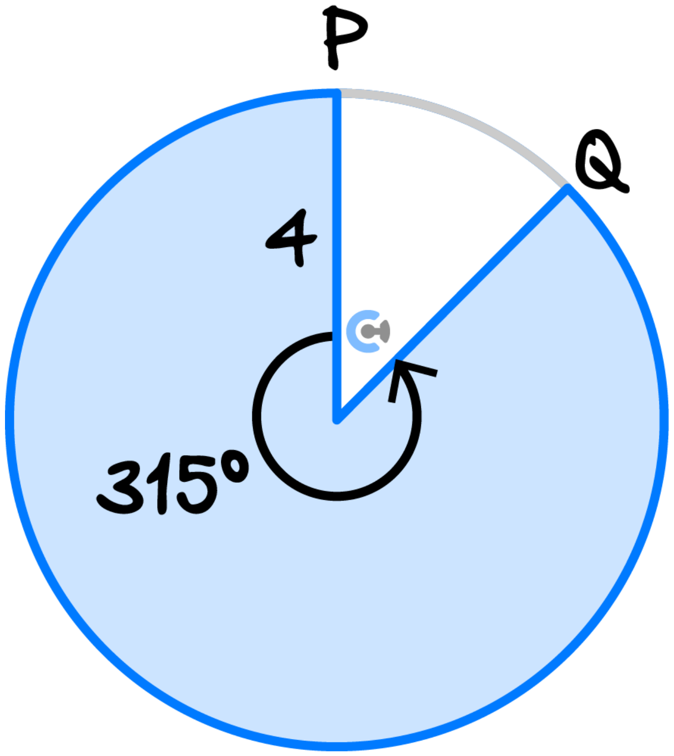
Sub-Section: [3.1.2] - Find Arc Lengths, Chord Lengths, Sector and Segment Areas

Question 4

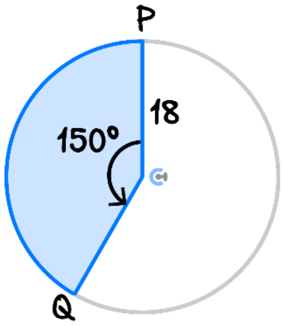
Consider the following circles:

a. Find the area of the shaded sector.



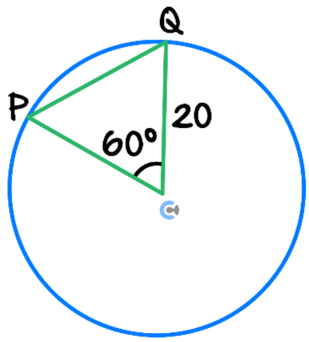
$$\text{Area} = \frac{\theta}{360} \times \pi r^2 = \frac{315}{360} \times \pi \times 4^2 = 14\pi$$

b. Find the length of the arc PQ.



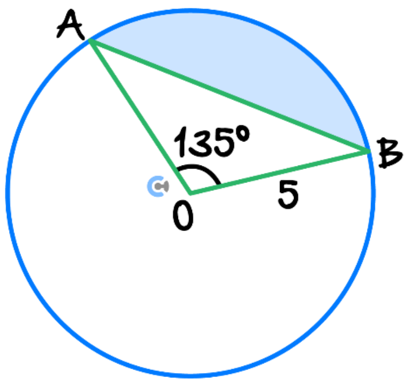
$$\text{Arc } PQ = \frac{\theta}{360} \times 2\pi r = \frac{150}{360} \times 2 \times \pi \times 18 = 15\pi$$

c. Find the length of the chord PQ.



$\because$The perpendicular from the centre of a circle to a chord bisects the chord.
Length of chord $PQ = 2 \times 20 \times \sin(30°) = 20$

d. Find the area of the shaded segment.

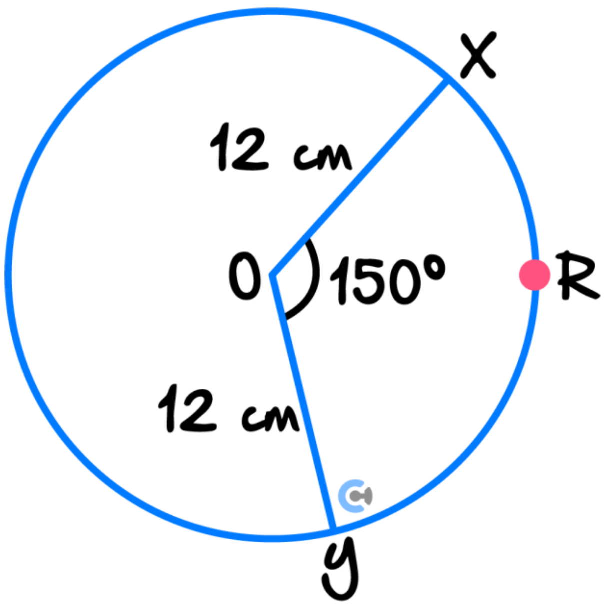


Area of segment = Area of sector − Area of triangle AOB

$$= \frac{135}{360} \times \pi \times 5^2 - \frac{1}{2} \times 5^2 \times \sin(135°) = \frac{75\pi}{8} - \frac{25\sqrt{2}}{4}$$

Question 5 Tech-Active.

A circle has a centre O and a radius of $12\ cm$. The angle subtended at O by arc XY has a magnitude of $150°$.



a. Find the exact length of the chord XY. [USE: $\sin(75°) = \frac{\sqrt{3}+1}{2\sqrt{2}}$]

$\because$The perpendicular from the centre of a circle to a chord bisects the chord.
Length of chord $PQ = 2 \times 12 \times \sin(75°) = 2 \times 12 \times \frac{\sqrt{3}+1}{2\sqrt{2}} = 6\sqrt{2}\left(\sqrt{3}+1\right)\, cm$

b. Find the exact length of the arc XY.

Arc $XY = \frac{\theta}{360} \times 2\pi r = \frac{150}{360} \times 2 \times \pi \times 12 = 10\pi$

c. Find the exact area of the minor sector XOY.

Area $= \frac{\theta}{360} \times \pi r^2 = \frac{150}{360} \times \pi \times 12^2 = 60\pi$

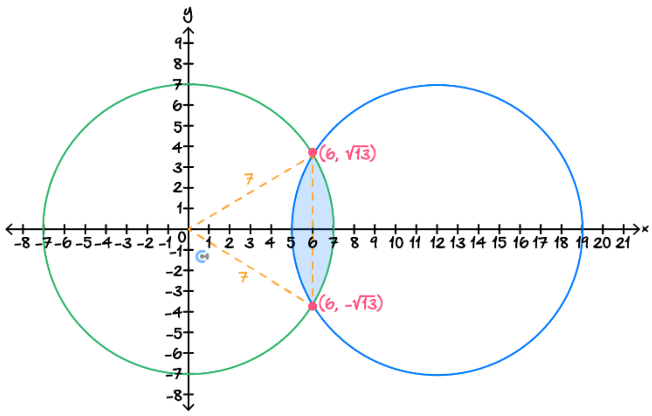
d. Find the magnitude of the angle XOR, in degrees, if the minor arc has a length of 5 cm. Give your answer correct to two decimal places.

Let $\angle XOR = \theta$ in radians. Then $12\theta = 5 \Rightarrow \theta = \left(\frac{5}{12}\right)^c$.

In degrees, $\frac{5}{12} \times \frac{180}{\pi} = \frac{75}{\pi} \approx 23.87°$

Question 6 Tech-Active.

Two circles, each with a radius of 7 cm, have their centres 12 cm apart. Calculate the exact area of the region common to both circles and then round this result to two decimal places.



The equation of circle will be $x^2 + y^2 = 7^2$ and $(x - 12)^2 + y^2 = 7^2$.

By symmetry the required area is double the area of one of the segments and we find that the circle intersect each other at $(6, \sqrt{13})$ and $(6, -\sqrt{13})$ by solving the equation of circles.

Isosceles triangle with side lengths 7, 7 and $2\sqrt{13}$. Then,

$$\cos(\theta) = \frac{49 + 49 - (2\sqrt{13})^2}{2 \times 7 \times 7} = \frac{23}{49}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{23}{49}\right)$$

then, $\sin^2(\theta) + \cos^2(\theta) = 1 \Rightarrow \sin(\theta) = \frac{12\sqrt{13}}{49}$

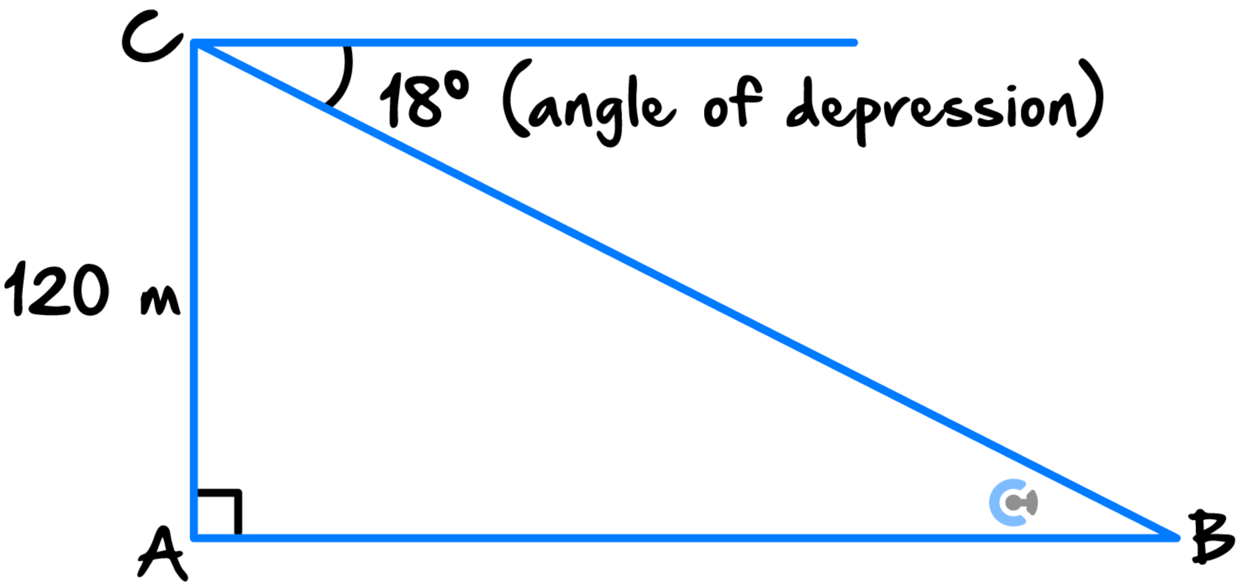
$$\text{Common area} = 2 \times \frac{1}{2} \times 7^2 \left(\cos^{-1}\left(\frac{23}{49}\right) - \frac{12\sqrt{13}}{49}\right)$$

$$= 49\cos^{-1}\left(\frac{23}{49}\right) - 12\sqrt{13} \approx 9.76\ cm^2$$

Sub-Section: [3.1.3] - Apply Angle of Elevation/Depression and Bearing to Solve Geometric Problems (Only 2D)

Question 7 Tech-Active.

a. A cliff is 120 m high. A person standing at the top of the cliff observes a boat in the ocean at an angle of depression of 18°. Calculate the horizontal distance between the boat and the base of the cliff, correct to the nearest metre.



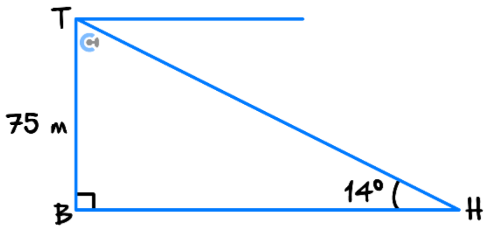
Angle of depression = Angle of elevation

So, $\angle ABC = 18°$

Now, $\tan(18°) = \frac{AC}{AB} = \frac{120}{AB}$

$$AB = \frac{120}{\tan(18°)} = 369\ m$$

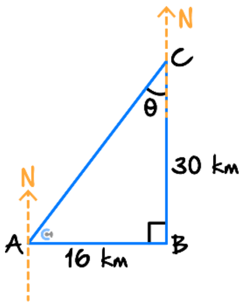
b. A tree stands $75\ m$ tall. A hiker on the ground observes the top of the tree at an angle of elevation of $14°$. Calculate the horizontal distance from the hiker to the base of the tree, correct to the nearest metre.



In ΔTBH, $\tan(14°) = \frac{75}{BH}$

$$BH = \frac{75}{\tan(14°)} = 301\ m$$

c. An airplane flies $16\ km$ due east from Point A to Point B. It then changes direction and flies $30\ km$ due north to Point C. Calculate the distance and bearing of Point C from Point A, giving your answers correct to two decimal places.



In ΔABC, $\tan\theta = \frac{BC}{AB} = \frac{30}{16} = \frac{15}{8}$

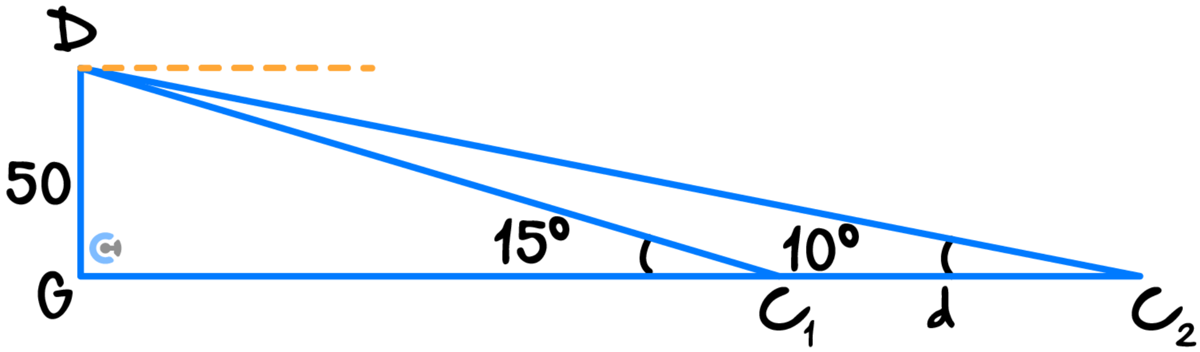
$$\theta = \tan^{-1}\left(\frac{15}{8}\right) = 61.93°$$

Thus the bearing is $180 + 61.93 = 241.93°$

Then, $AC^2 = 16^2 + 30^2 \Rightarrow AC = \sqrt{1156} = 34.00\ km$

Question 8 Tech-Active.

A drone is hovering at a constant height of 50 m above a straight road. It tracks two cars moving along the road, both in line with each other. The angles of depression to the cars are 10° and 15°. Calculate the distance between the two cars, giving your answer correct to two decimal places.



In $\Delta DGC_1, \tan(15°) = \frac{50}{GC_1} \Rightarrow GC_1 = \frac{50}{\tan(15°)} \approx 186.60$

In $\Delta DGC_2, \tan(10°) = \frac{50}{GC_1} \Rightarrow GC_2 = \frac{50}{\tan(10°)} \approx 283.56$

Thus, $d = GC_2 - GC_1 = 283.56 - 186.60 = 96.96$ metres

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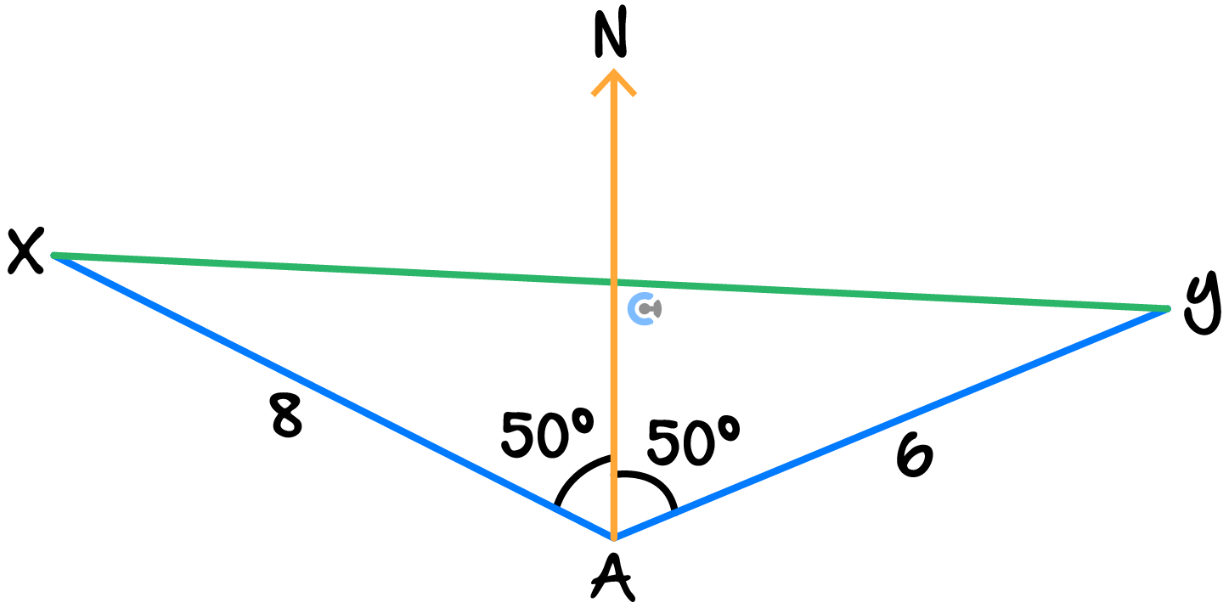
Question 9 Tech-Active.

From an air traffic control tower, two airplanes X and Y are on bearings of 310° and 050°, respectively. The distance XA (from the tower to airplane X) is 8 km, and the distance YA (from the tower to airplane Y) is 6 km. Find the distance XY, giving your answer correct to two decimal places.



From our diagram it is clear that we should use the cosine rule to find XY with $\theta = 50° + 50° = 100°$.

$$XY^2 = 8^2 + 6^2 - 2(8)(6)\cos(100°) = 116.6702251$$

$$XY = \sqrt{116.6702251} \approx 10.80\ km$$

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Section B: [3.2] - Trigonometry II (Checkpoints)

Question 10

Simplify the following expressions:

a. $\cos(\pi - x)$

Since $\pi - x$ is an angle in second quadrant, $\cos(\pi - x) = -\cos(x)$.

b. $\tan(\pi + x)$

Since, $\pi + x$ is an angle in third quadrant where tan values are positive,

$\tan(\pi + x) = \tan(x)$.

c. $\sin\left(x - \frac{\pi}{2}\right)$

$$\sin\left(x-\frac{\pi}{2}\right) = \sin\left(-\left(\frac{\pi}{2}-x\right)\right) = -\sin\left(\frac{\pi}{2}-x\right) = -\cos(x)$$

Question 11

If $\cos(x) = \frac{5}{13}$, where x is an angle in the first quadrant, evaluate the following:

a. $\cos(\pi - x)$

$$cos(\pi - x) = -\cos(x) = \frac{-5}{13}$$

b. $\sin(\pi + x)$

$$\sin(\pi + x) = -\sin(x) = \frac{-12}{13}$$

c. $\tan\left(\frac{\pi}{2} - x\right)$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot(x) = \frac{5}{12}$$

Question 12

If $\sin(x) = \frac{6}{11}$, where $\frac{\pi}{2} \leq x \leq \pi$, evaluate the following:

a. $\sin(\pi + x)$

$$\sin(\pi + x) = -\sin x = \frac{-6}{11}$$

b. $-\tan\left(\frac{\pi}{2}+x\right)$

$$-\tan\left(\frac{\pi}{2}+x\right) = -(-\cot x) = \cot x = \frac{-\sqrt{85}}{6}$$

c. $\cos(\pi - x)$

$$\cos(\pi - x) = -\cos x = -\left(\frac{-\sqrt{85}}{11}\right) = \frac{\sqrt{85}}{11}$$

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Question 13 Tech-Active.

If $\cos(x) = -\frac{8}{15}$, where x is an angle which lies in the third quadrant, evaluate $\sin(\pi + x)$.

$$\sin(\pi + x) = -\sin x = -\left(\frac{-\sqrt{161}}{15}\right) = \frac{\sqrt{161}}{15}$$

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Sub-Section: [3.2.2] - Find Particular and General Solutions

Question 14

Solve the following trigonometric equations over the specified domain:

a. $2\cos(3x) = -\sqrt{3}$, for $x \in [0, \pi]$.

$$\cos(3x) = \frac{-\sqrt{3}}{2} \to 3x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{18}, \frac{31\pi}{18}$$

$$x = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}$$

b. $\sqrt{2}\sin(2x) = 1$, for $x \in [0, 2\pi]$.

$$\sin(2x) = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

c. $4\tan(x) - 2 = 2, x \in [-\pi, \pi]$.

$$4\tan(x) = 4 \quad \rightarrow \quad \tan(x) = 1$$
$$x = -\frac{3\pi}{4}, \; x = \frac{\pi}{4}$$

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Question 15

Find the general solution to the following trigonometric equations:

a. $\sin\left(-3x+\frac{\pi}{4}\right)=\frac{1}{2}$

$$\sin\left(-3x+\frac{\pi}{4}\right)=\frac{1}{2}$$

$$-3x+\frac{\pi}{4}=\frac{\pi}{6}+2\pi n \text{ or } -3x+\frac{\pi}{4}=\frac{5\pi}{6}+2\pi n, \quad n\in Z$$

$$x=\frac{\pi}{36}-\frac{2\pi n}{3} \text{ or } x=-\frac{7\pi}{36}-\frac{2\pi n}{3}, \quad n\in Z$$

b. $2\cos\left(2x-\frac{\pi}{3}\right)=1$

$$2\cos\left(2x-\frac{\pi}{3}\right)=1$$

$$\cos\left(2x-\frac{\pi}{3}\right)=\frac{1}{2}$$

$$2x-\frac{\pi}{3}=\frac{\pi}{3}+2\pi n \text{ or } 2x-\frac{\pi}{3}=\frac{5\pi}{3}+2\pi n$$

$$x=\frac{\pi}{3}+\pi n \text{ or } x=\pi+\pi n, \quad n\in Z$$

c. $\tan\left(\frac{\pi}{3}x+\frac{\pi}{6}\right)=1$

$$\tan\left(\frac{\pi}{3}x+\frac{\pi}{6}\right)=1$$

$$\frac{\pi}{3}x+\frac{\pi}{6}=\frac{\pi}{4}+\pi n$$

$$x=3n+\frac{1}{4}, \quad n\in Z$$

Question 16

Consider the function $f(x) = \sqrt{3}\tan\left(2x + \frac{\pi}{4}\right) - 1$.

a. Find the general solution to $f(x) = 0$.

$$\tan\left(2x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{3}$$

$$2x + \frac{\pi}{4} = \frac{\pi}{6} + \pi n$$

$$x = -\frac{\pi}{24} + \frac{\pi n}{2}, \quad n \in Z$$

b. Hence, solve $f(x) = 0$ for $x \in \left[0, \frac{3\pi}{2}\right]$.

$$x = \frac{11\pi}{24}, \ x = \frac{23\pi}{24}, \ x = \frac{11\pi}{24}$$

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Question 17 Tech-Active.

Find the general solution to $\sqrt{2}\cos\big(\pi(x+1)\big) = 1$.

$$x = -\frac{5}{4} + 2n,\ x = -\frac{3}{4} + 2n, \quad n \in Z$$

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In[3]:= Solve[Sqrt[2] Cos[Pi (x + 1)] == 1] // Expand
Out[3]= {{x → -5/4 + 2 c1 if c1 ∈ ℤ}, {x → -3/4 + 2 c1 if c1 ∈ ℤ}}
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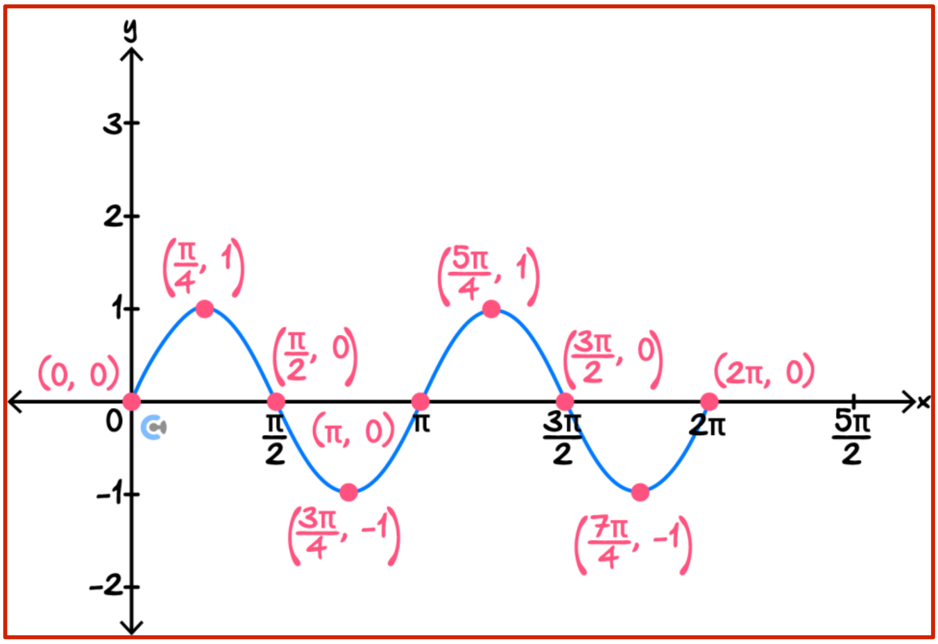
Sub-Section [3.2.3]: Graph Sine, Cosine and Tangent Functions

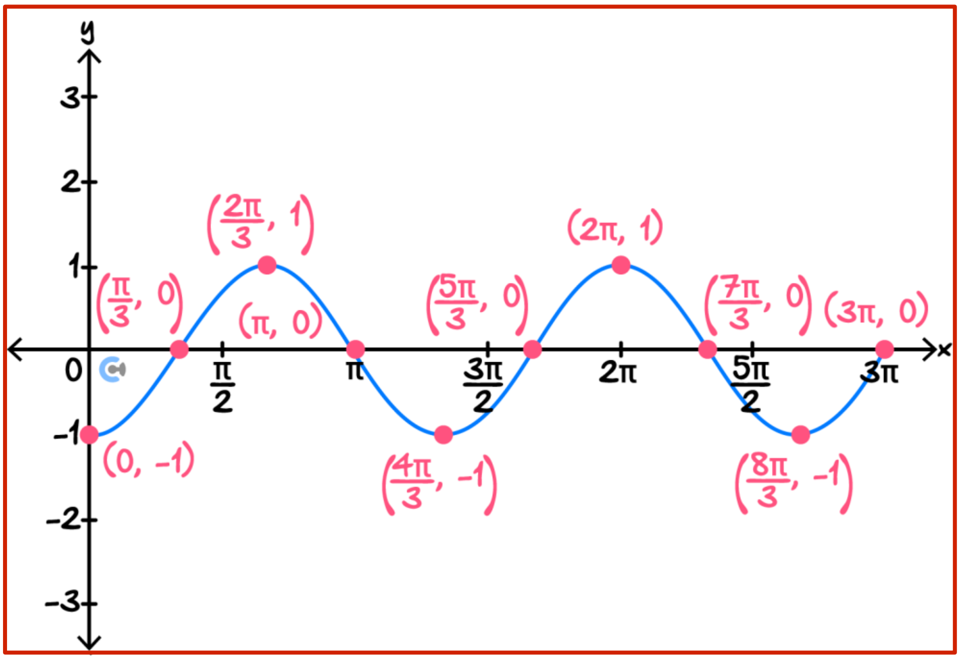
Question 18

Sketch the graphs of the following functions over the indicated domain. Label all axes intercepts and endpoints with coordinates, and label asymptotes with equations.

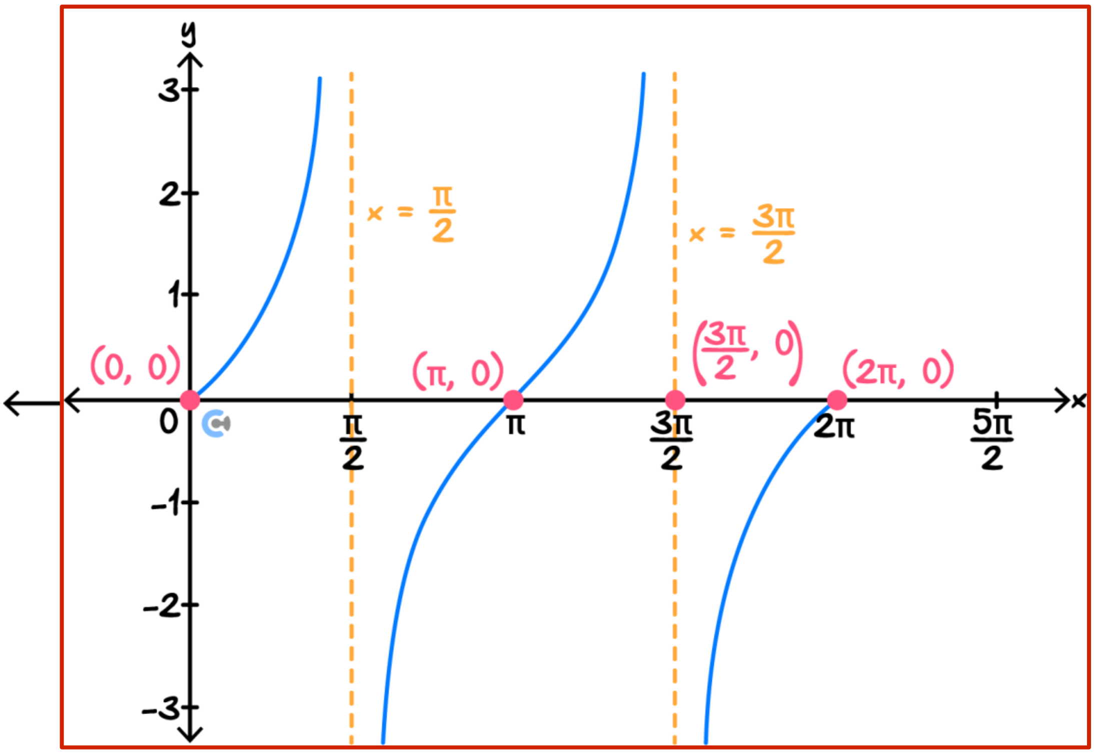
a. $y = \sin(2x), x \in [0, 2\pi]$.



b. $y = -\cos\left(\frac{3x}{2}\right), x \in [0, 3\pi]$.



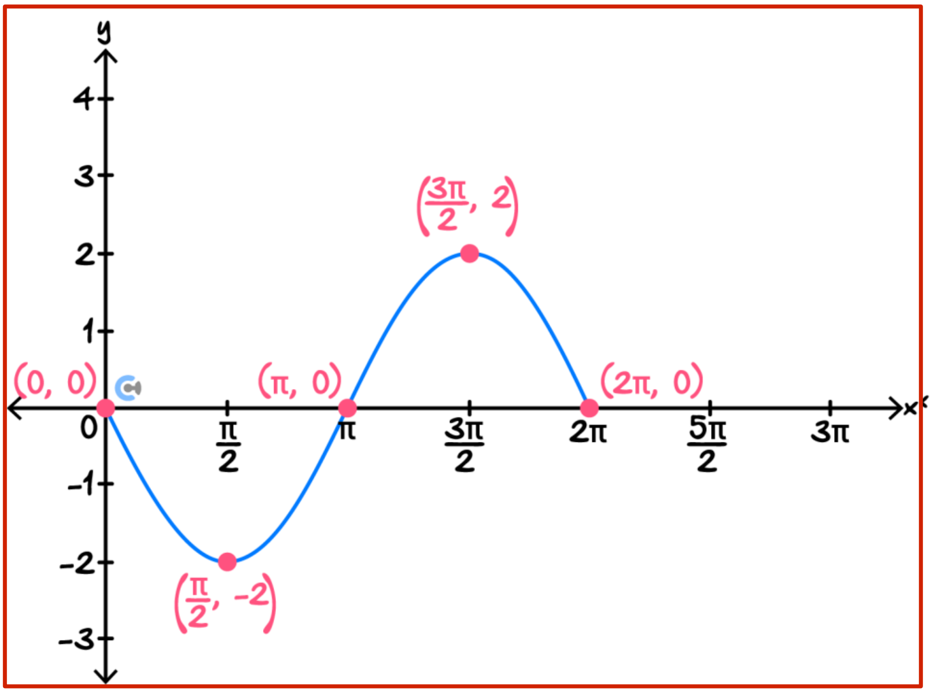
c. $y = \tan(x - \pi), x \in [0,2\pi]$.



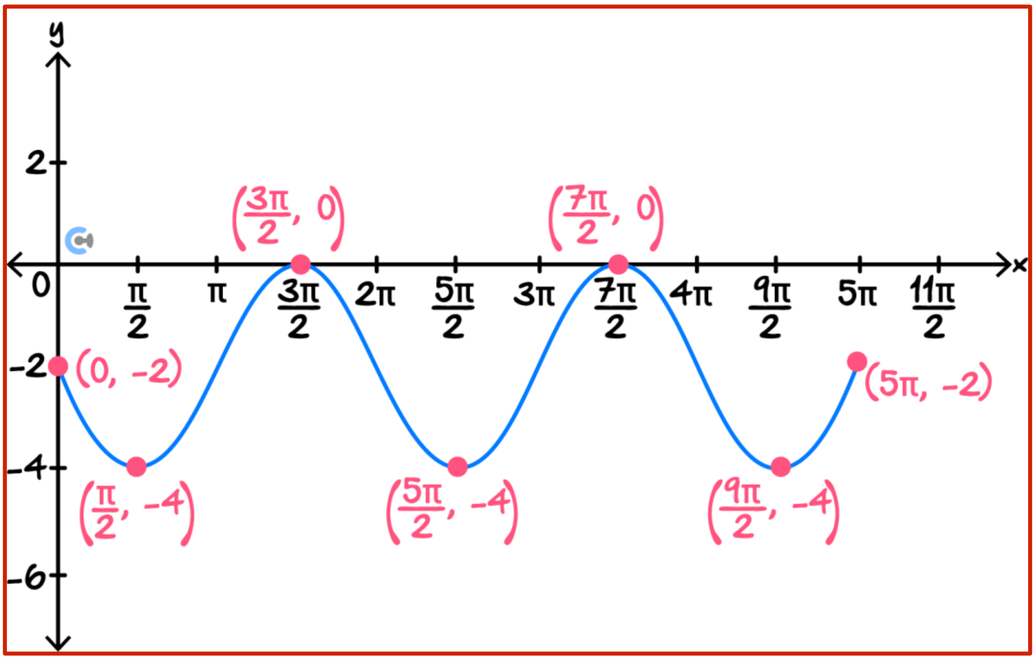
Question 19

Sketch the graphs of the following functions over the indicated domain. Label all axes intercepts, turning points and endpoints with coordinates, and label asymptotes with equations.

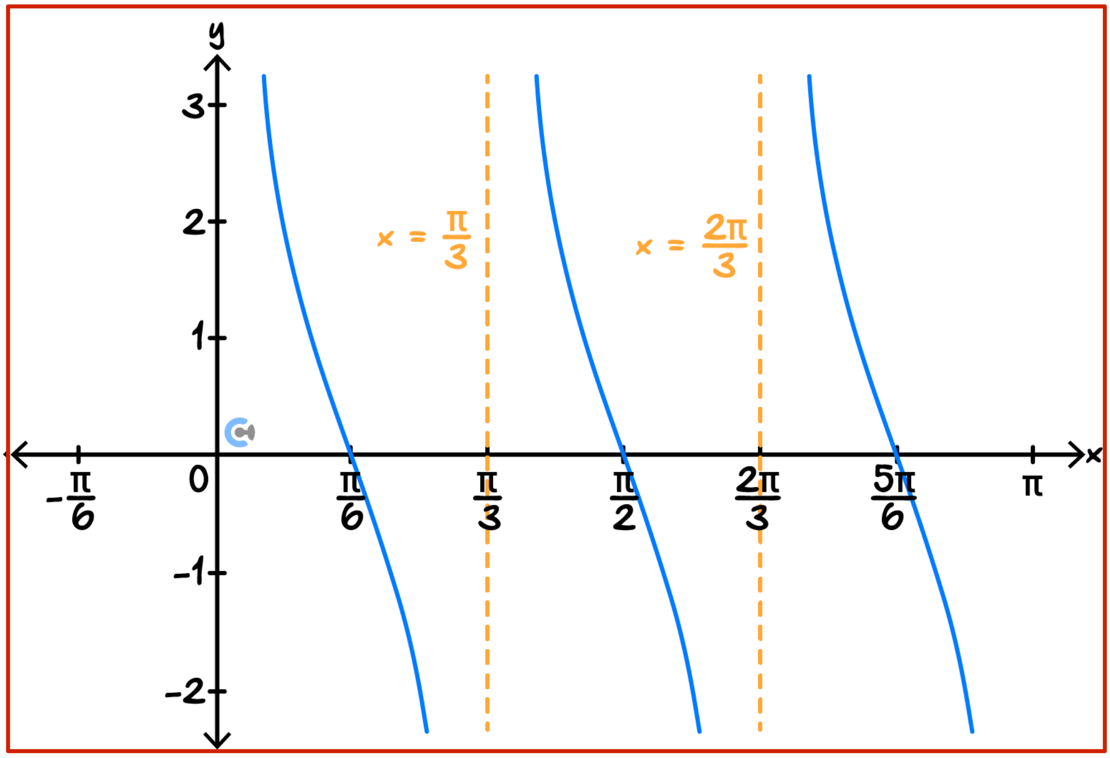
a. $y = 2\cos\left(x + \frac{\pi}{2}\right), x \in [0,2\pi]$.



b. $y = -2\sin(x) - 2, x \in [0, 5\pi]$.



c. $y = -2\tan\left(3x + \frac{\pi}{2}\right),\ x \in [0, \pi]$.



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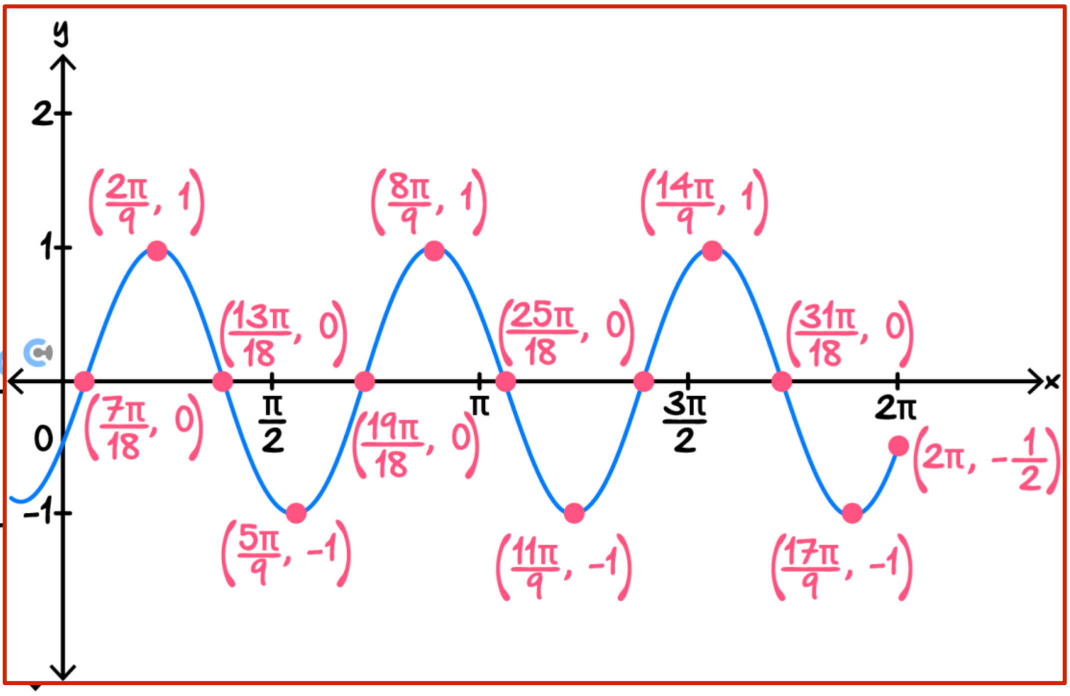
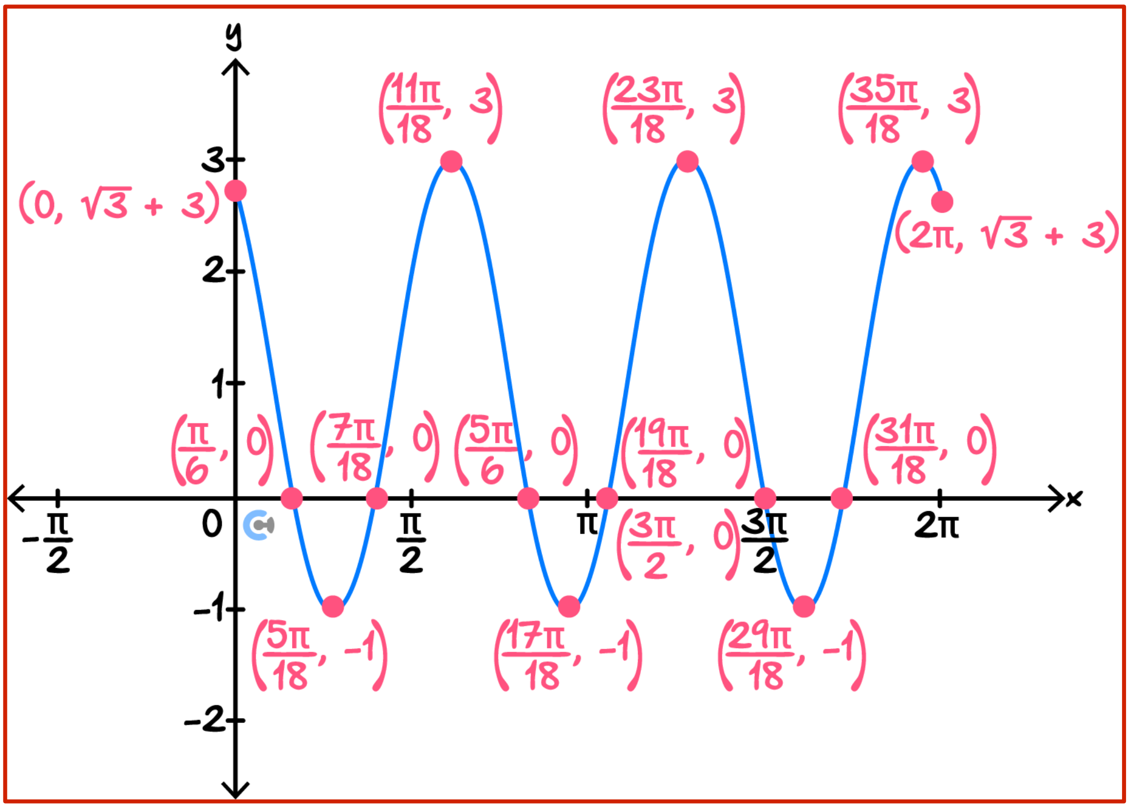
Question 20

Sketch the graphs of the following functions over the indicated domain. Label all axes intercepts, turning points and endpoints with coordinates, and label asymptotes with equations.

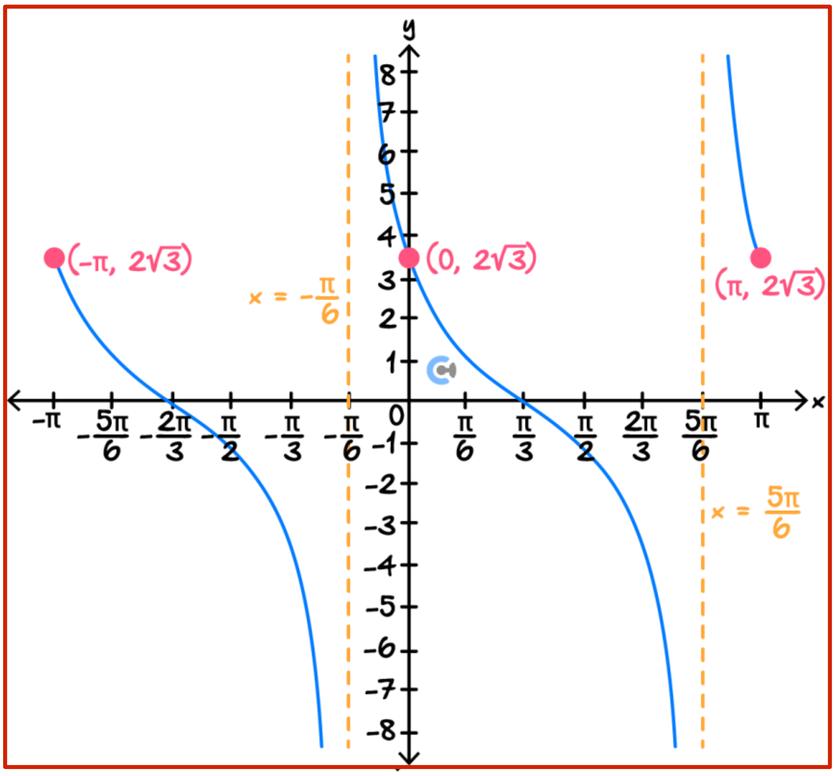
a. $y = \sin\left(3x - \frac{\pi}{6}\right), x \in [0, 2\pi]$.



b. $y = 2\sin\left(\frac{\pi}{3} - 3x\right) + 1, x \in [0, 2\pi]$.

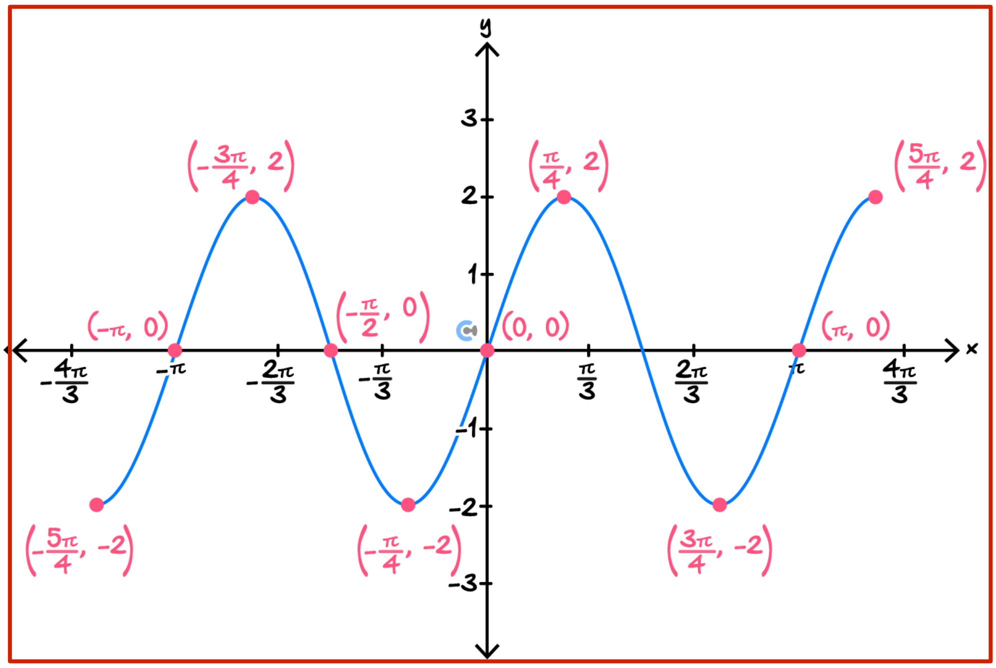


c. $y = 2\tan\left(\frac{\pi}{3} - x\right), x \in [-\pi, \pi]$.



Question 21 Tech-Active.

Sketch the graph of the equation $y = 2\cos\left(2x - \frac{\pi}{4}\right)$. Label all axes intercepts, turning points and endpoints with coordinates.

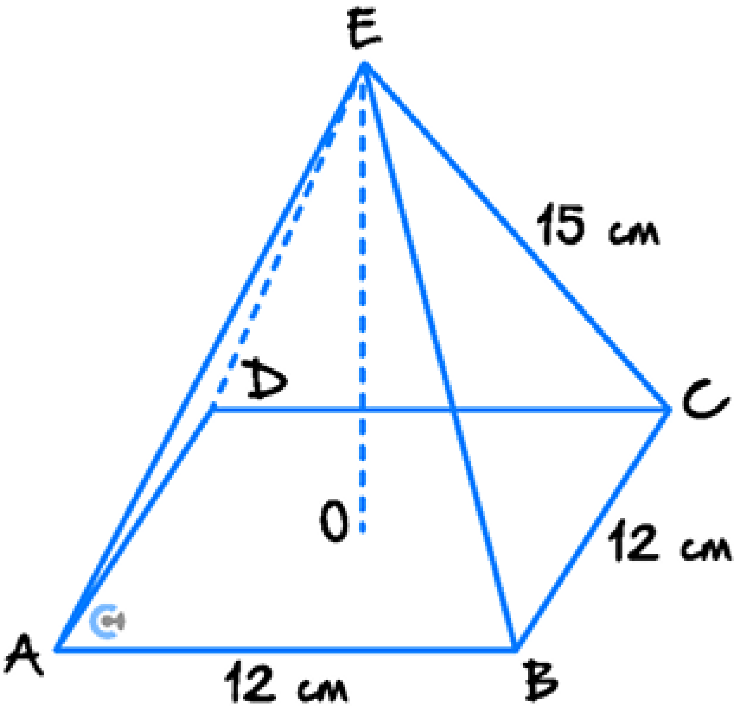


Section C: [3.3] – Trigonometry Exam Skills (Checkpoints)

Sub-Section [3.3.1] and [3.3.2]: Apply Trigonometry to Solve Problems in 3D and Find the Angle between Planes

Question 22

A square pyramid $PQRST$ stands on level horizontal ground. The vertex of the pyramid is at T. The points P, Q, R, S are the corners of a square of side 18 cm, whose diagonals intersect at the point O. Each of the sloping edges of the pyramid has a length of 24 cm.



a. Calculate the length OR.

By Pythagoras Theorem,
$OR^2 = 9^2 + 9^2 = 162 \rightarrow OR = 9\sqrt{2}\ cm$

b. Calculate the volume of the pyramid. (Recall: $V = \frac{1}{3} \times base \times height$)

$OT^2 = TR^2 - OR^2$

$OT^2 = 576 + 162 = 306 \;\rightarrow OT = \sqrt{738} = 3\sqrt{82}\; cm$

$Base\ Area = 18 \times 18 = 324\; sq.cm$

Therefore, $V = \frac{1}{3} \times 324 \times\ 3\sqrt{82} = 324\sqrt{82}\; cubic\ cm$

c. Calculate the total surface area of the pyramid.

Use Pythagoras theorem to find the heigh of the triangular face.

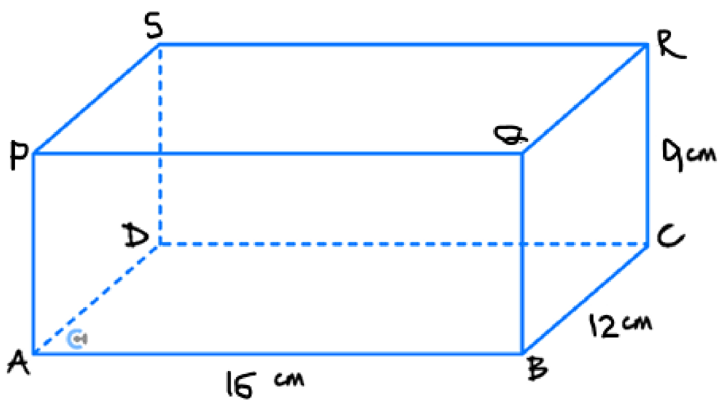
$(height)^2 = 24^2 - 9^2 = 495 \;\rightarrow\; height = 3\sqrt{55}\; cm$

$Then, face\ area = \frac{1}{2} \times 18 \times\ 3\sqrt{55} = 27\sqrt{55}\; cm$

$Surface\ Area = 324 + 4 \times 27\sqrt{55} = 324 + 108\sqrt{55}\; cm$

Question 23

The figure shows a cuboid $ABCDPQRS$ standing on level horizontal ground. The lengths of AB, BC and CR are 16 cm,12 cm and 9 cm, respectively.



a. Find the length of AR.

$$AR = \sqrt{16^2 + 12^2} = \sqrt{400} = 20\ cm$$
$$Then, AR = \sqrt{9^2 + 20^2} = \sqrt{481}\ cm$$

b. Calculate the angle AR makes with the ground, correct to two decimal places.

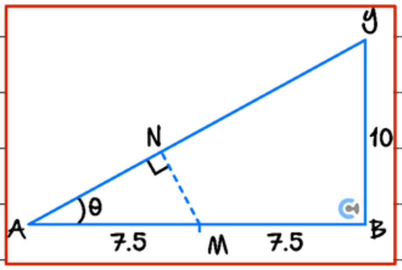
$$\tan(\alpha) = \frac{9}{20} \to \alpha \approx 24.23°$$

c. Determine the area of the triangle ABY.

$$AR^2 = AB^2 + BR^2$$
$$BR = \sqrt{481 - 256} = 15\ cm$$
$$Area(ABR) = \frac{1}{2} \times 15 \times 16 = 120\ sq.cm$$

The point M is the midpoint of AB and the point N lies on AR.

d. The point M is the midpoint of AB and the point N lies on AR. Calculate the length of MN, given that MN is perpendicular to AR. Give your answer correct to two decimal places.



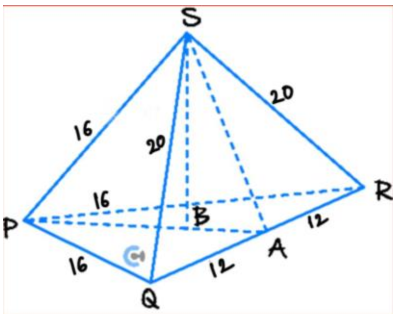
$\tan(\theta) = \frac{15}{16} \to \theta \approx 43.15°$

Then, $\sin(\theta) = \frac{NM}{7.5} \to NM = 5.13\ cm$

Question 24

A pyramid $PQRS$ has a triangular horizontal base PQR, where $PQ = PR = 16\ m$ and $RQ = 24\ m$. The vertex of the pyramid S lies directly above the level of PQR so that $SQ = SR = 20\ m$ and $SP = 16\ m$.

a. Show that the shortest distance of S from the base PQR is $2\sqrt{57}\ m$.



The shortest distance is perpendicular to the base. We must find SB.

So, $|AP| = \sqrt{16^2 - 12^2} = 4\sqrt{7}\ m$

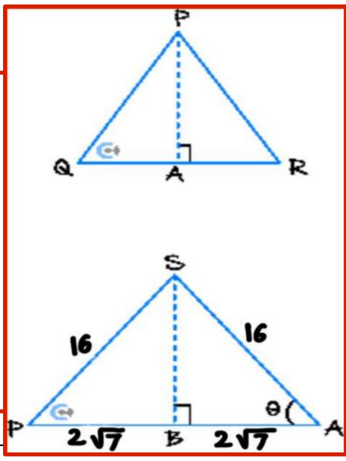
$|AS| = \sqrt{20^2 - 12^2} = 16\ m$

PQR is an isosceles triangle

So, B is the midpoint of AP so $PB = AB = 2\sqrt{7}\ m$

Hence, by Pythagoras theorem in SPB

$$SB = \sqrt{16^2 - (2\sqrt{7})^2} = \sqrt{228} = 2\sqrt{57}\ m$$



b. Calculate, in degrees correct to two decimal places, the acute angle between:

i. The plane SQR and the plane PQR.

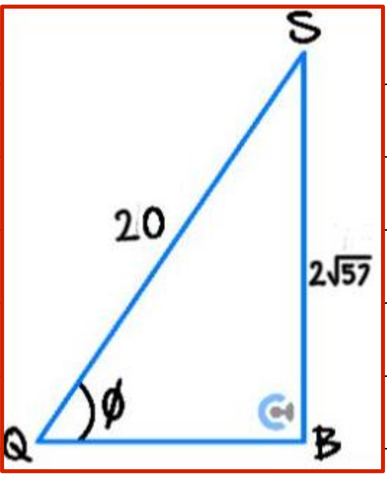
The angle in the diagram above.

$$\cos(\theta) = \frac{2\sqrt{7}}{16} \to \theta \approx 70.69°$$

ii. The edge SQ and the plane PQR.

$$\sin(\emptyset) = \frac{SB}{SQ} = \frac{2\sqrt{57}}{20} = \frac{\sqrt{57}}{10}$$

$$\to \emptyset \approx 40.92°$$



c. Determine, as an exact surd, the shortest distance of P from the plane SQR.

HINT: Compute the volume of the pyramid in two different ways.

$$\text{Area of } \Delta PQR = \frac{1}{2}|QR||AP| = \frac{1}{2} \times 24 \times 4\sqrt{7} = 48\sqrt{7}$$

$$Volume\ of\ Pyramid = \frac{1}{3} \times (base\ area) \times height$$

$$= \frac{1}{3} \times 48\sqrt{7} \times 2\sqrt{57}$$

$$= 32\sqrt{399} \text{ cubic m}$$

$$Area\ of\ \Delta SQR = \frac{1}{2}|QR||AS| = \frac{1}{2} \times 24 \times 16 = 192 \text{ sq.m}$$

$$Volume\ of\ Pyramid\ is\ also\ equal\ to:$$

$$= \frac{1}{3} \times (area\ of\ base\ SQR) \times (height\ from\ P\ to\ SQR)$$

$$\to \frac{1}{3} \times (192) \times h = 64h$$

$$64h = 32\sqrt{399}$$

$$h = \frac{1}{2}\sqrt{399}\ m$$

Section D: [3.4] - Advanced Trigonometric Functions (Checkpoints)

Sub-Section [3.4.1]: Trigonometric Identities and Solving Exact Values of Reciprocal Functions

Question 25

Evaluate the following:

a. $\sec\left(\frac{\pi}{4}\right)$

$$\frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

b. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$\frac{\pi}{6}$$

c. $\tan^{-1}(1)$

$$\frac{\pi}{4}$$

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Question 26

Evaluate the following:

a. $\cot\left(\frac{11\pi}{6}\right)$

$$\frac{1}{\tan\left(-\frac{\pi}{6}\right)} = -\sqrt{3}$$

b. $\text{cosec}\left(\frac{7\pi}{3}\right)$

$$\frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

c. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$$-\frac{\pi}{6}$$

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Question 27

Prove the identity $(\cot x + \operatorname{cosec} x)^2 = \frac{1+\cos x}{1-\cos x}$.

$$\begin{aligned}(\cot x + \operatorname{cosec} x)^2 &= \left(\frac{\cos x + 1}{\sin x}\right)^2\\ &= \frac{(\cos x + 1)^2}{\sin^2 x}\\ &= \frac{(\cos x + 1)^2}{1 - \cos^2 x}\\ &= \frac{(\cos x + 1)^2}{(1 + \cos x)(1 - \cos x)}\\ &= \frac{1 + \cos x}{1 - \cos x}\end{aligned}$$

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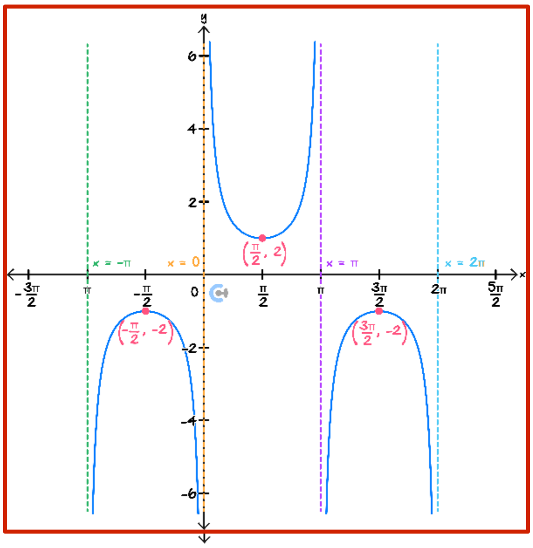
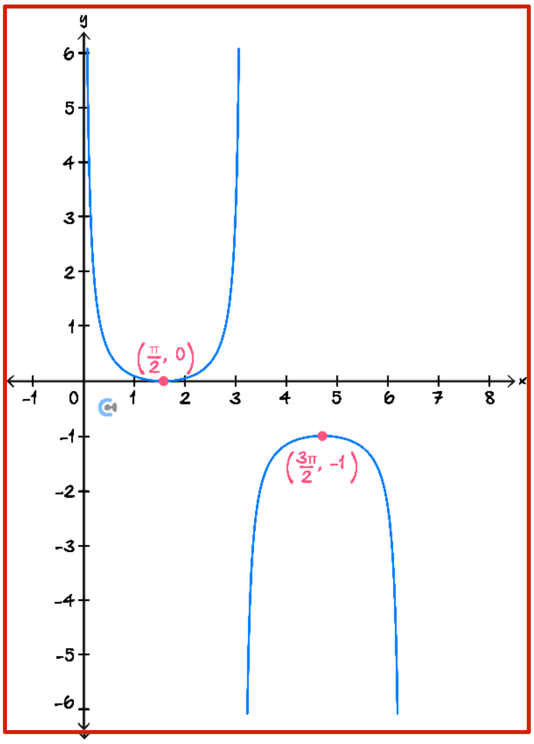
Sub-Section [3.4.2]: Graph Reciprocal Trigonometric Functions

Question 28

a. Sketch the graph of $y = 2\sec\left(x - \frac{\pi}{2}\right)$ for $-\pi < x < 2\pi$, labelling all stationary points, axes intercepts and asymptotes with their equations.

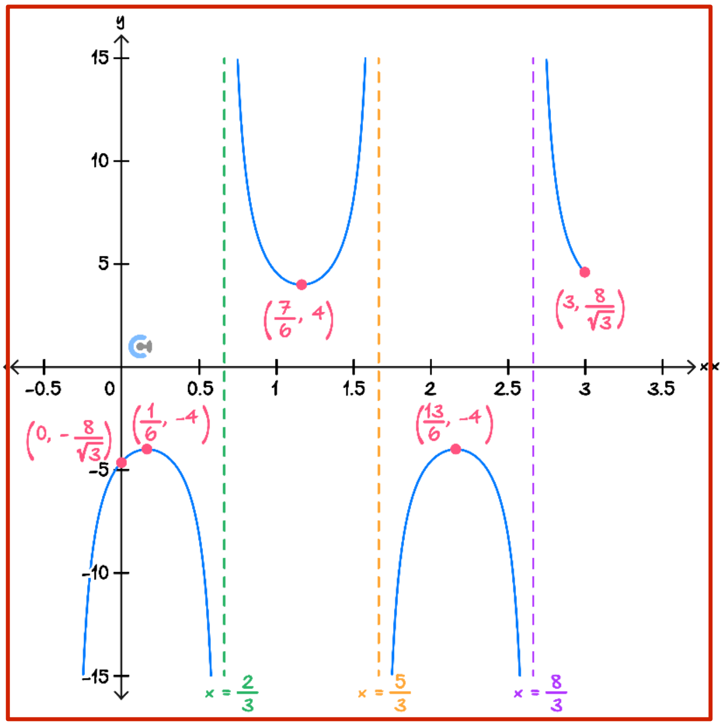


b. Sketch the graph of $\frac{\text{cosec}(x)}{2} - \frac{1}{2}$ for $0 < x < 2\pi$, labelling all stationary points, axes intercepts and asymptotes with their equations.

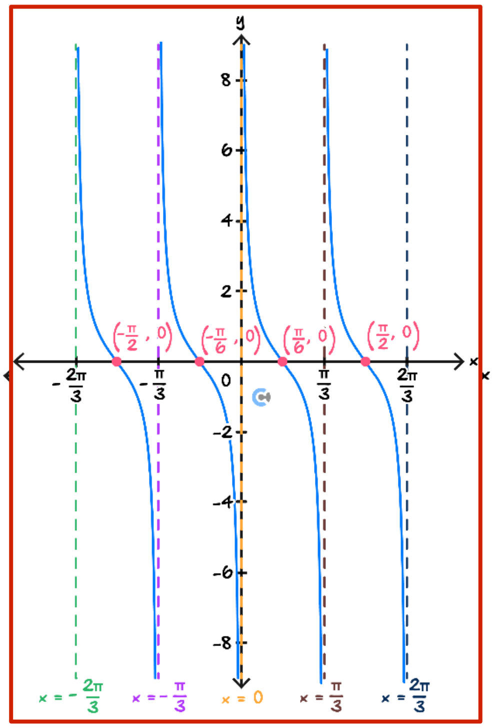


Question 29

a. Sketch the graph of $y = 4\text{cosec}\left(7\pi x - \frac{2\pi}{3}\right)$ for $-1 \leq x \leq 3$, labelling all stationary points, axes intercepts and asymptotes with their equations.

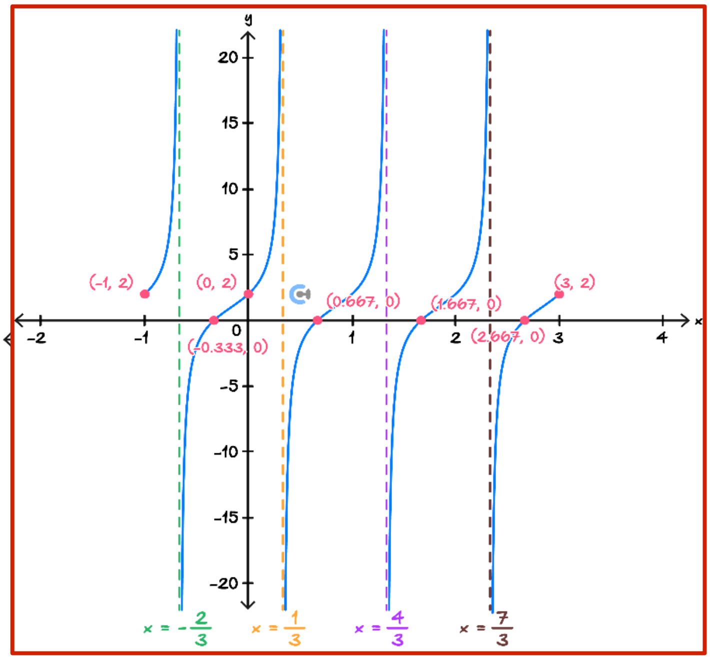


b. Sketch the graph of $y = -\cot(\pi - 3x)$ for $-\frac{2\pi}{3} < x < \frac{2\pi}{3}$, labelling all stationary points, axes intercepts and asymptotes with their equations.

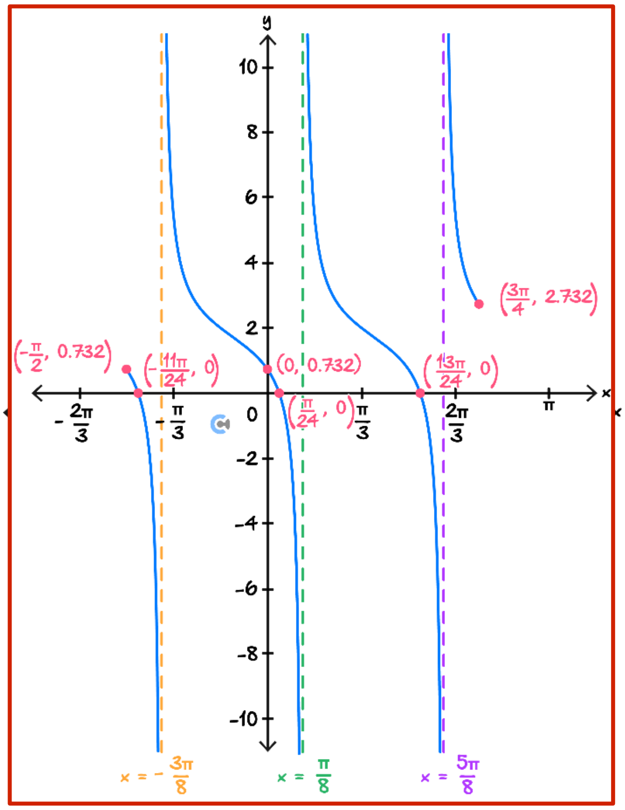


Question 30

a. Sketch the graph of $y = 1 - \sqrt{3}\cot\left(\pi x - \frac{\pi}{3}\right)$ for $-1 \leq x \leq 3$, labelling all stationary points, axes intercepts and asymptotes with their equations.



b. Sketch the graph of $y = \cot\left(2x - \frac{\pi}{4}\right) + \sqrt{3}$ for $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{4}$, labelling all stationary points, axes intercepts and asymptotes with their equations.



Sub-Section [3.4.3]: Apply Compound and Double Angle Formula to Solve Exact Values

Question 31

Use a compound angle formula to evaluate $\sin\left(\frac{5\pi}{12}\right)$.

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)\\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

Question 32

Use a double-angle formula to evaluate $\tan\left(-\frac{\pi}{8}\right)$.

Use the formula $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

$$\tan\left(-2\times\frac{\pi}{8}\right) = \frac{2\tan\left(-\frac{\pi}{8}\right)}{1-\tan^2\left(-\frac{\pi}{8}\right)}$$

$$-1 = \frac{2\tan\left(-\frac{\pi}{8}\right)}{1-\tan^2\left(-\frac{\pi}{8}\right)}$$

let $a = \tan\left(-\frac{\pi}{8}\right)$

$$\begin{aligned}-1+a^2 &= 2a\\ a^2-2a &= 1\\ (a-1)^2 &= 2\\ a-1 &= \pm\sqrt{2}\\ a &= 1-\sqrt{2}\end{aligned}$$

is the only solution since the solution must be < 0 because $-\frac{\pi}{8}$ is in the fourth quadrant
Therefore,

$$\tan\left(-\frac{\pi}{8}\right) = 1-\sqrt{2}.$$

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Question 33

Use a compound angle formula to evaluate $\cos\left(\frac{19\pi}{12}\right)$.

$$\begin{aligned}\cos\left(\frac{19\pi}{12}\right) &= \cos\left(\frac{15\pi}{12}\right) + \cos\left(\frac{4\pi}{12}\right)\\ &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)\\ &= -\frac{1}{\sqrt{2}}\times\frac{1}{2} + \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Question 34

Given that $\cos(x-y) = \frac{7}{25}$ and $\cot(x)\cot(y) = \frac{4}{3}$, find $\cos(x+y)$.

From the information given we have

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y) = \frac{7}{25},$$

$$\cos(x)\cos(y) = \frac{4}{3}\sin(x)\sin(y).$$

Therefore,

$$\begin{aligned}\frac{4}{3}\sin(x)\sin(y) + \sin(x)\sin(y) &= \frac{7}{25},\\ \implies \quad \sin(x)\sin(y) &= \frac{3}{25},\\ \implies \quad \cos(x)\cos(y) &= \frac{4}{3}\cdot\frac{3}{25} = \frac{4}{25},\end{aligned}$$

Now we have

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) = \frac{4}{25} - \frac{3}{25} = \frac{1}{25}$$

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Sub-Section [3.4.4]: Find Domain, Range and Rule of the Inverse Trigonometric Function

Question 35

Consider the function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R : f(x) = \frac{\tan(x)}{3}$.

a. State the domain of $f^{-1}(x)$.

$$\text{dom } f^{-1} = \text{ran } f = \mathbb{R}$$

b. State the range of $f^{-1}(x)$.

$$\text{ran } f^{-1} = \text{dom } f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

c. Hence, or otherwise, find the rule of $f^{-1}(x)$.

Swap x and y.

$x = \frac{1}{3}\tan(y) \implies y = \tan^{-1}(3x)$. Therefore,

$$f^{-1}(x) = \tan^{-1}(3x)$$

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Question 36

Consider the function $f : \left[-\frac{9\pi}{4}, \frac{3\pi}{4}\right] \to R : f(x) = 2\sin\left(\frac{x}{3} + \frac{\pi}{4}\right) - \sqrt{2}$.

a. State the domain of $f^{-1}(x)$.

$f\left(-\frac{9\pi}{4}\right) = -2 - \sqrt{2}$ and $f\left(\frac{3\pi}{4}\right) = 2 - \sqrt{2}$.
Therefore, dom f^{-1} = ran $f = \left[-2 - \sqrt{2}, 2 - \sqrt{2}\right]$

b. State the range of $f^{-1}(x)$.

ran f^{-1} = dom $f = \left[-\frac{9\pi}{4}, \frac{3\pi}{4}\right]$

c. Hence, or otherwise, find the rule of $f^{-1}(x)$.

Swap x and y.
$x = 2\sin\left(\frac{y}{3} + \frac{\pi}{4}\right) - \sqrt{2} \implies \frac{y}{3} + \frac{\pi}{4} = \sin^{-1}\left(\frac{x + \sqrt{2}}{2}\right)$. Therefore,

$$f^{-1}(x) = 3\sin^{-1}\left(\frac{x + \sqrt{2}}{2}\right) - \frac{3\pi}{4}$$

Question 37

Consider the function $f : \left[\frac{5\pi}{3}, \frac{8\pi}{3}\right] \to R : f(x) = \sqrt{5}\cos\left(x + \frac{\pi}{3}\right)$.

a. State the domain of $f^{-1}(x)$.

$f\left(\frac{5\pi}{3}\right) = \sqrt{5}$ and $f\left(\frac{8\pi}{3}\right) = -\sqrt{5}$

dom $f^{-1} =$ ran $f = [-\sqrt{5}, \sqrt{5}]$

b. State the range of $f^{-1}(x)$.

ran $f^{-1} =$ dom $f = \left[\frac{5\pi}{3}, \frac{8\pi}{3}\right]$

c. Hence, or otherwise, find the rule of $f^{-1}(x)$.

Swap x and y.

$$x = \sqrt{5}\cos\left(y + \frac{\pi}{3}\right)$$

$$y + \frac{\pi}{3} = 2\pi + \cos^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

$$y = \frac{5\pi}{3} + \cos^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

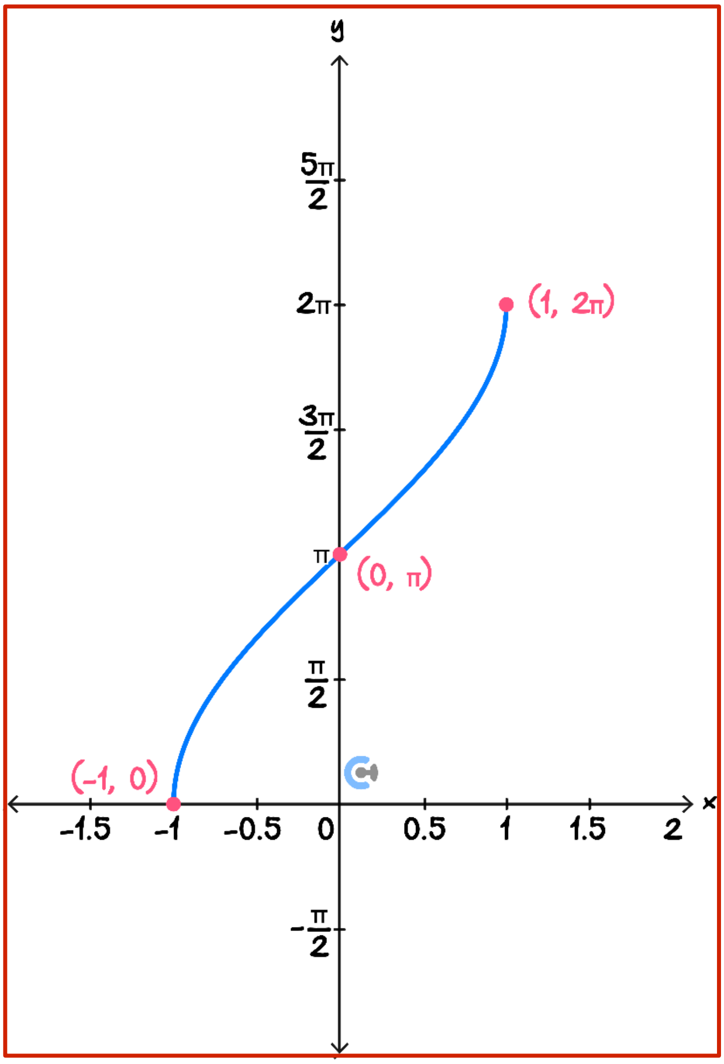
where we added the period 2π so that $y \in$ ran $f^{-1} = \left[\frac{5\pi}{3}, \frac{8\pi}{3}\right]$ Therefore,

$$f^{-1}(x) = \frac{5\pi}{3} + \cos^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

Sub-Section [3.4.5]: Graphing Inverse Trigonometric Functions

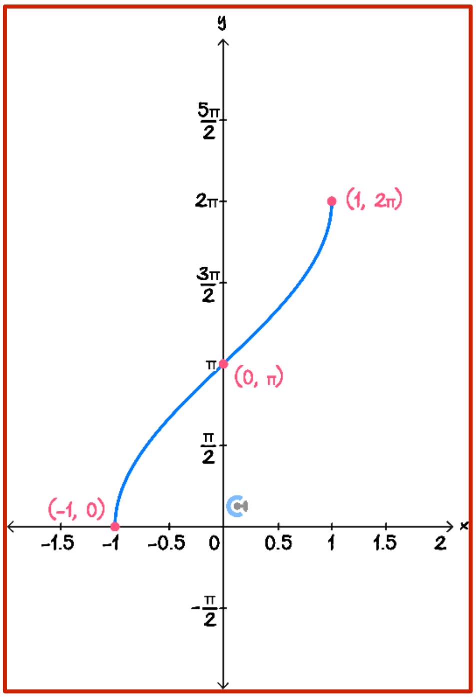
Question 38

a. Sketch the graph of $y = 2\sin^{-1}(x) + \pi$ on the axes below. Label all endpoints and axes intercepts.



b.

i. Sketch the graph of $y = 2\cos^{-1}(-x)$ below.



ii. What do you notice?

Same graph as **part a.**

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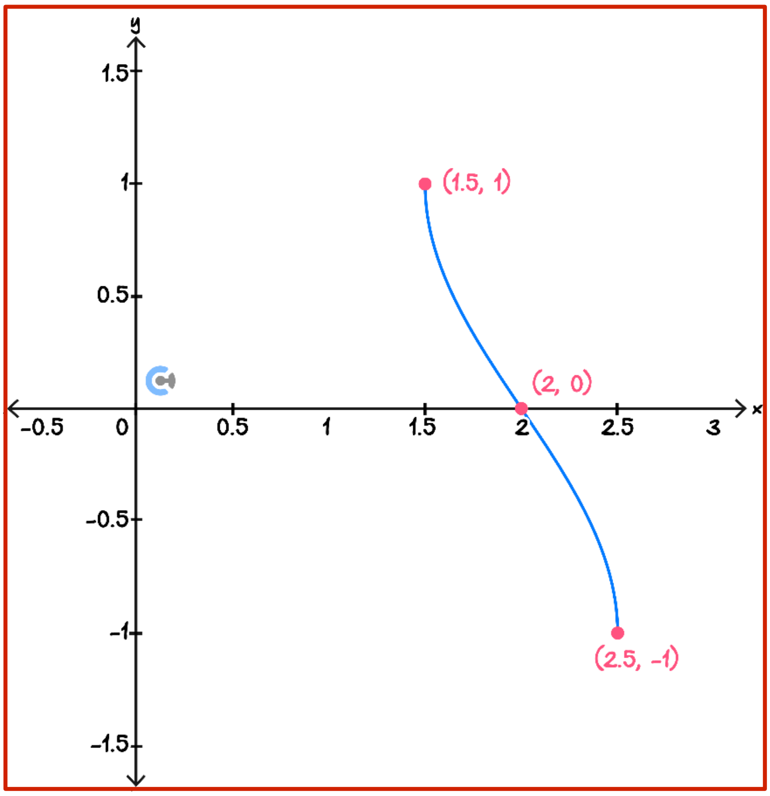
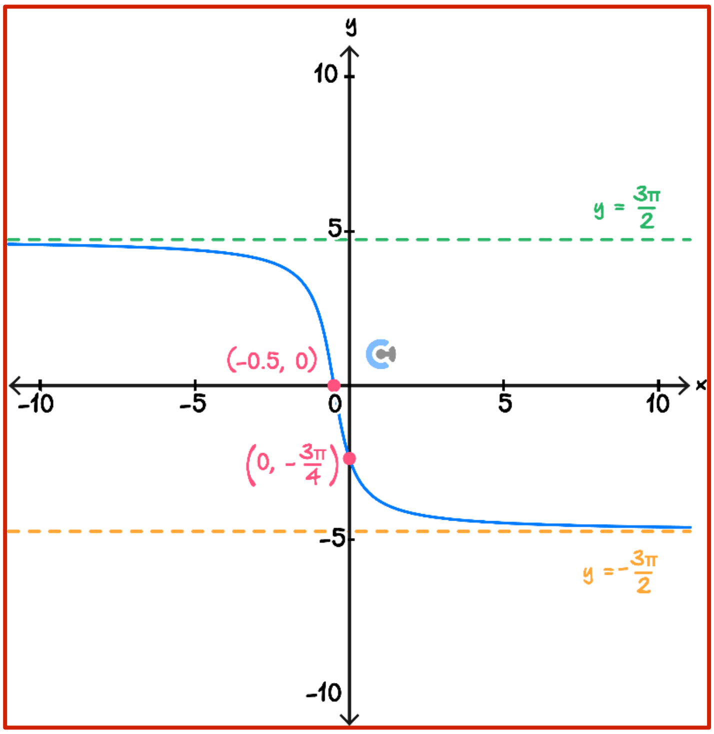
Question 39

a. Sketch the graph of $y = -\frac{2}{\pi}\cos^{-1}(4-2x)+1$ on the axes below, labelling all endpoints and axes intercepts.

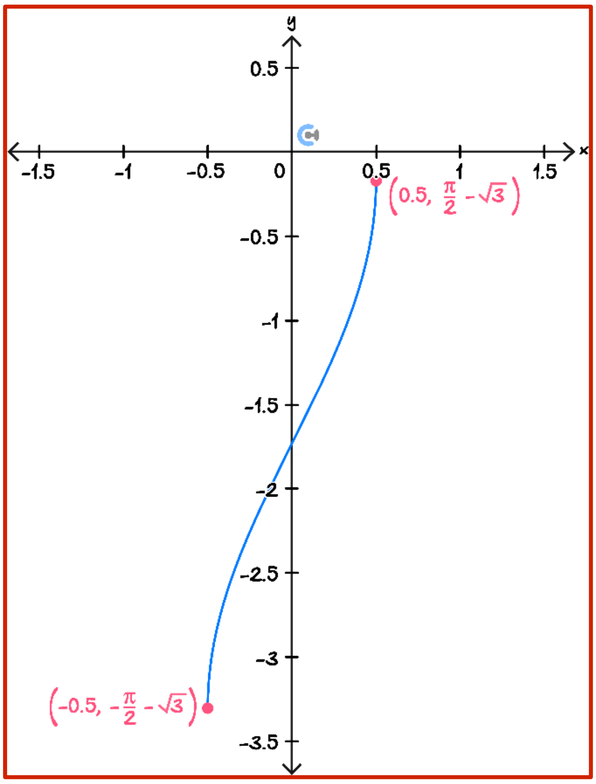


b. Sketch the graph of $y = -3\tan^{-1}(2x+1)$ below, labelling all key points and asymptotes.

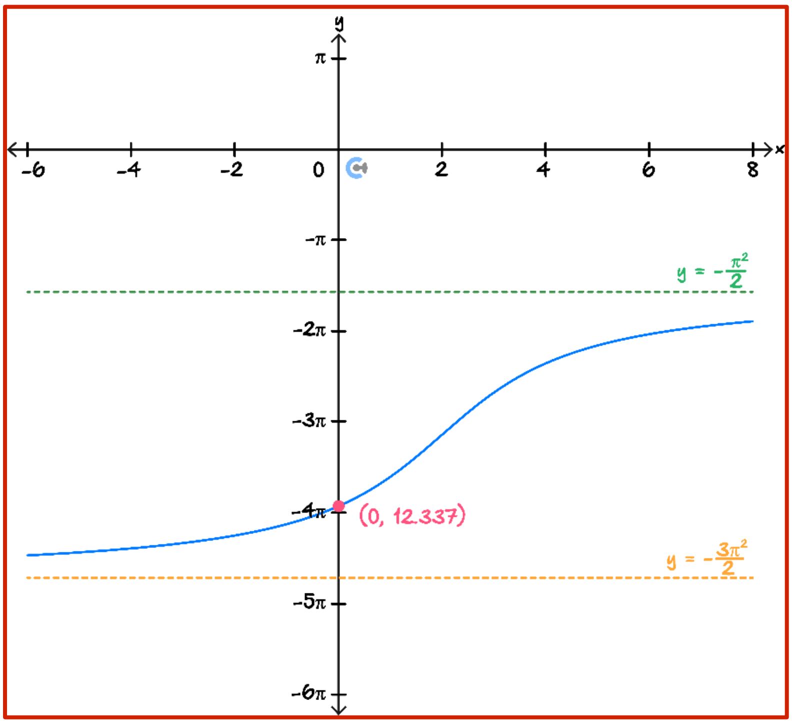


Question 40

a. Sketch the graph of $y = \sin^{-1}(2x) - \sqrt{3}$ on the axes below. Label all endpoints.



b. Sketch the graph of $y = \pi \tan^{-1}\left(\frac{x}{2} - 1\right) - \pi^2$ on the axes below. Label all axes intercepts and asymptotes with their equation.



Section E: [3.5] - Advanced Trigonometric Functions Exam Skills (Checkpoints)

Sub-Section [3.5.1]: Simplify the Composition of Inverse Trigonometric

Question 41

a. Simplify $\tan\left(\arctan\left(\frac{3}{4}\right)\right)$.

$$\tan\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{3}{4}$$

b. Simplify $\cos(\arctan(5))$.

Let $\theta = \arctan(5)$, right triangle with sides, $5, 1, \sqrt{26}$.

$$\cos(\arctan(5)) = \cos\left(\arccos\left(\frac{1}{\sqrt{26}}\right)\right) = \frac{1}{\sqrt{26}}$$

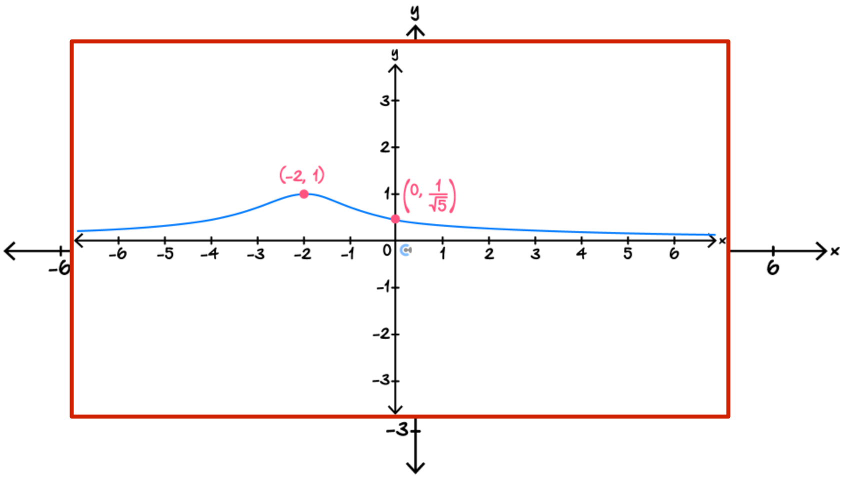
c. Simplify $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$.

Let $\theta = \arccos\left(\frac{5}{13}\right)$, right triangle with sides, $5, 12, 13$.

$$\sin\left(\arccos\left(\frac{5}{13}\right)\right) = \sin\left(\arcsin\left(\frac{12}{13}\right)\right) = \frac{12}{13}$$

Question 42

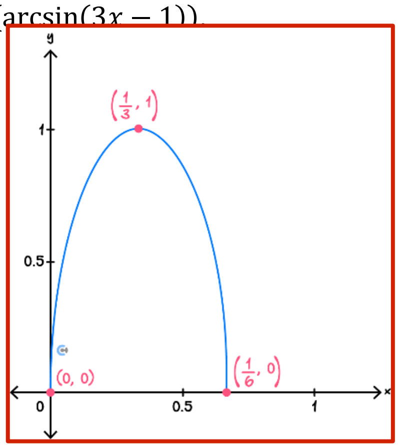
a. Simplify and sketch the graph of $\cos(\arctan(x + 2))$.



Let $\theta = \arctan(x + 2)$, right triangle with sides, $x + 2, 1, \sqrt{(x + 2)^2 + 1}$.

$$\cos(\arctan(x + 2)) = \cos\left(\arccos\left(\frac{1}{\sqrt{(x + 2)^2 + 1}}\right)\right) = \frac{1}{\sqrt{(x + 2)^2 + 1}}$$

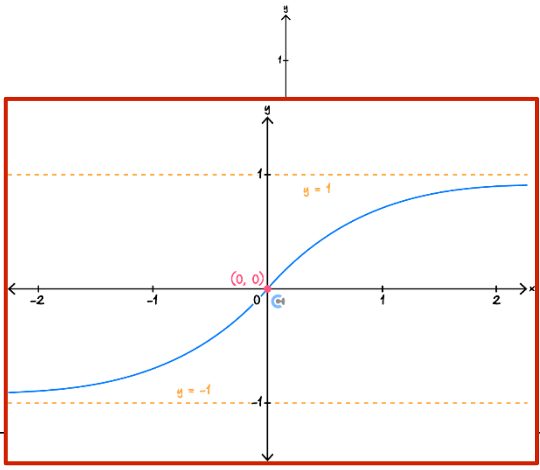
b. Simplify and sketch the graph of $\tan(\arcsin(3x-1))$.



Let $\theta = \arcsin(3x-1)$, right triangle with sides, $3x-1, \sqrt{1-(3x-1)^2}, 1$.

$$\tan(\arcsin(3x-1)) = \tan\left(\arctan\left(\frac{\sqrt{1-(3x-1)^2}}{1}\right)\right) = \sqrt{1-(3x-1)^2}$$

c. Simplify and sketch the graph of $\sin(\arctan(x))$.



Let $\theta = \arctan(x)$, right triangle with sides, $x, 1, \sqrt{x^2+1}$.

$$\sin(\arctan(x)) = \sin\left(\arcsin\left(\frac{x}{\sqrt{x^2+1}}\right)\right) = \frac{x}{\sqrt{x^2+1}}$$

Question 43

a. Simplify and determine the maximal domain of $g(x) = \cos(\arcsin(3x-2)) + \sin(\arctan(x+1))$.

$$g(x) = \cos(\arcsin(3x-2)) + \sin(\arctan(x+1))$$
$$= \sqrt{1-(3x-2)^2} + \frac{x+1}{\sqrt{(x+1)^2+1}}$$

$dom\ \cos(\arcsin(3x-2)) = \left[\frac{1}{3}, 1\right]$ and $dom\ \sin(\arctan(x+1)) = \mathbb{R}$

For the maximal domain we require both functions to be defending. Thus, $x \in \left[\frac{1}{3}, 1\right]$.

b. Simplify and determine the maximal domain of $g(x) = \sin(\arccos(2-x^2)) + \cos(\arcsin(x+2))$.

$$g(x) = \sin(\arccos(2-x^2)) + \cos(\arcsin(x+2))$$
$$= \sqrt{1-(2-x^2)^2} + \sqrt{1-(x+2)^2}$$

$dom\ \sin(\arccos(2-x^2)) = [-\sqrt{3}, -1] \cup [1, \sqrt{3}]$ *and* $dom\ \cos(\arcsin(x+2)) = [-3, -1]$

For the maximal domain we require both functions to be defined.

Thus, $x \in [-\sqrt{3}, -1]$.

c. Simplify and determine the maximal domain of $g(x) = \cos(\arcsin(3x+1)) + \sin(\arctan(2x))$.

$$g(x) = \cos(\arcsin(3x+1)) + \sin(\arctan(2x))$$
$$= \sqrt{1-(3x+1)^2} + \frac{2x}{\sqrt{4x^2+1}}$$

$dom\ \cos(\arcsin(3x+1)) = \left[-\frac{2}{3}, 0\right]$ *and* $dom\ \sin(\arctan(2x)) = \mathbb{R}$

F

For the maximal domain we require both functions to be defined.

Thus, $x \in \left[-\frac{2}{3}, 0\right]$.

Sub-Section [3.5.2]: Simplify $a\cos(x) + b\sin(x)$

Question 44

a. Express $3\sin(x) + 3\cos(x)$ in the form of $r\sin(x - \alpha)$.

Here $a = 3$ *and* $b = 3$

Then $r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$$\alpha = \arctan\left(\frac{3}{3}\right) = \frac{\pi}{4}$$

Thus, $3\sin(x) + 3\cos(x) = 3\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

b. Express $\cos(x) - \sin(x)$ in the form of $r\cos(x - \alpha)$.

Here $a = 1$ *and* $b = -1$

Then $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$$\alpha = \arctan\left(\frac{-1}{1}\right) = \frac{7\pi}{4}$$

Thus, $\cos(x) - \sin(x) = \sqrt{2}\cos\left(x - \frac{7\pi}{4}\right)$

c. Express $\sqrt{3}\sin(x) + 3\cos(x)$ in the form of $r\sin(x + \alpha)$.

Here $a = 3$ and $b = \sqrt{3}$

Then $r = \sqrt{3^2 + (\sqrt{3})^2} = 2\sqrt{3}$

$$\alpha = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

Thus, $\sqrt{3}\sin(x) + 3\cos(x) = 2\sqrt{3}\sin\left(x + \frac{\pi}{6}\right)$

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Question 45

a. Solve $\cos(x) + \sin(x) = 1$ for $0 \le x \le 2\pi$.

$\cos(x) + \sin(x) = 1$

$\cos(x) + \sin(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1$

$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow x = 0, \frac{\pi}{2}, 2\pi$

b. Solve $4\cos(x) + 4\sqrt{3}\sin(x) = 2$ for $0 \le x \le 2\pi$.

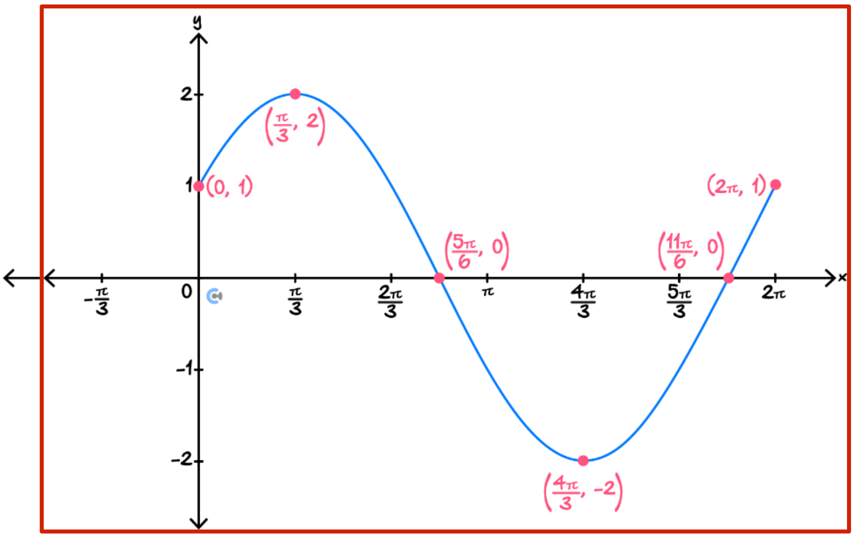
$2\cos(x) + 2\sqrt{3}\sin(x) = 2$

$2\cos(x) + 2\sqrt{3}\sin(x) = 4\sin\left(x + \frac{\pi}{3}\right) = 2$

$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{2}, \frac{11\pi}{6}$

c. Sketch the graph of $f(x) = \sqrt{3}\sin(x) + \cos(x)$ for $0 \le x \le 2\pi$. Label all turning points, endpoints, and axes intercepts with coordinates.



Question 46

a. Find the maximum and minimum values if $h(x) = 7\sin(x) + 24\cos(x)$.

$$r = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$$
$$h(x) = 25\sin(x + \alpha)$$
Thus, maximum value is 25 and minimum value is -25.

b. Solve $3\sin\left(x - \frac{\pi}{4}\right) + 3\sqrt{3}\cos\left(x - \frac{\pi}{4}\right) = 0$ for $0 \le x \le 2\pi$.

$$3\sin\left(x - \frac{\pi}{4}\right) + 3\sqrt{3}\cos\left(x - \frac{\pi}{4}\right) = 0$$
$$\Rightarrow 6\sin\left(x + \frac{\pi}{4}\right) = 0$$
$$\sin\left(x + \frac{\pi}{4}\right) = 0$$
$$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

c. Show that for $a > 0$, $a\sin(4x) - b\cos^2(2x) = \sqrt{4a^2 + b^2}\cos(2x)\sin(2x - \alpha)$, where $\beta = \arctan\left(\frac{b}{2a}\right)$.

$$a\sin(4x) - b\cos^2(2x) = 2a\sin(2x)\cos(2x) - b\cos^2(2x)$$
$$= \cos(2x)\left(2a\sin(2x) - b\cos(2x)\right)$$
$$= \cos(2x)\left(\sqrt{4a^2 + b^2}\sin(2x - \beta)\right), \text{ where } \beta = \arctan\left(\frac{b}{2a}\right)$$
$$= \sqrt{4a^2 + b^2}\cos(2x)\sin(2x - \beta)$$

Sub-Section [3.5.3]: Apply Product-to-Sum and Sum-to-Product Identities to Simplify Trigonometric Expressions

Question 47

a. Express $\sin(5\theta)\cos(3\theta)$ as a sum or difference.

$$\sin(5\theta)\cos(3\theta) = \frac{1}{2}[\sin(5\theta + 3\theta) + \sin(5\theta - 3\theta)]$$
$$= \frac{1}{2}[\sin(8\theta) + \sin(2\theta)]$$

b. Express $2\cos(4B)\cos(6B)$ as a sum or difference.

$$2\cos(4B)\cos(6B) = \cos(4B + 6B) + \cos(4B - 6B)$$
$$= \cos(10B) + \cos(2B)$$

c. Express $\sin(7A)\cos(4A)$ as a sum or difference.

$$\sin(7A)\cos(4A) = \frac{1}{2}[\sin(7A + 4A) + \sin(7A - 4A)]$$
$$= \frac{1}{2}[\sin(11A) + \sin(3A)]$$

d. Express $\sin(3\alpha) + \sin(4\alpha)$ as a product.

$$\sin(3\alpha) + \sin(4\alpha) = 2\sin\left(\frac{3\alpha + 4\alpha}{2}\right)\cos\left(\frac{3\alpha - 4\alpha}{2}\right)$$
$$= 2\sin\left(\frac{7\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)$$

e. Express $\cos(3x) + \cos(3y)$ as a product.

$$\cos(3x) + \cos(3y) = 2\cos\left(\frac{3x + 3y}{2}\right)\cos\left(\frac{3x - 3y}{2}\right)$$

f. Express $\cos(x+k) - \sin(x)$ as a product.

$$\cos(x+k) - \sin(x) = \cos(x+k) - \cos\left(\frac{\pi}{2} - x\right)$$

$$= 2\sin\left(\frac{x+k+\frac{\pi}{2}-x}{2}\right)\sin\left(\frac{x+k-\frac{\pi}{2}+x}{2}\right)$$

$$= 2\sin\left(\frac{2k+\pi}{4}\right)\sin\left(x+\frac{2k-\pi}{4}\right)$$

Space for Personal Notes

Question 48

a. Solve $\sin(2\theta) + \sin(4\theta) = 0$ for $0 \le \theta \le \pi$.

$$\sin(2\theta) + \sin(4\theta) = 0$$
$$2\sin(6\theta)\cos(2\theta) = 0$$
$$\sin(6\theta) = 0 \Rightarrow \theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$$
$$\cos(2\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Thus all solutions are $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{\pi}{4}, \frac{3\pi}{4}$

b. Solve $\cos(3x) - \cos(x) = 0$ for $0 \le \theta \le \pi$.

$$\cos(3x) - \cos(x) = 0$$
$$-2\sin(2x)\sin(x) = 0$$
$$\sin(2x) = 0 \Rightarrow x = 0, \frac{\pi}{2}, \pi$$
$$\sin(x) = 0 \Rightarrow x = 0, \pi$$

Thus all solutions are $x = 0, \frac{\pi}{2}, \pi$

c. Solve $\sin(z) - \sin\left(\frac{\pi}{3} - z\right) = 0$ for $0 \le \theta \le 2\pi$.

$$\sin(z) - \sin\left(\frac{\pi}{3} - z\right) = 0$$
$$2\cos\left(\frac{\pi}{6}\right)\sin\left(z - \frac{\pi}{6}\right) = 0$$
$$\sin\left(z - \frac{\pi}{6}\right) = 0 \Rightarrow z = \frac{\pi}{6}, \frac{7\pi}{6}$$

Question 49

a. Express $|b|\cos(2y) - |b|\cos(4y)$ as a product and hence, determine its minimum value in terms of b.

$|b|\cos(y) - |b|\cos(2y) = 2|b|\sin(3y)\sin(y)$
When $y = \frac{\pi}{2}$, we get a minimum of $-2|b|$.

b. If $p + q + r = \pi$, show that $\sin(p) + \sin(q) + \sin(r) = 4\cos\left(\frac{p}{2}\right)\cos\left(\frac{q}{2}\right)\cos\left(\frac{r}{2}\right)$.

$$\sin(p) + \sin(q) + \sin(r) = 2\sin\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right) + 2\sin\left(\frac{r}{2}\right)\cos\left(\frac{r}{2}\right)$$
$$= 2\sin\left(\frac{\pi}{2} - \frac{r}{2}\right)\cos\left(\frac{p-q}{2}\right) + 2\sin\left(\frac{r}{2}\right)\cos\left(\frac{r}{2}\right)$$
$$= 2\cos\left(\frac{r}{2}\right)\cos\left(\frac{p-q}{2}\right) + 2\sin\left(\frac{r}{2}\right)\cos\left(\frac{r}{2}\right)$$
$$= 2\cos\left(\frac{r}{2}\right)\left[\cos\left(\frac{p-q}{2}\right) + \sin\left(\frac{r}{2}\right)\right]$$
$$= 2\cos\left(\frac{r}{2}\right)\left[\cos\left(\frac{p-q}{2}\right) + \sin\left(\frac{\pi}{2} - \frac{p+q}{2}\right)\right]$$
$$= 2\cos\left(\frac{r}{2}\right)\left[\cos\left(\frac{p-q}{2}\right) + \cos\left(\frac{p+q}{2}\right)\right]$$
$$= 2\cos\left(\frac{r}{2}\right)\left[2\cos\left(\frac{p}{2}\right)\cos\left(\frac{q}{2}\right)\right]$$
$$= 4\cos\left(\frac{p}{2}\right)\cos\left(\frac{q}{2}\right)\cos\left(\frac{r}{2}\right)$$

c. Solve the equation $\cos(4x) + \cos(2x) - \cos(3x) = 0$ for $x \in [0, 2\pi]$.

$$\cos(4x) + \cos(2x) - \cos(3x) = 0$$
$$2\cos(3x)\cos(x) - \cos(3x) = 0$$
$$\cos(3x)(2\cos(x) - 1) = 0$$

So, $\cos(3x) = 0$ or $2\cos(x) - 1 = 0$

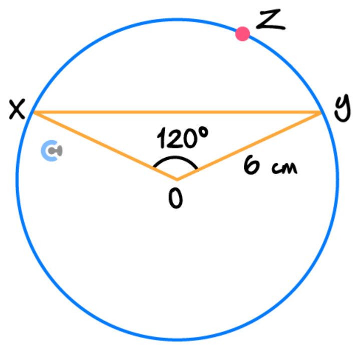
$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ or $x = \frac{\pi}{3}, \frac{5\pi}{3}$

Thus, all solutions to the equation are: $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{3}$

Section F: [3.1-3.5] - Exam 1 Overall (Checkpoints) (19 Marks)

Question 50

Consider a circle of radius 6 cm, with centre O. The angle subtended at O by the arc XY has a magnitude of 120°. Exact answers are required for all parts, and no CAS is allowed.



a.

i. Find the length of the chord XY.

∵The perpendicular from the centre of a circle to a chord bisects the chord.

Length of chord $PQ = 2 \times 6 \times \sin(60°) = 6\sqrt{3}\ cm$

OR

Using cosine rule:

$$XY^2 = 6^2 + 6^2 - 2(6)(6)\cos(120°) = 6^2 + 6^2 + 6^2 = 3 \times 6^2$$

$$XY = \sqrt{3 \times 6^2} = 6\sqrt{3}\ cm$$

ii. Find the length of the arc XY.

Length of arc $= \frac{\theta}{360} \times 2\pi r = \frac{120}{360} \times 2 \times \pi \times 6 = 4\pi\ cm$

b. Find the area of the minor segment formed by the chord XY.

$$\theta = 120° \Rightarrow \theta^c = \frac{2\pi}{3}$$

$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

$$= \frac{1}{2}(6^2)\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) = \left(12\pi - 9\sqrt{3}\right)\, cm^2$$

c. The point Z is located between X and Y such that it divides the arc XY in a $5:3$ ratio. Find the length of the arc ZY and the angle YOZ in degrees.

$$XY = 6 \times \frac{2\pi}{3} = 4\pi$$

Thus, $XZ = \frac{5}{5+3} \times 4\pi = \frac{5\pi}{2}$ and $ZY = \frac{3}{5+3} \times 4\pi = \frac{3\pi}{2}$.

Let $\theta = \angle YOZ$.

Then, $6 \times \theta = \frac{3\pi}{2}$

$$\theta = \frac{\pi}{4}$$

In degrees, $\theta = \frac{\pi}{4} \times \frac{180}{\pi} = 45°$

Space for Personal Notes

Question 51

Consider the function $f(x) = 3\sin\left(2x + \frac{\pi}{4}\right) + \cos\left(2x + \frac{3\pi}{4}\right) - 1$.

a. Express $f(x)$ in the form $f(x) = a\sin(2x + b) - 1$.

$\cos\left(2x + \frac{3\pi}{4}\right) = \sin\left(2x + \frac{3\pi}{4} + 2\pi\right) = -\sin\left(2x + \frac{\pi}{4}\right)$

So,

$f(x) = 2\sin\left(2x + \frac{\pi}{4}\right) - 1$

$$a = 2, b = \frac{\pi}{4}$$

b. Find the general solution to $f(x) = 0$.

$2\sin\left(2x + \frac{\pi}{4}\right) = 1$

$\sin\left(2x + \frac{\pi}{4}\right) = \frac{1}{2}$

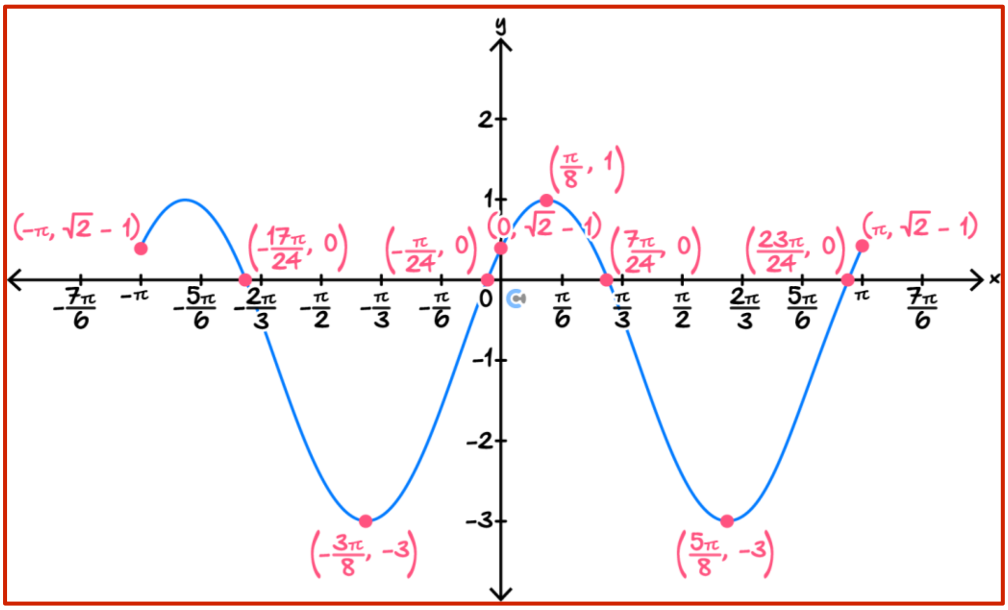
$2x + \frac{\pi}{4} = \frac{\pi}{6} + 2\pi n$ or $2x + \frac{\pi}{4} = \frac{5\pi}{6} + 2\pi n$, $n \in Z$

$x = \pi n - \frac{\pi}{24}$ or $x = \pi n + \frac{7\pi}{24}$, $n \in Z$

c. Find all solutions to $f(x) = 0$ for $x \in [-\pi, \pi]$.

$$x = -\frac{\pi}{24}, x = \frac{7\pi}{24}, x = \frac{23\pi}{24}, x = -\frac{17\pi}{24}$$

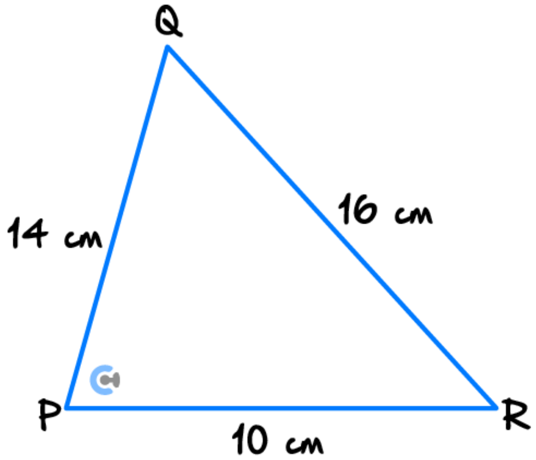
d. Sketch the graph of $y = f(x)$ on the axes below. Label all axes' intercepts, turning points and endpoints with coordinates.



Space for Personal Notes

Question 52

The figure below shows a triangle PQR where the following information is given. $|PQ| = 14\ cm, |QR| = 16\ cm, |PR| = 10\ cm$.



a. Find the size of the angle $\angle PRQ$ in degrees.

$Let\ \angle \text{PRQ} = \emptyset$

$\cos(\theta) = \frac{16^2 + 10^2 - 14^2}{320} = \frac{160}{320} = \frac{1}{2}$

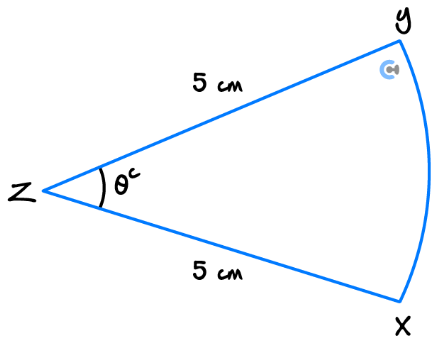
Therefore, $\emptyset = 60°$

b. Hence, determine as an exact surd the area of the triangle PQR.

$\text{Area} = \frac{1}{2} \times (10) \times (16) \times \sin(60°) = 80 \times \frac{\sqrt{3}}{2} = 40\sqrt{3}\ sq.cm$

Question 53

The figure below shows a circular sector XYZ of radius 5 cm subtending an angle θ radians at Z. Given that the perimeter of the sector is equal to the area of the sector, find the value of θ in radians.



Perimeter $= 10 + 5\theta$

Area $= \frac{1}{2} \times 5^2\theta = \frac{25}{2}\theta$

Solve $10 + 5\theta = \frac{25}{2}\theta \to \frac{25\theta}{2} = 10 \to 25\theta = 20$

$\theta = \frac{20}{25} = \frac{4}{5}$ radians

Question 54

Prove the identity: $\frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$

Given RHS $= \frac{\sin\alpha}{1+\cos\alpha}$

$= \frac{\sin\alpha}{1+\cos\alpha} \times \frac{1-\cos\alpha}{1-\cos\alpha}$ [Multiply the numerator and denominator by $1 - \cos\alpha$]

$= \frac{\sin\alpha(1-\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)}$

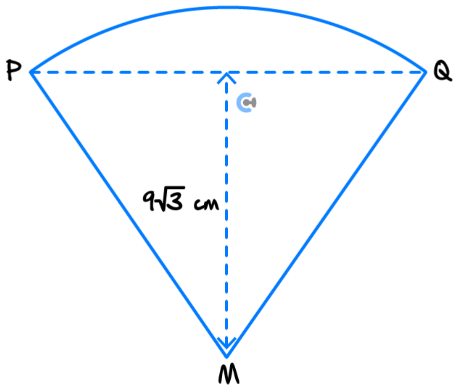
$= \frac{\sin\alpha(1-\cos\alpha)}{(1-\cos^2\alpha)}$

$= \frac{\sin\alpha(1-\cos\alpha)}{\sin^2\alpha}$

$= \frac{1-\cos\alpha}{\sin\alpha}$

Question 55

The figure above shows a badge in the shape of a circular sector MPQ, centred at M. The triangle MPQ is equilateral and its perpendicular height is $9\sqrt{3}\ cm$.



a. Find the length of MP.

$$\sin\left(\frac{\pi}{3}\right) = \frac{9\sqrt{3}}{MP} \rightarrow MP = \frac{2 \times 9\sqrt{3}}{\sqrt{3}} = 18\ cm$$

b. Determine in terms of π:

i. The area of the badge.

$$\text{Area} = \frac{1}{2}r^2\theta = \frac{1}{2} \times 18^2 \times \frac{\pi}{3} = 54\pi\ sq.cm$$

ii. The perimeter of the badge.

Length of arc $PQ = r\theta = 18 \times \frac{\pi}{3} = 6\pi\ cm$

Therefore Perimeter $= 36 + 6\pi\ cm$

Question 56

Consider the function $g(x) = 2\sin\left(3x - \frac{\pi}{4}\right) + 1$.

a. Find the general solution to $f(x) = 0$.

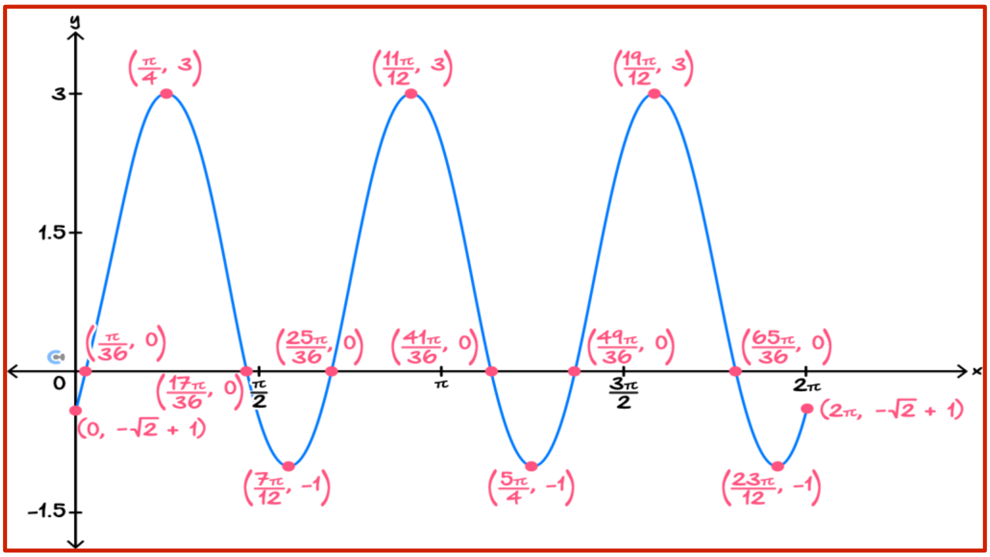
$$2\sin\left(3x - \frac{\pi}{4}\right) = -1$$

$$\sin\left(3x - \frac{\pi}{4}\right) = -\frac{1}{2}$$

$$3x - \frac{\pi}{4} = \frac{7\pi}{6} + 2\pi n,\ 3x - \frac{\pi}{4} = \frac{11\pi}{6} + 2\pi n \qquad n \in Z$$

$$x = \frac{2\pi n}{3} + \frac{17\pi}{36},\ x = \frac{2\pi n}{3} + \frac{25\pi}{36}, \qquad n \in Z$$

b. Sketch the graph of $y = f(x)$ for $x \in [0, 2\pi]$ on the axes below. Label all axes intercepts, turning points and endpoints with coordinates.



c. Find the values of x for which $f(x) > 2$.

We solve for $f(x) = 2 \rightarrow 2\sin\left(3x - \frac{\pi}{4}\right) = 1$

$$\sin\left(3x - \frac{\pi}{4}\right) = \frac{1}{2}$$

$$3x - \frac{\pi}{4} = \frac{\pi}{6} + 2\pi n,\ 3x - \frac{\pi}{4} = \frac{5\pi}{6} + 2\pi n, \qquad n \in Z$$

$$x = \frac{2\pi n}{3} + \frac{5\pi}{36},\ x = \frac{2\pi n}{3} + \frac{13\pi}{36}, \qquad n \in Z$$

$$For\ f(x) > 2 : we\ have, \frac{2\pi}{3} + \frac{5\pi}{36} < x < \frac{2\pi}{3}n + \frac{13\pi}{36}, \qquad n \in Z$$

d. The function $g(x)$ has an equivalent expression $g(x) = 2\cos\left(3x + \frac{a\pi}{4}\right) + 1$, where $0 < a < 10$.

State the value of a.

Note that $\sin\left(3x - \frac{\pi}{4}\right) = \cos\left(3x - \frac{\pi}{4} - \frac{\pi}{2}\right) = \cos\left(3x - \frac{3\pi}{4}\right)$

$$g(x) = 2\cos\left(3x - \frac{3\pi}{4}\right) + 1$$

$$-\frac{3\pi}{4} + 2\pi = \frac{5\pi}{4}$$

$$a = 5$$

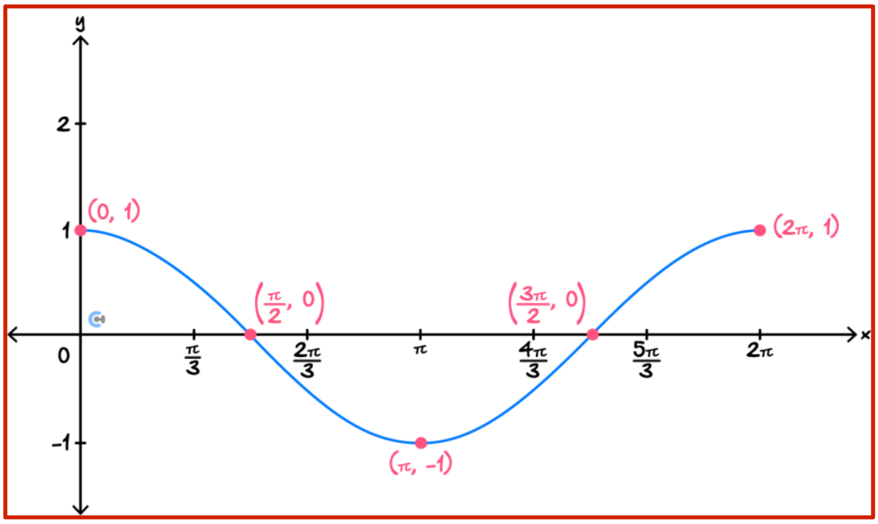
Question 57

a. Solve the equation $\sin(3x) + \sin(5x) = 0$, where $x \in [0, \pi]$.

$$\sin(3x) + \sin(5x) = 0$$
$$2\sin(4x)\cos(x) = 0$$
Thus, $\sin(4x) = 0 \Rightarrow x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$
$$\cos(x) = 0 \Rightarrow x = \frac{\pi}{2}$$
Thus, all solutions are $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

b. Consider the function $g : [0, 2\pi] \to R$, where $g(x) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{2}\right)$.

i. Sketch the graph of g on the axes below. Label all axes intercepts, turning points, and endpoints with coordinates.



ii. Use your sketch to solve the equation $g(x) = \frac{1}{2}$.

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

iii. Hence, find $\left\{x : g(x) > \frac{1}{2}\right\}$.

$x \in \left[0, \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right]$

Question 58 (3 marks)

Given that $\cos(x - y) = \frac{3}{5}$ and $\tan(x)\tan(y) = 2$, find $\cos(x + y)$.

Marks	0	1	2	3	Average
%	15	28	13	44	1.9

$-\frac{1}{5}$

This question was reasonably well answered, with most students using one or more trigonometric identities correctly. A large number of students had difficulty with the algebra but many persisted and answered correctly. Some used $\tan(x - y)$, which was not always successful. Of great concern was the number of students who gave answers for sine or cosine that were either less than -1 or greater than 1.

Space for Personal Notes

Question 59 (3 marks)

Given that $\cot(2x) + \frac{1}{2}\tan(x) = a\cot(x)$, use a suitable double angle formula to find the value of a, $a \in R$.

Marks	0	1	2	3	Average
%	10	6	19	66	2.4

$a = \frac{1}{2}$

This question was answered well, with most students converting $\cot(2x)$ into $\frac{1}{\tan(2x)}$ and using a double angle formula. A number of students used $\sin$ and $\cos$ but were less successful than those who used the more direct approach. A number of students thought that $\cot(2x)$ was equal to $\frac{1}{\cos(2x)}$. Some students gave their final answer as $a = 2$.

Question 60 (3 marks)

Find all real solutions of $\tan(2x) = -\tan(x)$.

$$x = n\pi, \pm\frac{\pi}{3} + n\pi, \ n \in Z$$

Space for Personal Notes

Question 61 (3 marks)

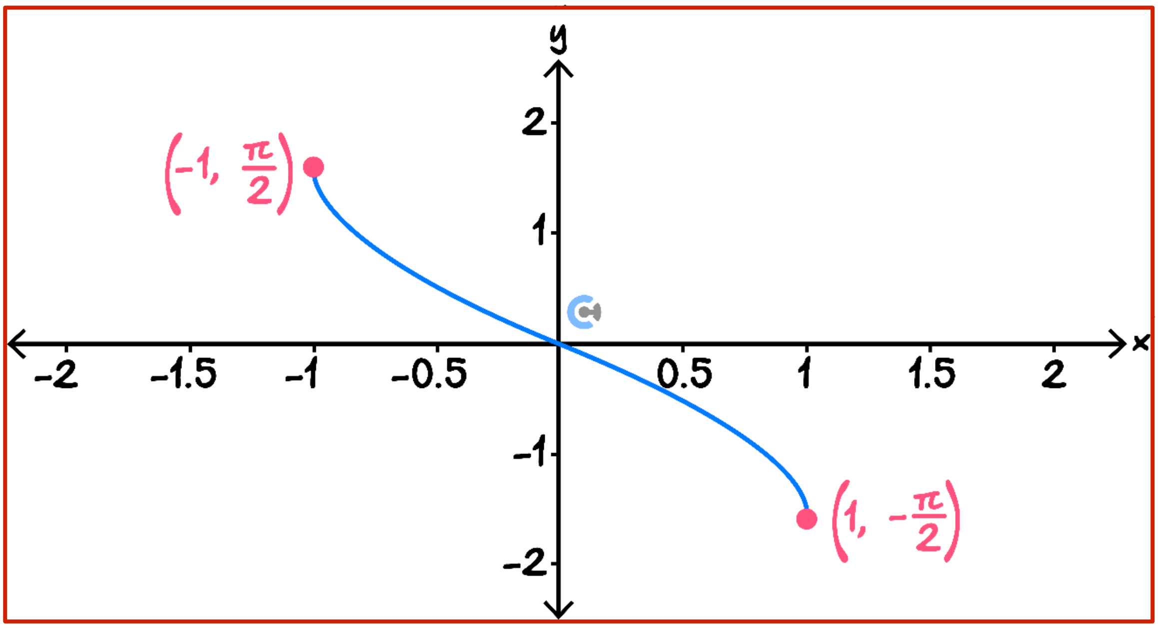
Find $\sin(t)$, given that $t = \arccos\left(\frac{12}{13}\right) + \arctan\left(\frac{3}{4}\right)$.

$\frac{56}{65}$

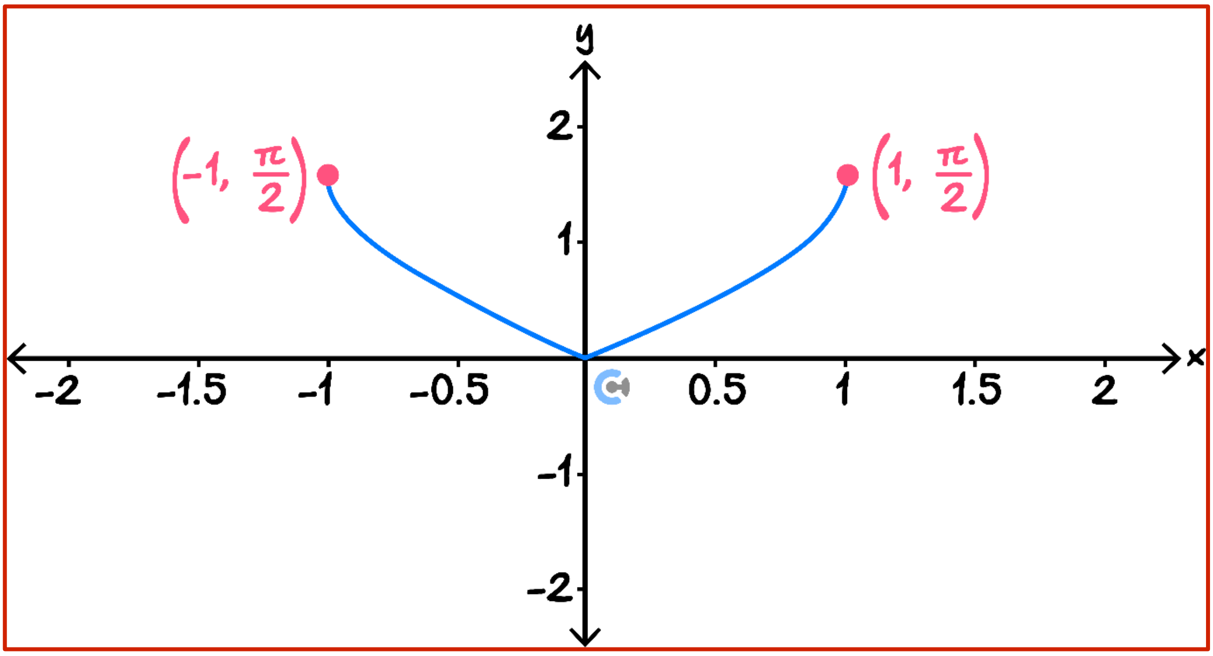
Question 62 (4 marks)

Consider the function $f : [-1, 1] \to R, f(x) = \arccos(x) - \frac{\pi}{2}$.

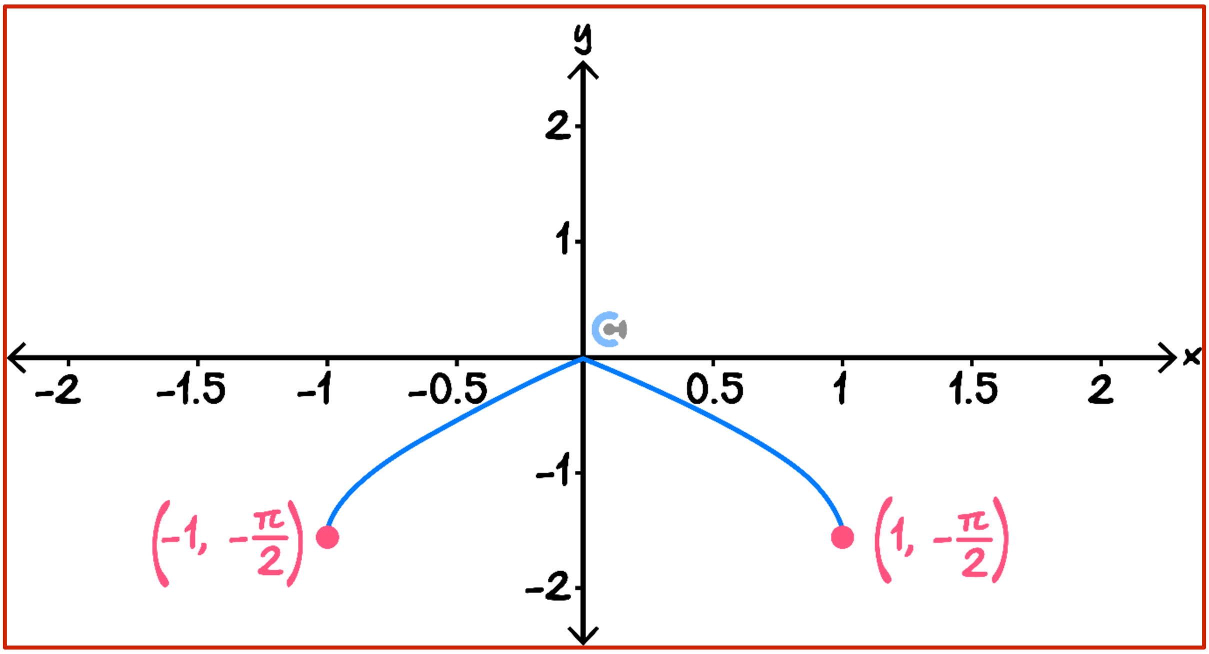
a. Sketch the graph of f on the axes below, labelling the endpoints with their coordinates. (2 marks)



b. Sketch the graph of $y = |f(x)|$ on the axes below. (1 mark)



c. Sketch the graph of $y = f(|x|)$ on the axes below. (1 mark)



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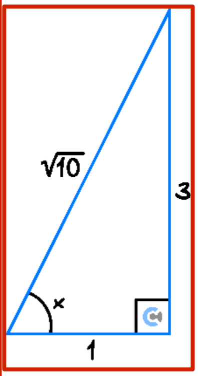
Question 63 (3 marks)

If $\sin(x) = 3\cos(x)$, find the value of $\sin(2x)$, where $x \in \left(0, \frac{\pi}{2}\right)$.

Since $\sin(x) = 3\cos(x)$, $\tan(x) = 3$. Consider the triangle:



Since $x \in \left(0, \frac{\pi}{2}\right)$, $\sin(x) = \frac{3}{\sqrt{10}}$ and $\cos(x) = \frac{1}{\sqrt{10}}$.

Then:

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$= 2 \cdot \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}}$$
$$= \frac{6}{10}$$
$$= \frac{3}{5}$$

Alternative approaches can be used.

Space for Personal Notes

Section G: [3.1-3.5] - Exam 2 Overall (Checkpoints) (36 Marks)

Question 64

A building is 72 m tall. From the top of the building, the angle of depression to a certain point on level ground is 30°. How far is that point from the base of the building?

A. $36\sqrt{3}\ m$

B. $48\sqrt{3}\ m$

C. $72\sqrt{3}\ m$

D. $60\sqrt{3}\ m$

Let the point from the base of the building which is 72 m tall, be x metres away.

So, in the right angled triangle, $\tan(30°) = \frac{72}{x} = \sqrt{3}$

$\rightarrow x = 72\sqrt{3}\ m$

Question 65

If $\cot(\theta) = -\frac{12}{5}$ and $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, then $\sin(\theta)$ is equal to:

A. $\frac{12}{13}$

B. $-\frac{5}{13}$

C. $-\frac{12}{13}$

D. $\frac{5}{13}$

$\cot(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-12}{5}$

$Let\ \cos(\theta) = -12k, \sin(\theta) = 5k$

$Now, \sin^2(\theta) + \cos^2(\theta) = 1 \rightarrow 144k^2 + 25k^2 = 1 \rightarrow k = \frac{1}{13}$

$Since\ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\ we\ have:\ \sin(\theta) = \frac{5}{13}$

Question 66

The solutions of the equation: $2\sin\left(2x - \frac{\pi}{4}\right) + 1 = 0$ are:

A. $x = \pi n + \frac{17\pi}{24}\ \ or\ \ x = \pi n - \frac{25\pi}{24}, n \in \mathbb{Z}.$

B. $x = \pi n + \frac{17\pi}{24}\ \ or\ \ x = \pi n + \frac{25\pi}{24}, n \in \mathbb{Z}.$

C. $x = \pi n - \frac{17\pi}{24}\ \ or\ \ x = \pi n + \frac{25\pi}{24}, n \in \mathbb{Z}.$

D. $x = \pi n - \frac{17\pi}{24}\ \ or\ \ x = \pi n - \frac{25\pi}{24}, n \in \mathbb{Z}.$

$2\sin\left(2x - \frac{\pi}{4}\right) + 1 = 0$

$\sin\left(2x - \frac{\pi}{4}\right) = -\frac{1}{2}$

$2x - \frac{\pi}{4} = 2\pi n + \frac{7\pi}{6},\ 2x - \frac{\pi}{4} = 2\pi n + \frac{11\pi}{6},\quad n \in \mathbb{Z}$

$x = \pi n + \frac{17\pi}{24}\ \ or\ \ x = \pi n + \frac{25\pi}{24},\quad n \in \mathbb{Z}$

Question 67

Let $\sin(\theta) = \frac{-7}{13}$ and $\cos^2(\alpha) = \frac{81}{169}$, where $\theta \in \left[\pi, \frac{3\pi}{2}\right]$ and $\alpha \in \left[\frac{3\pi}{2}, 2\pi\right]$.
The value of $\sin(\theta) + \cos(\alpha)$ is:

A. $\frac{2}{13}$

B. $-\frac{2}{13}$

C. $\frac{16}{13}$

D. $-\frac{16}{13}$

$\sin(\theta) = \frac{-7}{13}$

$\cos^2(\alpha) = \frac{81}{169} \rightarrow \cos\alpha = \frac{9}{13}$ *since α is in fourth quadrant.*

So $\sin(\theta) + \cos(\alpha) = \frac{2}{13}$

Question 68

Jack's line of sight, while looking at a bird on top of a tree, makes a 30° angle of elevation. He then walks 150 metres toward the tree to observe the bird more closely, causing his line of sight to make a 45° angle of elevation. How far was Jack from the tree initially?

A. $\frac{150\sqrt{3}}{\sqrt{3}+1}$

B. $\frac{150\sqrt{3}}{\sqrt{3}-1}$

C. $50\sqrt{3}$

D. $75\sqrt{3}$

Let d be the initial horizontal distance from Jack to the tree's base, and
Let h be the height of the tree

Let the initial distance be d. Then, from the 30° angle : $\tan(30°) = \frac{d}{\sqrt{3}}$

After walking 150 m, his distance becomes d − 150 and

with a 45° angle, $\tan 45° = \frac{h}{d-150} = 1$ so $h = d - 150$

Now equating both : $\frac{d}{\sqrt{3}} = d - 150$

we get: $d = \frac{150\sqrt{3}}{\sqrt{3}-1}$

Question 69 (1 mark)

The implied domain of $y = \arccos\left(\frac{x-a}{b}\right)$, where $b > 0$ is:

A. $[-1, 1]$

B. $[a - b, a + b]$

C. $[a - 1, a + 1]$

D. $[a, a + b\pi]$

E. $[-b, b]$

$-1 \leq \frac{x-a}{b} \leq 1$

Question 70 (1 mark)

The implied domain of $f(x) = 2\cos^{-1}\left(\frac{1}{x}\right)$ is:

A. R

B. $[-1, 1]$

C. $(-\infty, -1] \cup [1, \infty)$

D. $R\backslash\{0\}$

E. $[-1, 1]\backslash\{0\}$

$-1 \le \frac{1}{x} \le 1 \Rightarrow x \in (-\infty, -1] \cup [1, \infty)$

Question 71 (1 mark)

The solutions to $\cos(x) > \frac{1}{4}\operatorname{cosec}(x)$ for $x = (0, 2\pi)\backslash\{\pi\}$ are given by:

A. $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\frac{5\pi}{12}, \frac{13\pi}{12}\right) \cup \left(\frac{17\pi}{12}, 2\pi\right)$

B. $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\frac{13\pi}{12}, \frac{17\pi}{12}\right)$

C. $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\pi, \frac{13\pi}{12}\right) \cup \left(\frac{13\pi}{12}, 2\pi\right)$

D. $x \in \left(\frac{\pi}{12}, \frac{13\pi}{12}\right) \cup \left(\frac{17\pi}{12}, 2\pi\right)$

E. $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\pi, \frac{13\pi}{12}\right) \cup \left(\frac{17\pi}{12}, 2\pi\right)$

The solve and graphing capabilities of a CAS could have been used to find the correct answer.

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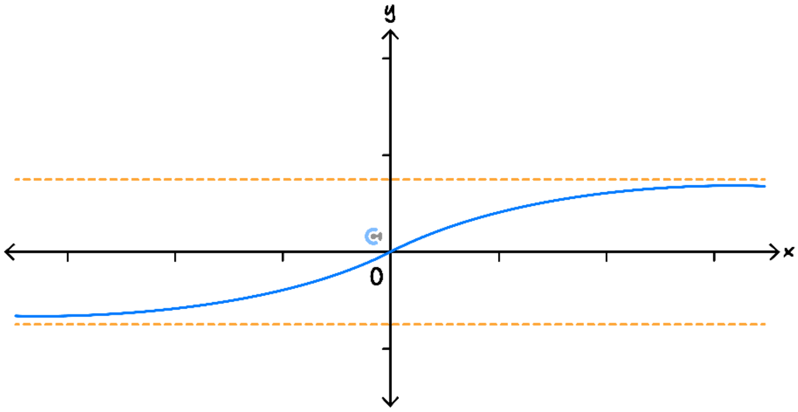
Question 72 (1 mark)

Part of the graph of $y = \frac{1}{2}\tan^{-1}(x)$ is shown below:



The equations of its asymptotes are:

A. $y = \pm\frac{1}{2}$

B. $y = \pm\frac{3}{4}$

C. $y = \pm 1$

D. $y = \pm\frac{\pi}{2}$

E. $y = \pm\frac{\pi}{4}$

Question 73 (1 mark)

Consider the function f with rule $f(x) = -\frac{1}{\sqrt{\sin^{-1}(cx+d)}}$, where $c, d \in R$ and $c > 0$. The domain of f is:

A. $x > -\frac{d}{c}$

B. $-\frac{d}{c} < x \le \frac{1-d}{c}$

C. $\frac{-1-d}{c} \le x \le \frac{1-d}{c}$

D. $x \in R \backslash \left\{-\frac{d}{c}\right\}$

E. $x \in R$

Question 74 (1 mark)

If $\cos(x) = -a$ and $\cot(x) = b$, where $a, b > 0$, then cosec $(-x)$ is equal to:

A. $\frac{b}{a}$

B. $-\frac{b}{a}$

C. $-\frac{a}{b}$

D. $\frac{a}{b}$

E. $-ab$

Question 75 (1 mark)

The implied domain of the function with rule $f(x) = 1 - \sec\left(x + \frac{\pi}{4}\right)$ is:

A. R

B. $[0, 2]$

C. $R \backslash \left\{\frac{(4n-1)\pi}{4}\right\}, n \in Z$

D. $R \backslash \left\{\frac{(4n+1)\pi}{4}\right\}, n \in Z$

E. $R \backslash \left\{\frac{(2n-1)\pi}{2}\right\}, n \in Z$

Question 76 (1 mark)

A function f has the rule $f(x) = |b\cos^{-1}(x) - a|$, where $a > 0$, $b > 0$ and $a < \frac{b\pi}{2}$. The range of f is:

A. $[-a, b\pi - a]$

B. $[0, b\pi - a]$

C. $[a, b\pi - a]$

D. $[0, b\pi + a]$

E. $[a - b\pi, a]$

Use transformations on $g(x) = \cos^{-1}(x)$

$a < \frac{b\pi}{2} \Rightarrow b\pi > 2a$

Question 77 (1 mark)

Let $f(x) = \frac{\sqrt{x-1}}{x}$ over its implied domain and $g(x) = \text{cosec}^2 x$ for $0 < x < \frac{\pi}{2}$.

The rule for $f(g(x))$ and the range, respectively, are given by:

A. $f(g(x)) = \text{cosec}^2\left(\frac{\sqrt{x-1}}{x}\right), [1, \infty)$

B. $f(g(x)) = \text{cosec}^2\left(\frac{\sqrt{x-1}}{x}\right), [2, \infty)$

C. $f(g(x)) = \sin(x)\cos(x), [-0.5, 0.5]\backslash\{0\}$

D. $f(g(x)) = \sin(x)\cos(x), \left(0, \frac{1}{2}\right)$

E. $f(g(x)) = \frac{1}{2}\sin(2x), \left(0, \frac{1}{2}\right]$

Options C, D and E have the correct rule but only option E has the correct range. The range must include $\frac{1}{2}$ as $\frac{1}{2}\sin(2x)$ is a maximum at $x = \frac{\pi}{4}$.

Question 78 (1 mark)

Let $f(x) = \frac{1}{\sec(3x)+\frac{3}{2}}$.

The number of asymptotes that the graph of f has in the interval $\left[-\frac{\pi}{6}, \pi\right]$ is:

A. 2

B. 3

C. 4

D. 5

E. 6

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Question 79 (1 mark)

The implied domain of the function with rule $f(x) = \cos^{-1}(\log_e(bx))$, $b > 0$ is:

A. $(0, 1]$

B. $[1, e]$

C. $\left[\frac{1}{b}, \frac{e}{b}\right]$

D. $\left[\frac{1}{b}, \frac{e^{\pi}}{b}\right]$

E. $\left[\frac{1}{be}, \frac{e}{b}\right]$

Question 80 (1 mark)

The expression $1 - \frac{4\sin^2(x)}{\tan^2(x)+1}$ simplifies to:

A. $\sin(x)\cos(x)$

B. $1 - 2\cos^2(2x)$

C. $2\sin(2x)$

D. $2\sin^2(2x)$

E. $\cos^2(2x)$

Question 81 (1 mark)

In the interval $-\pi \leq x \leq \pi$, the graph of $y = a + \sec(x)$, where $a \in R$, has two x-intercepts when:

A. $0 \leq a \leq 1$

B. $-1 < a < 1$

C. $a \leq -1$ or $a > 1$

D. $-1 \leq a < 0$

E. $a < -1$ or $a \geq 1$

Question 82 (1 mark)

Given that $\sin(x) = a$, where $x \in \left(\frac{3\pi}{2}, 2\pi\right)$, the $\cos\left(\frac{x}{2}\right)$ is equal to:

A. $-\frac{\sqrt{1+\sqrt{1-a^2}}}{\sqrt{2}}$

B. $\frac{\sqrt{1-\sqrt{a^2-1}}}{\sqrt{2}}$

C. $\frac{\sqrt{1+\sqrt{1-a^2}}}{\sqrt{2}}$

D. $-\frac{\sqrt{\sqrt{1-a^2}-1}}{\sqrt{2}}$

$\cos^2(x) + \sin^2(x) = 1$

$\Rightarrow \cos(x) = \sqrt{1-a^2}$

$\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$\sqrt{1-a^2} = 2\cos^2\left(\frac{x}{2}\right) - 1$

$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\sqrt{1-a^2}+1}{2}}$

$x \in \left(\frac{3\pi}{2}, 2\pi\right) \therefore \frac{x}{2} \in \left(\frac{3\pi}{4}, \pi\right) \therefore \cos\left(\frac{x}{2}\right) < 0 \therefore$ take negative root

Take positive root since $x \in \left(\frac{3\pi}{2}, 2\pi\right)$ so $\cos x > 0$

Question 83 (1 mark)

For the function $f: R \to R, f(x) = k\arctan(ax - b) + c$, where $k > 0$, $c > 0$ and $a, b = R$, $f(x) > 0$ if:

A. $c < \frac{k\pi}{2}$

B. $c \geq \frac{k\pi}{2}$

C. $x > \frac{b}{a}$

D. $c + k > \frac{\pi}{2}$

E. $c \geq \frac{\pi}{2}$

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Question 84 (1 mark)

If $\sin(\theta + \emptyset) = a$ and $\sin(\theta - \emptyset) = b$, then $\sin(\theta)\cos(\emptyset)$ is equal to:

A. ab

B. $\sqrt{a^2 + b^2}$

C. $\sqrt{ab}$

D. $\sqrt{a^2 - b^2}$

E. $\frac{a+b}{2}$

Question 85 (1 mark)

Let $f(x) = \text{cosec}\,(x)$. The graph of f is transformed by:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 1 unit horizontally to the right, followed by,
- A dilation by a factor of $\frac{1}{2}$ from the y-axis.

The rule of the transformed graph is:

A. $g(x) = 2\text{cosec}(3x + 1)$

B. $g(x) = 3\text{cosec}(2x - 1)$

C. $g(x) = 3\text{cosec}(2(x - 1))$

D. $g(x) = 2\text{cosec}\left(\frac{x}{3} - 1\right)$

E. $g(x) = 3\text{cosec}\left(\frac{x-1}{2}\right)$

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Question 86 (1 mark)

Let $f(x) = \frac{\sqrt{x+1}}{x}$ and $g(x) = \tan^2(x)$, where $0 < x < \frac{\pi}{2}$. $f(g(x))$ is equal to:

A. $\sin(x)\sec^2(x)$

B. $\sec(x)\tan^2(x)$

C. $\cos(x)\cot^2(x)$

D. $\cos(x)\text{cosec}^2(x)$

E. $\text{cosec}(x)\cos^2(x)$

Question 87 (1 mark)

The implied domain of the function with rule $f(x) = \frac{3x}{\frac{\pi}{2} - \arccos(2-x)}$ is:

A. $[1, 3]$

B. $[-1, 1]$

C. $[0,1] \cup (1, 2]$

D. $[-1,0) \cup (0, 1]$

E. $[1, 2) \cup (2, 3]$

Question 88 (1 mark)

The maximal domain and range of the function $f(x) = a\cos^{-1}(bx) + c$, where a, b, and c are real constants with $a > 0$, $b < 0$ and $c > 0$, are respectively:

A. $[0, \pi]$ and $[-a, a]$

B. $[0, \pi]$ and $[-a + c, a + c]$

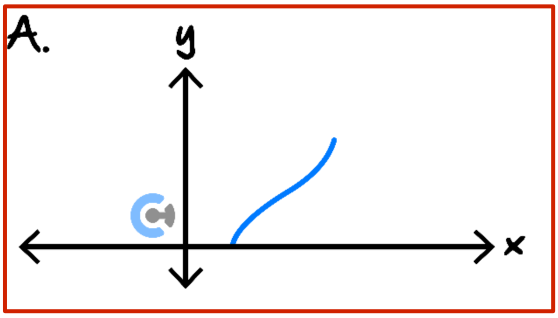
C. $\left[-\frac{1}{b}, \frac{1}{b}\right]$ and $[c, a\pi + c]$

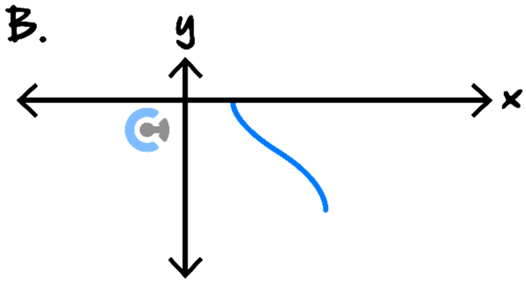
D. $\left[\frac{1}{b}, -\frac{1}{b}\right]$ and $[c, a\pi + c]$

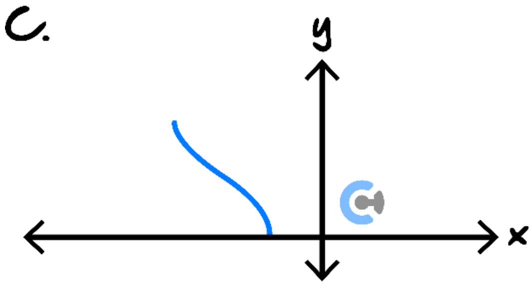
E. $\left[\frac{1}{b}, -\frac{1}{b}\right]$ and $[-a\pi + c, a\pi + c]$

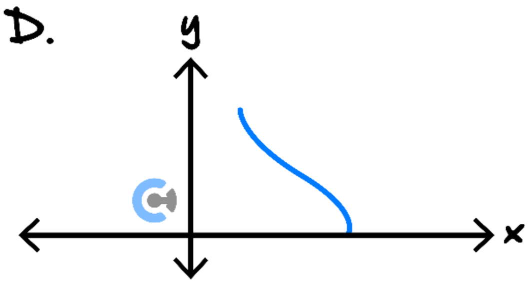
Question 89 (1 mark)

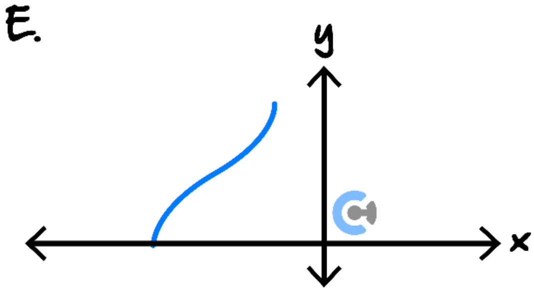
The graph of $y = \cos^{-1}(2 - bx)$, where b is a positive real constant, could be:











Question 90 (1 mark)

If the implied domain of $y = \sin(\cos^{-1}(ax - 1))$, where $a \in R \setminus \{0\}$, is the same as the range, then the value of a is:

A. -2

B. -1

C. 1

D. 2

E. 3

Question 91 (1 mark)

The implied domain and range of $f(x) = \sin(\cos^{-1}(1 - 2x))$ are respectively:

A. $[0, 1]$ and $[0, 1]$

B. $[-1, 0]$ and $[0, 1]$

C. R and $[-1, 1]$

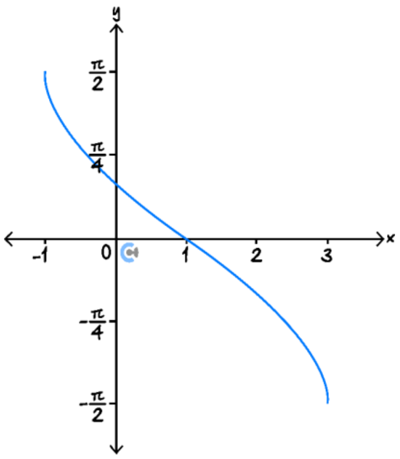
D. $[0, 1]$ and $[1, 1]$

E. R and $[0, 1]$

Question 92 (1 mark)

Let $f(x) = \arcsin(x)$ and $g(x) = ax + b$, where $a, b \in R$.

The graph of $y = f(g(x))$ is shown below.



The values of a and b are, respectively:

A. $\frac{1}{2}$ and $\frac{1}{2}$

B. $-\frac{1}{2}$ and $-\frac{1}{2}$

C. $-\frac{1}{2}$ and $\frac{1}{2}$

D. $\frac{1}{2}$ and 1

E. $-\frac{1}{2}$ and 1

Question 93 (1 mark)

The solutions of $\frac{1+5\sin(x)\cos(x)}{\cos^2(x)} - 7 = 0$ can be found by solving:

A. $(\tan(x) - 2)(\tan(x) + 3) = 0$

B. $(\tan(x) - 1)(\tan(x) + 7) = 0$

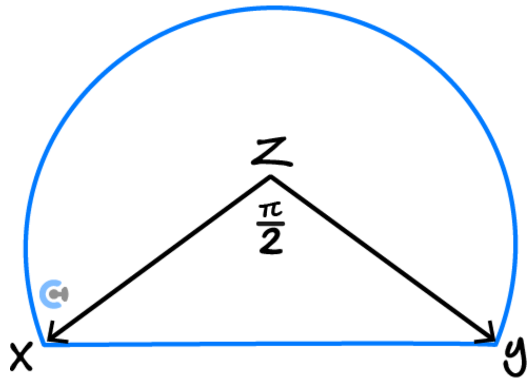
C. $(\tan(x) - 3)(\tan(x) - 2) = 0$

D. $(\tan(x) - 1)(\tan(x) + 6) = 0$

E. $(\tan(x) + 1)(\tan(x) + 6) = 0$

Question 94

The figure below shows the cross-section of a railway tunnel, modelled as the major segment of a circle, centre at Z and radius of 5 m. The angle $\angle XZY$ is $\frac{\pi}{2}$ radians.



a. Find the exact length of XY.

$$AB = 2 \times 5 \times \sin\left(\frac{\pi}{4}\right) = 5\sqrt{2}\ m$$

b. Determine the area of the triangle ACB.

$$\text{Area} = \frac{1}{2} \times 5 \times 5 \times \sin\left(\frac{\pi}{2}\right) = \frac{25}{2}\ sq.m$$

c. Find the cross-sectional area of the tunnel.

$$\text{Area} = \text{major sector} + \text{triangle}$$

$$\text{Area} = \frac{1}{2} \times 25 \times \frac{\pi}{2} \times + \frac{25}{2}$$

$$= \frac{25\pi}{4} + \frac{25}{2}\ sq.m$$

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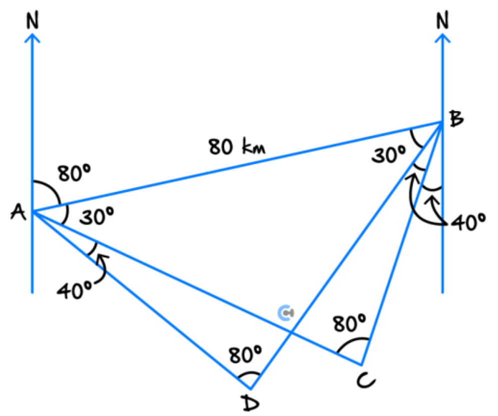
Question 95

The distance between the town of Algebraville (A) and the town of Baseville (B) is 80 km. Baseville is on a bearing of 80° from Algebraville.
The village of Contourville (C) is on a bearing of 110° from Algebraville and on a bearing of 190° from Baseville. The village of Desmosville (D) is on a bearing of 150° from Algebraville and on a bearing of 230° from Baseville.

a. Find, correct to one decimal place where appropriate, the distance between:

i. Baseville and

Our first step should be to draw a diagram and fill in as much information as we can.



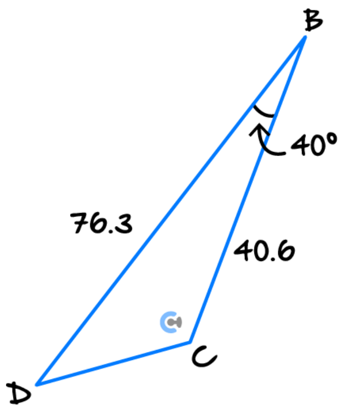
Then $\frac{80}{\sin(80)} = \frac{BC}{\sin(30)} \rightarrow BC = \frac{40}{\sin(80)} \approx 40.6\ km$

ii. Baseville and

$$\frac{80}{\sin(80)} = \frac{BD}{\sin(70)} \rightarrow BD \approx 76.3\ km$$

iii. Contourville and Desmosville.

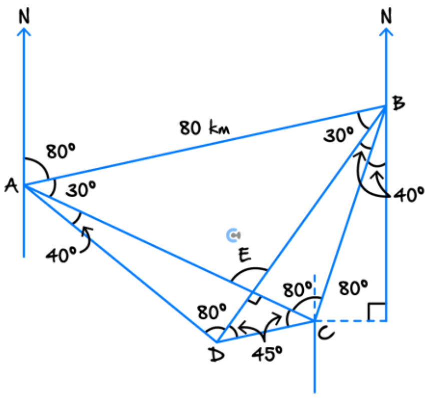
Draw a diagram and use the cosine rule:



$|DC| = \sqrt{|DB|^2 + |BC|^2 - 2|DB||BC|\cos(40°)} \approx 52.2\ km$

b. Find the bearing of Desmosville from C

A diagram and noting that DEC is a right angled isosceles triangle will help us to get the bearing as:
$360 - (70 + 30) = 260°$



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Question 96

The population of birds in a particular location varies according to the rule:

$$b(t) = 1200 + 300\cos\left(\frac{\pi t}{6}\right),$$

where b is the number of birds and t is the number of months after 1 April 2020.

a. Find the period and amplitude of the function $b(t)$.

$$b(t) = 1200 + 300\cos\left(\frac{\pi t}{6}\right)$$

comparing with $b(t) = A + B\cos\left(\frac{\pi t}{C}\right)$,

where $B =$ Amplitude and $2C =$ Period.

So, Period $= 12$ and Amplitude $= 300$.

b. Find the maximum and minimum populations of birds in this location.

Maximum when $\cos\left(\frac{\pi t}{6}\right) = 1 \rightarrow 1200 + 300 \times 1 = 1500$

Minimum when $\cos\left(\frac{\pi t}{6}\right) = -1 \rightarrow 1200 + 300 \times (-1) = 900$

Maximum Population $= 1500$

Minimum Population $= 900$

c. Find $b(4)$.

$$b(4) = 1200 + 300\cos\left(\frac{2\pi}{3}\right)$$
$$b(4) = 1200 + 300\cos\left(-\frac{1}{2}\right)$$
$$b(4) = 1050$$

d. Over the 10 months from 1 April 2020, find the fraction of time when the population of birds in this location was less than $b(4)$.

We find the population is greater than or equal to $b(4) = 1050$ for $t \in [0,4] \cup [4,8]$.
So, the fraction of time the population is less than $b(4)$ is $\frac{4}{8} = \frac{1}{2}$.

Question 97 (5 marks)

Consider the function $f: D \to R$, where $f(x) = 2\arcsin(x^2 - 1)$.

a. Determine the maximal domain D and the range of f. (2 marks)

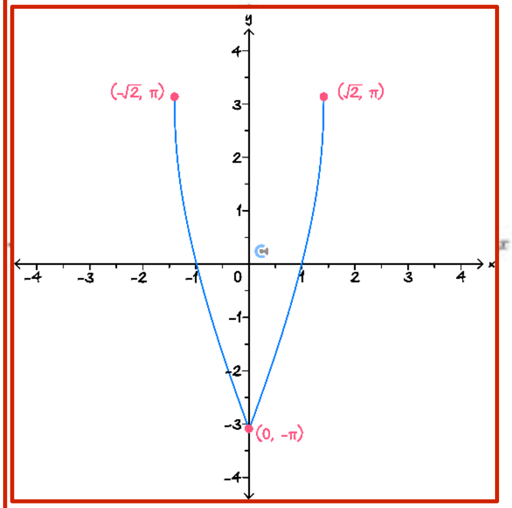
Marks	0	1	2	Average
%	16	16	68	1.5

Domain $\left[-\sqrt{2}, \sqrt{2}\right]$, Range $[-\pi, \pi]$

This question was generally handled well. Common errors included: giving open endpoints with round brackets on the intervals, decimal approximations rather than exact values and failing to state the range. Students should read questions carefully and ensure that all required information is supplied in their responses.

b. Sketch the graph of $y = f(x)$ on the axes below, labelling any endpoints and the y-intercept with their coordinates. (3 marks)

Marks	0	1	2	3	Average
%	6	13	32	50	2.3



This question required students to label any endpoints and the y-intercept with their coordinates. Not doing this or giving incorrect coordinates frequently caused students to miss out on marks.

Students' graphs were not always precise and accurate as required. For example, many graphs had an obvious turning point at the y-intercept rather than the required shape. Other incorrect responses had endpoints in the incorrect location.

Students are advised to set viewing windows on technology to a scale that closely matches the scale provided on the examination.

Question 98 (6 marks)

a.

i. Use an appropriate double-angle formula with $t = \tan\left(\frac{5\pi}{12}\right)$ to deduce a quadratic equation of the form $t^2 + bt + c = 0$, where b and c are real values. (2 marks)

$$t^2 - 2\sqrt{3}t - 1 = 0$$

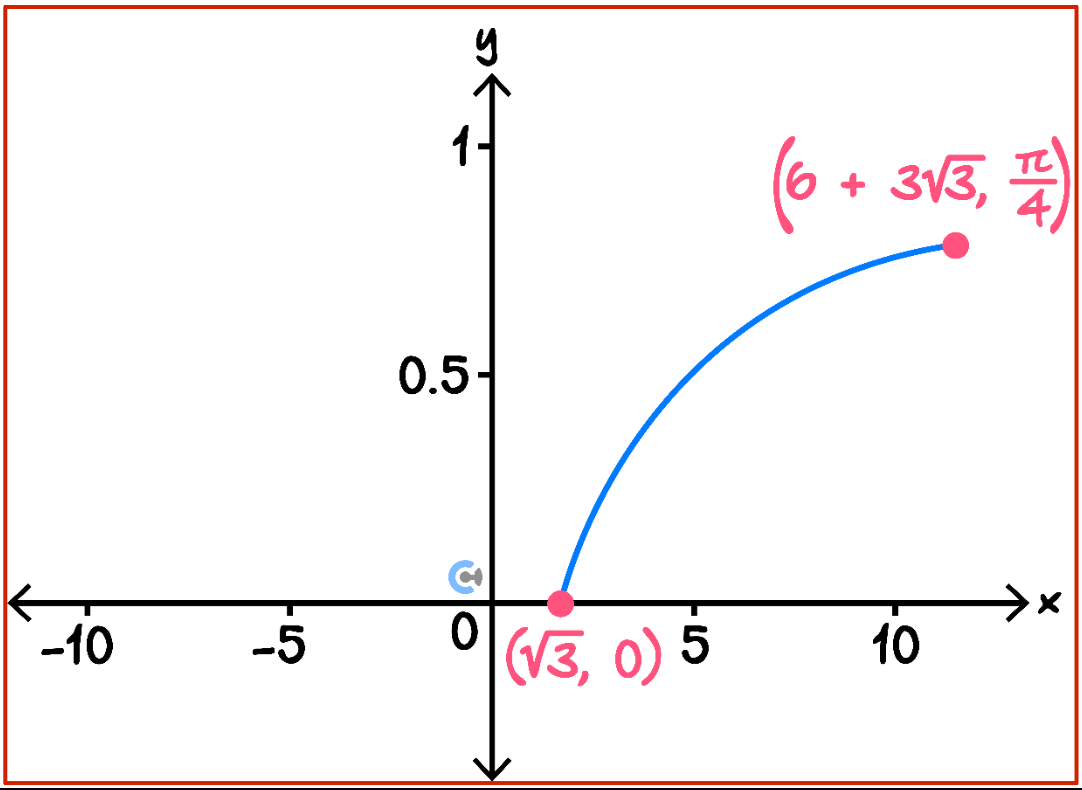
ii. Hence, show that $\tan\left(\frac{5\pi}{12}\right) = 2 + \sqrt{3}$. (1 mark)

Solve quadratic equation and identify correct root.

$\tan\left(\frac{5\pi}{12}\right) = \sqrt{3} + 2$ * Answer given

Consider $f: [\sqrt{3}, 6 + 3\sqrt{3}] \to R, f(x) = \arctan\left(\frac{x}{3}\right) - \frac{\pi}{6}$.

b. Sketch the graph of f on the axes below, labelling the endpoints with their coordinates. (3 marks)



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VCE Specialist Mathematics ½

Free 1-on-1 Consults

What Are 1-on-1 Consults?

- **Who Runs Them?** Experienced Contour tutors (45 + raw scores and 99 + ATARs).
- **Who Can Join?** Fully enrolled Contour students.
- **When Are They?** 30-minute 1-on-1 help sessions, after school weekdays, and all day weekends.
- **What To Do?** Join on time, ask questions, re-learn concepts, or extend yourself!
- **Price?** Completely free!
- **One Active Booking Per Subject:** Must attend your current consultation before scheduling the next. :)

SAVE THE LINK, AND MAKE THE MOST OF THIS (FREE) SERVICE!

Booking Link

bit.ly/contour-specialist-consult-2025