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VCE Specialist Mathematics ½
Proofs Exam Skills [2.3]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [2.3.1]** - Solve problems using AM-GM inequalities
- ☐ **SM12 [2.3.2]** - Solve Arithmetic and Geometric Series Proofs
- ☐ **SM12 [2.3.3]** - Prove divisibility with induction

Section A: Recap

(All notes from [2.2] - Proofs II & [2.1] - Proofs I!)

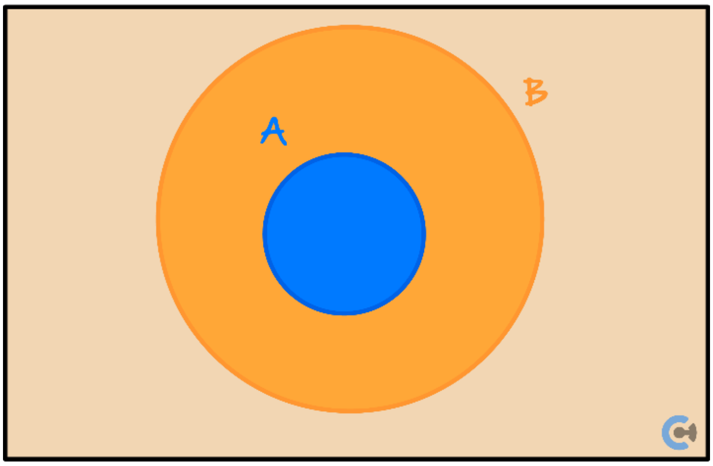
If you were here last week skip to Section B Warmup test.

Please revise 2.1 & 2.2 for proofs content.

Exploration: Visualisation of Conditional Statements

$$A \Rightarrow B$$

Visualisation of $A \Rightarrow B$



If A, then B.

Conditional Statements

- Conditional Statement: Hypothesis Implies Conclusion.
- Note: Notation for "implies": $\Longrightarrow$

$$\text{"}\textit{Hypothesis} \Longrightarrow \textit{Conclusion}\text{"}$$

NOTE: Order Matters!

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Question 1

Write a conditional statement for the following:

Doing strength training grows your muscles.

If you do strength training, your muscles will get bigger.

Proving Conditional Statements

1. Direct Proof.

2. Proof by Contrapositive.

3. Proof by Contradiction.

REMINDER

- From last week, we found that an even number can be written in the form:

$$2k, \text{ where } k \in Z.$$

- An odd number can be written in the form:

$$2k+1, \text{ where } k \in Z.$$

- If a number is divisible by 3, then it can be written in the form:

$$3k, \text{ where } k \in Z.$$

- If a number is not divisible by 3, then it can be written in the form:

$$3k+1 \text{ or } 3k+2, \text{ where } k \in Z.$$

- For proofs involving divisibility, we often need to split into cases.

Method 1: Direct Proof

- The simplest method of proof. These are the proofs we have been working on in [2.1].
- Here, we are not altering the statement we need to prove.

Question 2

Prove that for all integers m and n, if m is divisible by 5 and n is divisible by 2, then $10m + 3n$ is even.

Solution: Let $m = 5k$ and $n = 2j$ for $k, j \in \mathbb{Z}$. Therefore, $10m + 2n = 50k + 4j = 2(25k + 2j) = 2p$ where $p = 25k + 2j \in \mathbb{Z}$.

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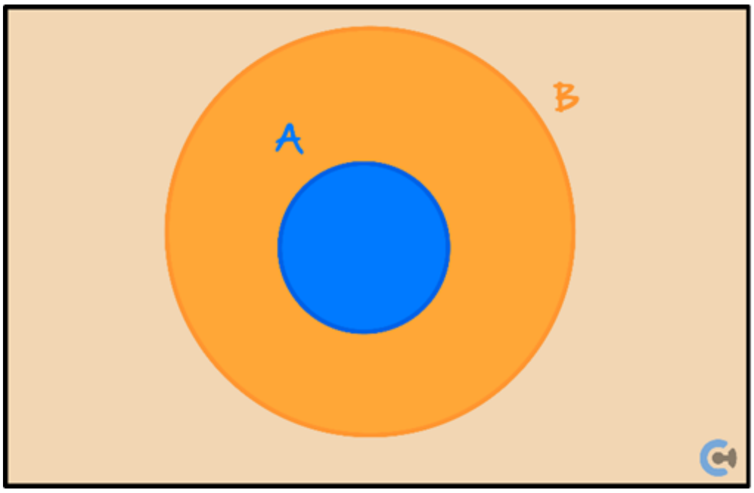
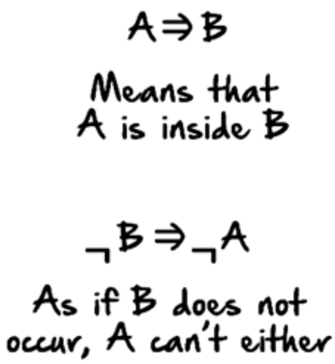
What is a contrapositive statement?

Contrapositive Statement





(Contrapositive Statement of $A \Rightarrow B$) is $\neg B \Rightarrow \neg A$

NOTE: Swap the order and negate the statements.

Question 3

Write down the contrapositive of the following statement:

If all the sides of a triangle have equal length, then the triangle is equilateral.

If a triangle is not equilateral, not all of its sides have equal length.

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Method 2: Proof by Contrapositive (Indirect Proof)

- Instead of proving $A \Rightarrow B$, we can prove its contrapositive $\neg B \Rightarrow \neg A$.

Prove a contrapositive statement instead.

- Considered to be an indirect proof, as the original statement is altered.
- **Steps**:
  1. Set up the contrapositive statement.
  2. Prove the contrapositive statement to be true.
  3. Conclude by saying "As contrapositive is true, the original statement is true."

Question 4

Prove the following conditional statement using contrapositive:

$$\text{Let } x \in R. \text{ If } x \text{ is irrational, then } \sqrt{x+\frac{1}{5}} \text{ is irrational.}$$

Solution: We prove the contrapositive statement "If $\sqrt{x+1/5}$ is rational, then x is rational." Indeed, since $\sqrt{x+1/5}$ is rational, $\sqrt{x+1/5} = p/q$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. Solving for x, we find that $x = (p/q)^2 - 1/5 = (5p^2 - q^2)/5q^2 = r/s$ where $r = 5p^2 - q^2 \in \mathbb{Z}$, $s = 5q^2 \in \mathbb{Z}$ and $s \neq 0$. Hence, as the contrapositive statement holds, the original statement is also holds.

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What is a contradicting statement?

Contradicting Statement

(Contradicting Statement of $A \Rightarrow B$) is $A \Rightarrow \neg B$

- Negate the conclusion.

Question 5

State the contradicting statement of the following:

If x is rational, then x^2 is rational.

If x is rational, then x^2 is irrational.

Method 3: Proof by Contradiction (Indirect Proof)

To Prove $A \Rightarrow B$

Assume $A \Rightarrow \neg B$ is true

And show that the assumption is FALSE.

- Steps:
  1. First, assume that the contradicting statement is true.
  2. Show that the assumption has a contradiction, and is hence false.
  3. Conclude by saying "Since the contradicting statement is false, the original statement is true."
- Considered to be an indirect proof, as the original statement is altered.

<u>Active Recall:</u> De Morgan's Law

$\neg(A \wedge B) =$ $\neg(A \wedge B) = \neg A \vee \neg B$

$\neg(A \vee B) =$ $\neg(A \vee B) = \neg A \wedge \neg B$

Question 6

Let $a, b \in R$. Prove that if $a + b > 150$, then $a > 75$ or $b > 75$.

STEP 1. Set up the contradicting statement (Assume opposite is true).

It's going to be, AND due to De Morgan's law $(A \cup B)' = A' \cap B'$.

STEP 2. Prove it to be FALSE → Original statement is TRUE.

Solution: Assume for contradiction that $a + b > 150$ but $a \leq 75$ **and** $b \leq 75$. Then $a + b \leq 150$, which is a contradiction. Therefore the original statement holds.

Solution: Alternatively, $a > 150 - b$ so that $75 > a > 150 - b$ implies that $b > 75$, which contradicts the fact that $b \leq 75$. Therefore the original statement holds.

<u>Key Takeaways</u>

- ☑ Direct proof involves proving without changing the conditional statement.
- ☑ The contrapositive of a statement $A \to B$ is given by $\neg B \to \neg A$.
- ☑ Proof by contrapositive involves proving the contrapositive of the conditional statement instead.
- ☑ Proof by contradiction requires first assuming that the contradicting statement is true, then showing contradiction.

Converse Statements

- **Definition**: Conditional statement that flows in the opposite direction.

$$(\text{Converse Statement of } A \Rightarrow B) \text{ is } B \Rightarrow A$$

Question 7

For the following statement, write down the converse statement, and conclude whether the converse is true:

If x is divisible by 2 and 5, then it is divisible by 10.

If x is divisible by 10, then its divisible by 2 and 5: TRUE

Equivalent Statements (Biconditional)

- It is a biconditional statement where if the original is ____proven to be true____, its converse is ____ALWAYS true____.

$$A \Rightarrow B \text{ and } B \Rightarrow A$$

$$A \Leftrightarrow B$$

$$A \text{ is true, if and only if } B$$

- In description, B is true ____If and Only If____ A is true.
- To prove equivalent statements, we prove each direction separately.

NOTE: For if and only if (equivalent statement), we must prove both converse statements.

Question 8

Let n be an integer. Prove that n is odd, if and only if n^2 is odd.

=> If n is odd, n^2 is odd

Let $n \in \mathbb{Z}$, where $n \geq 0$.

Suppose n is odd.

By definition, $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Then n^2

$$= (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

Also, $(2k^2 + 2k) \in \mathbb{Z}$ since $k \in \mathbb{Z}$.

Thus, n^2 is odd by definition.

<= if n^2 odd, n is odd (prove by contrapositive)

P: n odd
Q: n^2 odd
$Q \Rightarrow P$
Contrapositive: $\sim P \Rightarrow \sim Q$
$\sim P$: n even
$\sim Q$: n^2 even

Proof: Let $n \in \mathbb{Z}$, where $n \geq 0$.

($\Leftarrow$): Suppose n is even.

By definition, $n = 2k$ for some $k \in \mathbb{Z}$.

Then n^2

$$= (2k)^2$$
$$= 4k^2$$
$$= 2(2k^2)$$

Also, $(2k^2)$ is an integer since $k \in \mathbb{Z}$.

Thus, n^2 is even by definition.

Key Takeaways

- ☑ The converse statement of $A \to B$ is given by $B \to A$.
- ☑ An equivalent statement is when a statement and its converse both are proved to be true.
- ☑ "If and only if" stands for an equivalent statement.

Universal Quantifiers

- Universal quantifier is a way to represent all members of a given set.

For all real numbers x, x^2 is never negative.

- Notation: ∀ (Universal Quantifier) "For All".

$$\forall x \in R, x^2 \geq 0$$

Question 9

Rewrite the following statement using the universal quantifier:

For all integers n, $n^2 - 4n$ is an integer.

Solution: $\forall n \in \mathbb{Z}, n^2 - 4n \in \mathbb{Z}$

Existence Quantifiers

- Existence Quantifier is a way to represent certain members of a given set.

There exists an integer such that $x^2 - x - 12 = 0$.

- Notation: ∃ (Existential Quantifier) "There Exists".

$$\exists x \in \mathbb{Z}, x^2 - x - 12 = 0.$$

Question 10

Rewrite the following statement using the existence quantifier.

There exists a positive whole number n such that $2^n = 4$.

Solution: $\exists n \in \mathbb{N}, 2^n = 4$

Negation of Universal and Existence Statements

- They are opposites of each other (with opposite conclusions).

$\neg$Universal Statement = Existence Statement with Opposite Conclusion

- And vice versa.

Question 11

Write down the negation for the following statements below.

a. If n is a natural number, then $n + 1 > n$.

$$\neg(\forall n \in N, n + 1 > n) = \exists n \in N, n \leq n + 1$$

b. There exists an integer k such that $k^2 = k + 4$.

$$\neg(\exists k \in Z, k^2 = k + 4) = \forall k \in Z, k^2 \neq k + 4$$

Disproving Universal Statement

- We prove the *opposite* (negation) existence statement.
- We call this proof by counter example.
  - Giving a counterexample will be proving an opposite existence statement.

Question 12

Disprove the following statement: **For all positive integers m, if m is prime then $m^2 + 4$ is also prime.**

Counter example:
Assume $m = 2$ which is a positive, prime integer.
Then it follows that:

$$\begin{aligned} m^2 + 4 &= (2)^2 + 4 \\ &= 4 + 4 \\ &= 8 \end{aligned}$$

However, since 8 is a composite number, not a prime number, this is a contradiction and we have shown the claim to be false by counter example.

NOTE: It is important to understand this concept as it will be used for contrapositive, contradiction method.

NOTE: We simply give ourselves a counter-example to disprove a universal statement.

Disproving Existence Statements

- We prove the *opposite* (negation) universal statement.

NOTE: To disprove their existing statement, you must show the opposite universal statement as true.

Question 13

Disprove the following statements:

There exists a real number x, such that $10 + 3x^2 = 3 + x^2$.

Negation: For all real numbers, $10+3x^2 \neq 3+x^2$

$10+3x^2 \neq 3+x^2$

$2x^2 \neq -7$

$x^2 \neq -\frac{7}{2}$

True

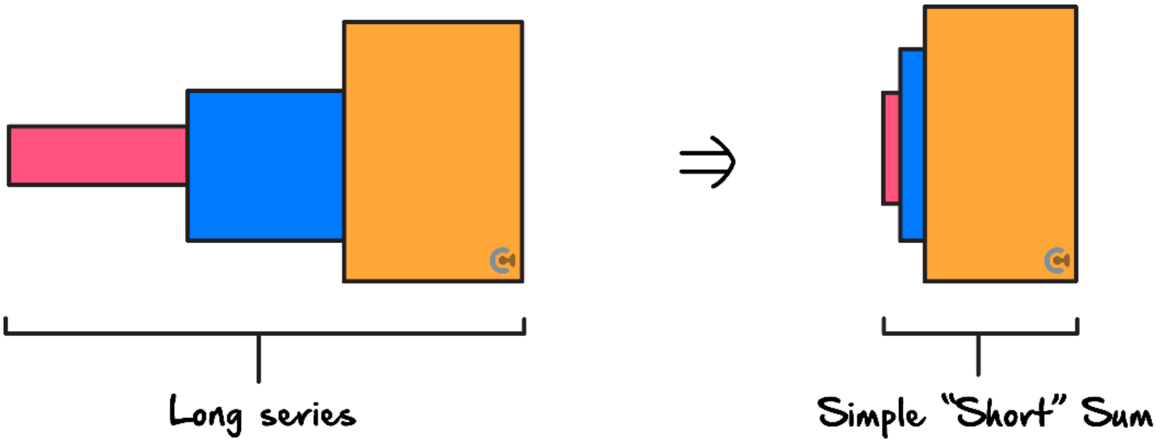
$\therefore$ There exist a real no such that $10+3x^2 = 3+x^2$ is a false statement

Key Takeaways

- There exists a quantifier ∃.
- For all quantifiers ∀.
- To prove a there exists statement, simply give an example.
- To prove for all statements, you must prove for all values.
- To disprove a there exist statement, prove the opposite for all statements.
- To disprove a for-all statement, prove the opposite there exist statement.

Telescopic Cancelling

- Telescoping series is a series (sum of terms) where all terms cancel out except for the first and last one.
- The name comes from the visualisation as shown below.



- Proofs of telescoping series can be done via telescopic cancelling: Shortening the telescope.

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Question 14

a. Using partial fractions, find the values of a and b such that:

$$\frac{3}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}$$

$a=3$ and $b=-3$.

b. Hence, prove that:

$$\frac{3}{1 \times 2} + \frac{3}{2 \times 3} + \cdots + \frac{3}{n \times (n+1)} = \frac{3n}{n+1}$$

b We use the result from part **a** to expand each term of the series:

$$\frac{3}{1\cdot 2} + \frac{3}{2\cdot 3} + \cdots + \frac{3}{(n-1)n} + \frac{3}{n(n+1)}$$

$$= \left(\frac{3}{1} - \frac{3}{2}\right) + \left(\frac{3}{2} - \frac{3}{3}\right) + \cdots + \left(\frac{3}{n-1} - \frac{3}{n}\right) + \left(\frac{3}{n} - \frac{3}{n+1}\right)$$

$$= \frac{3}{1} + \left(-\frac{3}{2} + \frac{3}{2}\right) + \left(-\frac{3}{3} + \frac{3}{3}\right) + \cdots + \left(-\frac{3}{n} + \frac{3}{n}\right) - \frac{3}{n+1} \quad \text{(regrouping)}$$

$$= 3 - \frac{3}{n+1} \quad \text{(cancelling)}$$

Cancel

$$= \frac{3n}{n+1}$$

<u>**Proof by Induction**</u>

- We will be following the given steps:" TAPE ".
  - **Step T:** Test the statement for its first possible value.
  - **Step A:** Assume that the statement is true for $n = k$.
  - **Step P:** Prove that if the statement holds true for $n = k$, It also holds true for $n = k + 1$.
    - Add "By Assumption".
  - **Step E:** "By the principle of mathematical induction, the statement is true for a set of values."

Question 15

Prove that for all $n \in N$, we have:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

NOTE: $1 + 2 + \cdots + n$ can be written as:

$$\sum_{i=1}^{n} i$$

1. Test for $n = 1$. (As $n \in \mathbb{N}$: First Domino)

LHS : 1.

RHS: $\frac{1(1+1)}{2} = \frac{2}{2} = 1$ ∴ Statement holds

2. Assume $P(k)$ is true

$1 + 2 + \cdots k = \frac{k(k+1)}{2}$

3. Prove $P(k+1)$ is true

$1 + 2 + \cdots k + k + 1 = \frac{(k+1)(k+2)}{2}$.

LHS: $\underbrace{1 + 2 + \cdots k} + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = RHS$

∴ Proven

4. Explain

By principle of mathematical induction, the statement is true for $n = 1, 2, 3, \ldots$

Section B: Warmup (15 Marks)

Question 16 (4 marks)

Let $n \in N$. Consider the statement

A B

If $3^n - 1$ is prime, then n is odd.

a. Write down the contraposite of the statement. (1 mark)

$\neg B \Rightarrow \neg A$

If n is even then $3^n - 1$ is not prime

b. Prove that the contrapositive is true. (3 marks)

n is even, so let $n = 2k$, $k \in N$

Then, $3^n - 1$

$= 3^{2k} - 1$

$= (3^k)^2 - 1^2$

$= (3^k - 1)(3^k + 1)$

Since $k \in N$, $3^k - 1$, $3^k + 1 > 1$

$\therefore$ $3^n - 1$ is not prime since it has factors other than itself and 1.

Hence, the contrapositive statement is true.

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Question 17 (3 marks)

Prove that if x is irrational then, $\sqrt{x-3}$ is irrational. Use the contradiction method in your proof.

For contradiction, suppose x is irrational but $\sqrt{x-3}$ is rational.

Then, $\sqrt{x-3} = p/q$, $p, q \in \mathbb{Z}$ & $q \neq 0$

$x - 3 = p^2/q^2$

$x = \dfrac{p^2 + 3q^2}{q^2}$

But $p^2 + 3q^2 \in \mathbb{Z}$ & $q^2 \in \mathbb{Z}$ & $q^2 \neq 0$

$\therefore$ x is rational.

This is a contradiction, so original statement is true.

Space for Personal Notes

Question 18 (4 marks)

Prove that the following is true for all positive integers n: n is odd if and only if $n^2 + 4$ is odd.

A $\iff$ B

(1) $A \Rightarrow B$: n is odd $\Rightarrow n^2+4$ is odd

n is odd, $\therefore n = 2k+1, k \in \mathbb{Z}$

Then, $n^2+4 = (2k+1)^2+4$

$= 4k^2+4k+1+4$

$= 2(2k^2+2k+2)+1$

$\therefore n^2+4$ is odd, as $2k^2+2k+2 \in \mathbb{Z}$.

(2) $B \Rightarrow A$: n^2+4 is odd $\Rightarrow n$ is odd

Prove instead the contrapositive; that is,

If n is even, then n^2+4 is even

$n = 2k, k \in \mathbb{Z}$

Then, $n^2+4 = 4k^2+4$

$= 2(2k^2+2)$

$= 2p, \; p = 2k^2+2 \in \mathbb{Z}$

$\therefore n^2+4$ is even.

Hence, (1) & (2) are true, so the original statement is true.

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$\frac{k+1}{6k+15}$

Question 19 (4 marks)

Prove that $\frac{1}{3\times5}+\frac{1}{5\times7}+\frac{1}{7\times9}+\cdots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{6n+9}$ for all $n\in N$.

$P(n): \frac{1}{3(5)}+\frac{1}{5(7)}+\frac{1}{7(9)}+\dots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{6n+9}$

① Test Base Case : $n=1$

LHS $=\frac{1}{3(5)}=\frac{1}{15}$, RHS $=\frac{1}{6(1)+9}=\frac{1}{15}$

$\therefore$ LHS = RHS so $P(1)$ true.

② Assumption: Let $k\in N$, and assume $P(k)$ is true.

$\frac{1}{3(5)}+\frac{1}{5(7)}+\dots\frac{1}{(2k+1)(2k+3)}=\frac{k}{6k+9}$

③ Prove:

Then LHS of $P(k+1)=\frac{1}{3(5)}+\frac{1}{5(7)}+\dots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}$

$=\frac{k}{6k+9}+\frac{1}{(2k+3)(2k+5)}$

$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)}$

$=\frac{k(2k+5)+3}{3(2k+3)(2k+5)}$

$=\frac{2k^2+5k+3}{3(2k+3)(2k+5)}$

$=\frac{(2k+3)(k+1)}{3(2k+3)(2k+5)}$

$=\frac{k+1}{6k+15}$

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④ Explain: 'Since $P(1)$ true, & $P(k)\Rightarrow P(k+1)$, then $P(n)$ true for all $n\in N$.'

Section C: Exam Skills

Sub-Section: Solve Problems Using AM-GM Inequalities

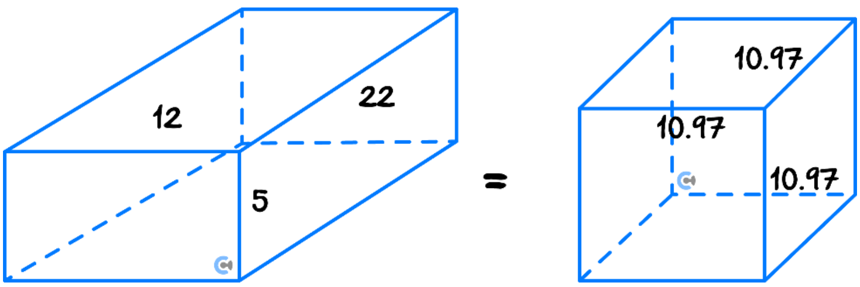
Context: Different Types of Means

- Normally, when we talk about a mean, we are talking about the 'arithmetic mean,' where we add the numbers and divide by the amount of numbers. In a sense, we do repeat addition (multiplication) and then undo that with division.
- There is also a geometric mean, where we multiply the different numbers, then undo repeated multiplication (exponentiation) by rooting the result.

Exploration: AM vs GM

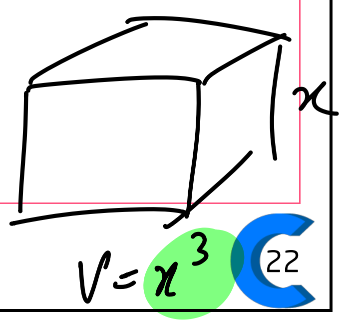
- Consider the numbers $a_1, a_2, a_3 \ldots a_n$.
  - The arithmetic mean AM = $\dfrac{a_1 + a_2 + \ldots + a_n}{n}$.
  - The geometric mean GM = $(a_1 a_2 a_3 \ldots a_n)^{\frac{1}{n}}$

Discussion: Interpreting Geometric Mean

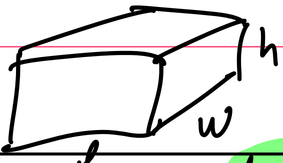


- Imagine that 3 numbers were the sides of a box. Then, imagine we squished the box into a cube, where the sides are the geometric mean. How are these two things related?

Same Volume



- Geometric mean = Average side length of a same-volume cube.



$V = x^3$

$x^3 = lwh \Rightarrow x = (lwh)^{\frac{1}{3}}$

$V = lwh$

Question 20 Walkthrough.

Calculate the arithmetic and geometric mean of 2, 4 and 8.

$AM = \frac{2+4+8}{3} = \frac{14}{3}$

$GM = (2 \times 4 \times 8)^{\frac{1}{3}} = (64)^{\frac{1}{3}}$

$= 4$

Question 21

Evaluate the geometric mean of 2^3, 3^3 and 5^3.

$GM = (2^3 \times 3^3 \times 5^3)^{\frac{1}{3}}$

$= 2 \times 3 \times 5 = 30$

Exploration: AM, GM Inequality

$\frac{a+b}{2} \geq \sqrt{ab}$

- For any collection of non-negative real numbers, AM $\geq$ GM.
- This can be proven with a special type of induction but is beyond the content of VCE!

Solve AM – GM Inequalities

Definition

- Steps:
  1. Rearrange the inequality into the form of an AM - GM inequality, noting your steps.
  2. Rewrite the steps backwards, starting from the AM - GM inequality and ending with the original inequality.

Question 22 Walkthrough.

Show that if $a, b > 0$ then $\frac{a}{b} + \frac{b}{a} \geq 2$.

$$AM = \frac{a/b + b/a}{2}$$

$$GM = \sqrt{\frac{a}{b} \times \frac{b}{a}}$$

$$AM \geq GM$$

$$\frac{a/b + b/a}{2} \geq \sqrt{\frac{a}{b} \times \frac{b}{a}}$$

$$\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right) \geq 1$$

$$\therefore \frac{a}{b} + \frac{b}{a} \geq 2$$

So, the statement is true.

Question 23

If $xyz = 27$, what is the minimum value of $x + y + z$?

$$GM = (xyz)^{1/3} = (27)^{1/3} = 3$$

$$AM \geq GM$$

$$\frac{x+y+z}{3} \geq 3$$

$$x + y + z \geq 9$$

$\therefore$ Min value $= 9$

Question 24 Additional Question.

If $wxyz = 16$, what is the minimum value of $w + x + y + z$?

$GM = (wxyz)^{\frac{1}{4}}$
$= 16^{\frac{1}{4}}$
$= 2$

$AM \geq GM$

$\frac{w+x+y+z}{4} \geq 2$

$\therefore w+x+y+z \geq 8$

$\therefore$ Min Value $= 8$.

Space for Personal Notes

Sub-Section: Solve Arithmetic and Geometric Series Proofs

Context

- We saw how we can prove arithmetic series results using the general formula, but we can also prove these with results through induction!

Exploration: Proving the General Arithmetic Series Formula using Induction

First Term: a_1
Common diff: d

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

- Test: $S_1 = \frac{1}{2}(2a_1 + 0) = a_1$. $\therefore$ Base Case True.
- Assume: $S_k = \frac{k}{2}(2a_1 + (k-1)d)$.
- Prove: We need to prove $a_1 + a_2 + a_3 + \dots + a_k + a_{k+1} = \frac{k+1}{2}(2a_1 + kd)$

LHS = $a_1 + a_2 + \dots + a_k + a_{k+1}$

$= \frac{k}{2}(2a_1 + (k-1)d) + a_{k+1}$, by assumption of $P(k)$

$= \frac{k}{2}(2a_1 + (k-1)d) + a_1 + kd$

$= ka_1 + \frac{k^2 d}{2} - \frac{kd}{2} + a_1 + kd$

$= a_1(k+1) + kd\left(\frac{k}{2} - \frac{1}{2} + 1\right)$

$= a_1(k+1) + \frac{kd}{2}(k+1)$

$= \frac{k+1}{2}(2a_1 + kd) = RHS$

- Explain: Hence, we have proven the arithmetic series formula by induction.

Now, let's look at particular examples!

Question 25 Walkthrough.

Using induction, prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \in N$.

$P(n): 1 + 3 + 5 + \ldots + (2n-1) = n^2$

- Test : Test base case, $n = 1$ — LHS $= 1$, RHS $= 1^2 = 1$ $\therefore$ $P(1)$ True
- Assume : Assume for $k \in N$, that $P(k)$ true, that is

$$1 + 3 + \ldots + (2k-1) = k^2$$

- Prove : LHS of $P(k+1)$ $= 1 + 3 + 5 + \ldots + (2k-1) + (2k+1)$

$= k^2 + 2k + 1$ (by assumption of $P(k)$)

$= (k+1)^2$

$=$ RHS of $P(k+1)$

- Explain : "Since $P(1)$ true, & $P(k) \Rightarrow P(k+1)$, then $P(n)$ true for all $n \in N$ by induction."

$\therefore$ LHS of $P(k+1) = (k+1)(k+2)$

Question 26

Prove by induction that $2+4+6 \ldots + 2n = n(n+1)$ for all $n \in N$.

Let $P(n): 2+4+6+\ldots+2n = n(n+1)$

Base Case: $P(1) = 2 = 1(1+1) = 2 \quad \therefore P(1)$ true

Induction: Assume $P(k)$ is true for $k \in N$, that is

$$2+4+6+\ldots+2k = k(k+1)$$

Then LHS of $P(k+1) = 2+4+6+\ldots+2k+2(k+1)$

$= k(k+1)+2(k+1)$ (by Assumption)

$= (k+1)(k+2)$

$=$ RHS of $P(k+1)$

$\therefore P(1)$ true & $P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for all $n \in N$ by Induction.

Question 27 Additional Question.

Prove by induction that $3 + 9 + 27 + \cdots + 3^n = \frac{3(3^n-1)}{2}$ for all $n \in N$.

$P(n): 3 + 9 + 27 + \ldots + 3^n = \frac{3(3^n-1)}{2}$

- Base Case: $n = 1$

$LHS = 3$, $RHS = \frac{3(3^1-1)}{2} = \frac{3(2)}{2} = 3$ $\therefore LHS = RHS$

& $P(1)$ is true

- Assume for $k \in N$ that $P(k)$ is true, that is

$3 + 9 + 27 + \ldots + 3^k = \frac{3(3^k-1)}{2}$

- Then, LHS of $P(k+1) = 3 + 9 + 27 + \ldots + 3^k + 3^{k+1}$

$= \frac{3(3^k-1)}{2} + 3^{k+1}$

$= \frac{3^{k+1} - 3^k + 2(3^{k+1})}{2}$

$= \frac{3^{k+1} - 3 + 2(3^{k+1})}{2}$

$= \frac{3(3^{k+1}) - 3}{2}$

$= \frac{3(3^{k+1}-1)}{2}$ = RHS of $P(k+1)$

$\therefore P(1)$ true & $P(k) \Rightarrow P(k+1)$, $P(n)$ true for all $n \in N$ by POMI.

Sub-Section: Prove Divisibility with Induction

We've learnt to prove divisibility directly. What about through induction?

Definition

Proving that $a^n + b$ is Divisible by k using Induction

1. Let $a^n + b =$ km where m is a natural number.
2. Rearrange so that $a^n =$ km − b.
3. Replace a^n with km − b when proving.

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Question 28 Walkthrough.

Prove that $7^n + 5$ is divisible by 6 for all $n \in N$.

Let $P(n): 7^n + 5 = 6m, \quad m \in N$

(1) Base Case: $n = 1$

$7^1 + 5 = 12 = 6(2) \quad \therefore P(1)$ True.

(2) Assume that $P(k)$ true for $k \in N$.

$7^k + 5 = 6m, \quad m \in N.$

(3) Then $7^{k+1} + 5 = 7(7^k) + 5$

$= 7(6m - 5) + 5$ (by assumption)

$= 42m - 35 + 5$

$= 42m - 30$

$= 6(7m - 5)$

$= 6l, \quad l = 7m - 5 \in N$

$\therefore P(k+1)$ is true.

(4) Since $P(1)$ True & $P(k) \Rightarrow P(k+1)$

then $P(n)$ true for all $n \in N$.

Question 29

Prove that $11^n - 1$ is divisible by 10 for all $n \in N$.

Let $P(n): 11^n - 1 = 10m, m \in N$

Base case: $n = 1 \Rightarrow 11^1 - 1 = 10 = (1)(10)$

$\therefore P(1)$ True

Assume for $k \in N$ that $P(k)$ true, that is,

$11^k - 1 = 10m, m \in N$

Then, $11^{k+1} - 1 = 11(11^k) - 1$

$= 11(10m+1) - 1$, by assumption

$= 110m + 11 - 1$

$= 110m + 10$

$= 10(11m+1)$

$= 10\ell, \ell = 11m+1 \in N$

$\therefore P(k+1)$ true.

$P(1)$ true & $P(k) \Rightarrow P(k+1)$, so $P(n)$ true for all $n \in N$ by the principles of mathematical induction

Question 30 Additional Question.

Prove that $7^n - 3^n$ is divisible by 4 for all $n \in N$.

$P(n): 7^n - 3^n = 4m, \ m \in N$

• Base Case: $n = 1$

$7^1 - 3^1 = 4$, which is divisible by 4.

$\therefore P(1)$ True

• Assume for $k \in N$ that $P(k)$ is true. That is,

$7^k - 3^k = 4m, \ m \in N$

Then, $7^{k+1} - 3^{k+1}$

$= 7(7^k) - 3(3^k)$

$= 3(7^k) - 3(3^k) + 4(7^k)$

$= 3(7^k - 3^k) + 4(7^k)$

$= 3(4m) + 4(7^k)$ by assumption of $P(k)$

$= 4(3m + 7^k)$

$= 4\ell, \ \ell = 3m + 7^k \in N$

$\therefore P(k+1)$ true,

$\therefore P(n)$ true for all $n \in N$ by POMI.

Section D: Exam 1 (24 Marks)

Question 31 (4 marks)

Prove that $3^{2n} - 1$ is divisible by 8 for all $n \in N$.

$P(n): 3^{2n} - 1 = 8m, m \in N$ for all $n \in N$

Base Case: $n = 1 \Rightarrow 3^2 - 1 = 8 = 8(1)$ $\therefore P(1)$ True

Assume $P(k)$ is true for $k \in N$. That is,

$3^{2k} - 1 = 8m, m \in N$

Then, $3^{2(k+1)} - 1 = 3^{2k+2} - 1$

$= 9(3^{2k}) - 1$

$= 9(8m+1) - 1$, by assumption

$= 72m + 8$

$= 8(9m+1)$

$= 8p, p = 9m + 1 \in N$

$\therefore P(k+1)$ is true.

$\therefore$ The result is true by POMI.

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Question 32 (3 marks)

If x and y are positive real numbers, determine the minimum value of:

$$(2x+3y)\left(\frac{8}{x}+\frac{3}{y}\right)$$

$(2x+3y)\left(\frac{8}{x}+\frac{3}{y}\right) = 16+9+\frac{6x}{y}+\frac{24y}{x}$

$= 25 + \frac{6x}{y}+\frac{24y}{x}$

Then using AM-GM inequality: $\frac{\frac{6x}{y}+\frac{24y}{x}}{2} \geq \sqrt{\frac{6x}{y}\cdot\frac{24y}{x}}$

$\frac{1}{2}\left(\frac{6x}{y}+\frac{24y}{x}\right) \geq \sqrt{144}$

$\frac{6x}{y}+\frac{24y}{x} \geq 24$

$\therefore 25+\frac{6x}{y}+\frac{24y}{x} \geq 25+24 = 49$

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$\therefore$ Min Value is 49.

Question 33 (3 marks)

Prove by contradiction that if p and q are positive integers, then $\frac{2p}{q} + \frac{2q}{p} \geq 4$.

For contradiction, assume that $\frac{2p}{q} + \frac{2q}{p} < 4$, $p, q \in \mathbb{N}$

then $\frac{2p^2 + 2q^2}{pq} < 4$

$2p^2 + 2q^2 < 4pq$

$p^2 - pq + q^2 < 0$

$(p-q)^2 < 0$

Which is a contradiction since $p - q \in \mathbb{Z}$ & the square of an integer is ≥ 0

$\therefore$ The original statement is true.

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Question 34 (5 marks)

Prove using induction that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

Base Case: If $n = 1$:

$$\text{LHS} = 1^2 = 1$$

$$\text{RHS} = \frac{1}{3}(2 \cdot 1 - 1)(2 \cdot 1 + 1) = \frac{1}{3} \cdot 1 \cdot 3 = 1.$$

The result holds for $n = 1$.

Assume for $k \in N$ that the result is true.

$1^2 + 3^2 + \ldots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

Then, LHS for $n = k+1$ is

$1^2 + 3^2 + \ldots + (2k-1)^2 + (2k+1)^2$

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$

$= \frac{1}{3}(2k+1)\left(k(2k-1) + 3(2k+1)\right)$

$= \frac{1}{3}(2k+1)\left[2k^2 + 5k + 3\right]$

$= \frac{1}{3}(2k+1)(k+1)(2k+3)$

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$= \frac{1}{3}(k+1)(2k+1)(2k+3)$

$= RHS$

Conclusion: If the result holds for $n = k \in \mathbb{N}$, it also holds for $n = k + 1$. Since the result holds for $n = 1$, it must hold for all $n \in \mathbb{N}$ by the principal of mathematical induction.

Question 35 (4 marks)

Consider the sequence $t_n = \frac{3}{7}(8^n - 1), n \in N$.

Prove by induction that every term of this sequence is an integer.

• Base Case: $t_1 = \frac{3}{7}(8^1 - 1) = \frac{3}{7}(7) = 3 \in Z$

$\therefore$ Base Case True

• Assume for $k \in N$ that $t_k \in Z$, that is

$\frac{3}{7}(8^k - 1) \in Z$

i.e $\frac{3}{7}(8^k - 1) = m, \quad m \in Z$

$\therefore 8^k = \frac{7m}{3} + 1$

• Then, $t_{k+1} = \frac{3}{7}(8^{k+1} - 1)$

$= \frac{3}{7}(8(8^k) - 1)$

$= \frac{3}{7}\left(8\left(\frac{7m}{3} + 1\right) - 1\right)$ (by assumption)

$= \frac{3}{7}\left(\frac{56m}{3} + 7\right)$

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$= 8m + 3 \in Z$

$\therefore t_{k+1} \in Z$

$\therefore$ Result is true for $n = k+1$.

So true for all $n \in N$ by POMI.

Question 36 (5 marks)

Prove using induction that $2^n > n^2$ for all integers $n \geq 5$.

Use the result: $n^2 > 2n+1$ for $n \geq 5$

① Base Case $n=5$

$2^5 = 32$, $5^2 = 25$

Since $32 > 25$, then base case is true.

② Assume for $n=k, k \in N$, that $2^k > k^2$

③ Need to show $2^{k+1} > (k+1)^2$

$$\begin{aligned} LHS &= 2^{k+1} \\ &= 2(2^k) \\ &= 2^k + 2^k \\ &> k^2 + k^2 \quad \text{by assumption} \\ &> k^2 + 2k + 1 \quad \text{since } k^2 > 2k+1 \text{ for } k \geq 5 \\ &= (k+1)^2 \\ &= RHS \end{aligned}$$

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$\therefore$ Statement is true for $n = k+1$

$\therefore$ True for all $n \in N$ by POMI

Section E: Exam 2 (5 Marks)

Question 37 (1 mark)

Which of the following best demonstrates a proof by contradiction?

A. Assume a statement is false and show that this leads to a logical inconsistency.

B. Use specific examples to verify the truth of a statement.

C. Start from a known fact and logically deduce the conclusion.

D. Test a hypothesis by substituting values into an equation.

Question 38 (1 mark)

A student claims: "The sum of any two odd numbers is even." To prove this, which of the following approaches would work?

A. Test the statement with one example, such as $3 + 5$.

B. Test the statement with multiple examples, such as $1 + 3$ and $7 + 9$.

C. Represent two odd numbers as $2n + 1$ and $2m + 1$, then show their sum is even.

D. Assume the statement is false and attempt to find a counter-example.

Question 39 (1 mark)

Which of the following is a valid proof for the statement: "If a number is divisible by 4, then it is even"?

A. Check if the statement is true for one number divisible by 4.

B. Write any number divisible by 4 as $4n$, and note that $4n$ is also divisible by 2.

C. Write the number as $4n + 1$ and verify it is divisible by 4.

D. None of the above.

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Question 40 (1 mark)

A rectangle has a fixed perimeter of 20 units. Which of the following values for the side lengths maximises its area?

A. Side lengths 4 and 6.

B. Side lengths 3 and 7.

C. Side lengths 5 and 5.

D. Side lengths 7 and 4.

Question 41 (1 mark)

Consider the statement: "If a number is divisible by 6, then it is divisible by 3." What is the contrapositive of this statement?

A. If a number is divisible by 3, then it is not divisible by 6.

B. If a number is not divisible by 3, then it is not divisible by 6.

C. If a number is not divisible by 6, then it is not divisible by 3.

D. If a number is divisible by 6, then it is not divisible by 3.

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Contour Check

Learning Objective: [2.3.1] - Solve Problems using AM-GM Inequalities

Key Takeaways

- The geometric mean of a_1, a_2, a_3 is $(a_1 a_2 a_3)^{1/3}$.
- When dealing with AM and GM, AM ≥ GM.

Learning Objective: [2.3.2] - Solve Arithmetic and Geometric Series Proofs

Key Takeaways

- Geometric and arithmetic series results can be proven using induction.

Learning Objective: [2.3.3] - Prove Divisibility with Induction

Key Takeaways

- To prove $a^n + b$ is divisible by something, we have to rearrange for a^n.

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