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VCE Specialist Mathematics ½
Proofs Exam Skills [2.3]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [2.3.1]** - Solve problems using AM-GM inequalities
- ☐ **SM12 [2.3.2]** - Solve Arithmetic and Geometric Series Proofs
- ☐ **SM12 [2.3.3]** - Prove divisibility with induction

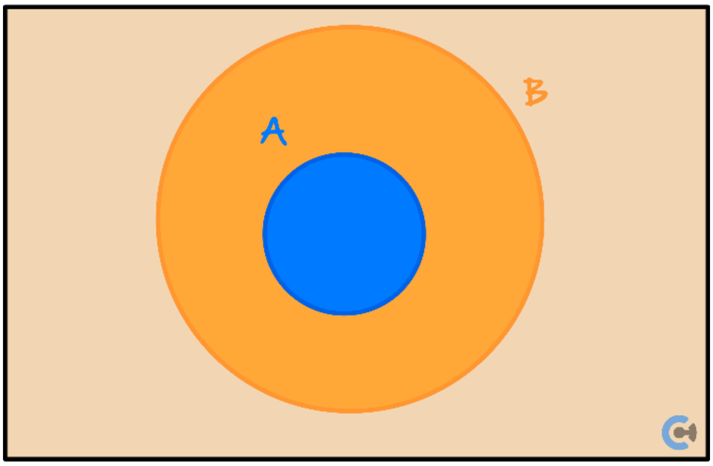
Section A: Recap

If you were here last week skip to Section B Warmup test.

Exploration: Visualisation of Conditional Statements

$$A \Rightarrow B$$

Visualisation of $A \Rightarrow B$



If A, then B.

Conditional Statements

- Conditional Statement: ____.
- Note: Notation for "implies": $\Longrightarrow$

$$\text{"}Hypothesis \Longrightarrow Conclusion\text{"}$$

NOTE: Order Matters!

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Question 1

Write a conditional statement for the following:

Doing strength training grows your muscles.

Proving Conditional Statements

1. Direct Proof.

2. Proof by Contrapositive.

3. Proof by Contradiction.

REMINDER

- From last week, we found that an even number can be written in the form:

- An odd number can be written in the form:

- If a number is divisible by 3, then it can be written in the form:

- If a number is not divisible by 3, then it can be written in the form:

- For proofs involving divisibility, we often need to split into ________________.

Method 1: Direct Proof

- The simplest method of proof. These are the proofs we have been working on in $[2.1]$.
- Here, we are not altering the statement we need to prove.

Question 2

Prove that for all integers m and n, if m is divisible by 5 and n is divisible by 2, then $10m + 3n$ is even.

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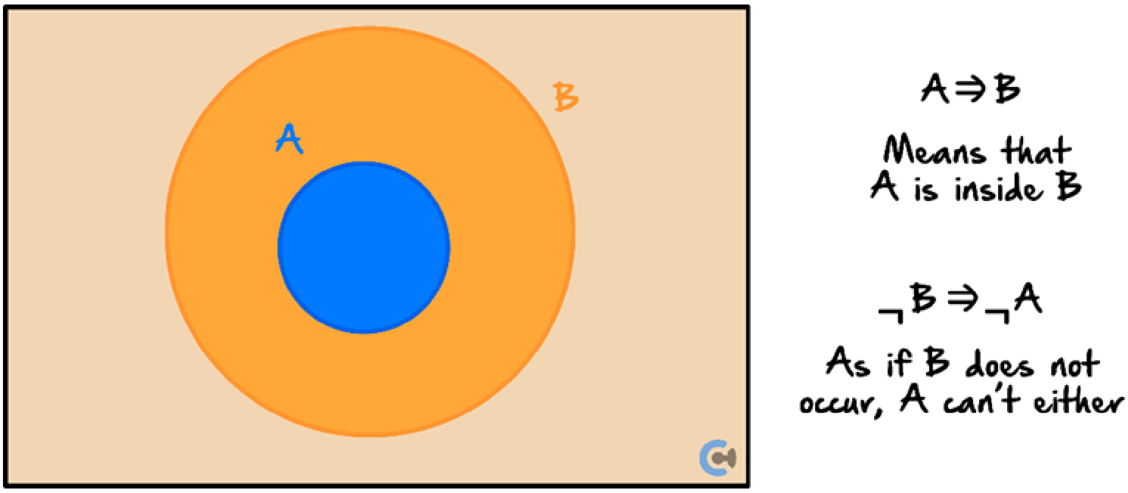
What is a contrapositive statement?

Contrapositive Statement



(Contrapositive Statement of $A \Rightarrow B$) is $\neg B \Rightarrow \neg A$

NOTE: Swap the order and negate the statements.

Question 3

Write down the contrapositive of the following statement:

If all the sides of a triangle have equal length, then the triangle is equilateral.

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<u>Method 2: Proof by Contrapositive (Indirect Proof)</u>

- Instead of proving $A \Rightarrow B$, we can prove its contrapositive $\neg B \Rightarrow \neg A$.

Prove a contrapositive statement instead.

- Considered to be an ________________________, as the original statement is altered.
- **Steps**:
  1. Set up the contrapositive statement.
  2. Prove the contrapositive statement to be true.
  3. Conclude by saying "As contrapositive is true, the original statement is true."

Question 4

Prove the following conditional statement using contrapositive:

$$\text{Let } x \in R. \text{ If } x \text{ is irrational, then } \sqrt{x + \frac{1}{5}} \text{ is irrational.}$$

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What is a contradicting statement?

Contradicting Statement

(Contradicting Statement of $A \Rightarrow B$) is $A \Rightarrow \neg B$

- Negate the ____________________.

Question 5

State the contradicting statement of the following:

If x is rational, then x^2 is rational.

Method 3: Proof by Contradiction (Indirect Proof)

To Prove $A \Rightarrow B$

Assume $A \Rightarrow \neg B$ is true

And show that the assumption is FALSE.

- Steps:

1. First, assume that the contradicting statement is true.
2. Show that the assumption has a contradiction, and is hence false.
3. Conclude by saying "Since the contradicting statement is false, the original statement is true."

- Considered to be an ____________________, as the original statement is altered.

Active Recall: De Morgan's Law

$$\neg(A \wedge B) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\neg(A \vee B) = \_\_\_\_\_\_\_\_\_\_\_\_$$

Question 6

Let $a, b \in R$. Prove that if $a + b > 150$, then $a > 75$ or $b > 75$.

Key Takeaways

- ☑ Direct proof involves proving without changing the conditional statement.
- ☑ The contrapositive of a statement $A \rightarrow B$ is given by $\neg B \rightarrow \neg A$.
- ☑ Proof by contrapositive involves proving the contrapositive of the conditional statement instead.
- ☑ Proof by contradiction requires first assuming that the contradicting statement is true, then showing contradiction.

Definition

Converse Statements

- **Definition**: Conditional statement that flows in the opposite direction.

$$(\text{Converse Statement of } A \Rightarrow B) \text{ is } B \Rightarrow A$$

Question 7

For the following statement, write down the converse statement, and conclude whether the converse is true:

If x is divisible by 2 and 5, then it is divisible by 10.

Equivalent Statements (Biconditional)

- It is a biconditional statement where if the original is ______________________________, its converse is _________________________.

$$A \Rightarrow B \text{ and } B \Rightarrow A$$

$$A \Leftrightarrow B$$

$$A \text{ is true, if and only if } B$$

- In description, B is true _________________________ A is true.
- To prove equivalent statements, we prove each direction separately.

NOTE: For if and only if (equivalent statement), we must prove both converse statements.

Question 8

Let n be an integer. Prove that n is odd, if and only if n^2 is odd.

Key Takeaways

- ☑ The converse statement of $A \to B$ is given by $B \to A$.
- ☑ An equivalent statement is when a statement and its converse both are proved to be true.
- ☑ "If and only if" stands for an equivalent statement.

Universal Quantifiers

- ______________________________ is a way to represent all members of a given set.

For all real numbers x, x^2 is never negative.

- Notation: ∀ (Universal Quantifier) ____________________.

$$\forall x \in R, x^2 \geq 0$$

Question 9

Rewrite the following statement using the universal quantifier:

For all integers n, $n^2 - 4n$ is an integer.

Existence Quantifiers

- _________________________________ is a way to represent certain members of a given set.

There exists an integer such that $x^2 - x - 12 = 0$.

- Notation: ∃ (Existential Quantifier) ________________________________.

$$\exists x \in \mathbb{Z}, x^2 - x - 12 = 0.$$

Question 10

Rewrite the following statement using the existence quantifier.

There exists a positive whole number n such that $2^n = 4$.

<u>**Negation of Universal and Existence Statements**</u>

- They are opposites of each other (with opposite conclusions).

$\neg$Universal Statement = Existence Statement with Opposite Conclusion

- And vice versa.

Question 11

Write down the negation for the following statements below.

a. If n is a natural number, then $n + 1 > n$.

b. There exists an integer k such that $k^2 = k + 4$.

Disproving Universal Statement

- We prove the *opposite* (negation) existence statement.
- We call this proof by ________________________________.
  - Giving a counterexample will be proving an opposite existence statement.

Question 12

Disprove the following statement: **For all positive integers m, if m is prime then $m^2 + 4$ is also prime.**

NOTE: We simply give ourselves a counter-example to disprove a universal statement.

Disproving Existence Statements

- We prove the *opposite* (negation) ___________________________.

NOTE: To disprove their existing statement, you must show the opposite universal statement as true.

Question 13

Disprove the following statements:

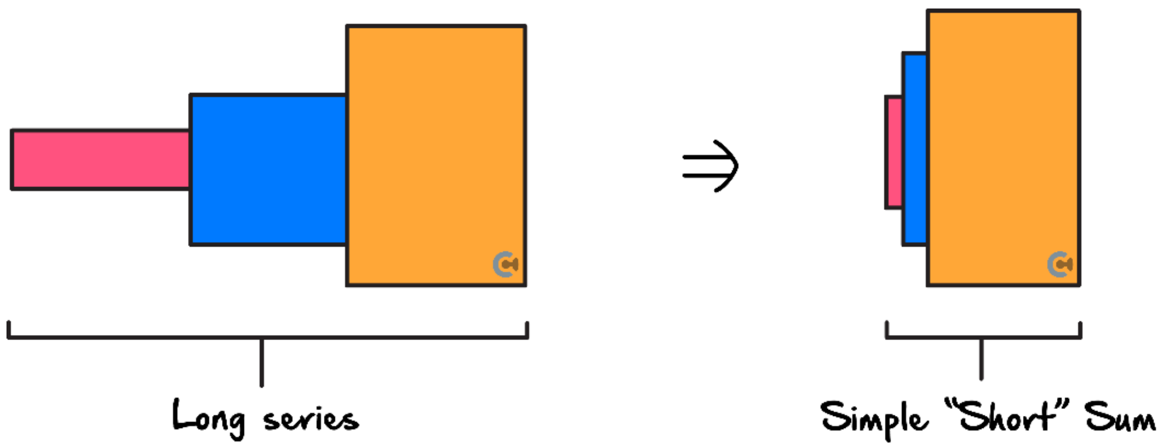
There exists a real number x, such that $10 + 3x^2 = 3 + x^2$.

Key Takeaways

- There exists a quantifier $\exists$.
- For all quantifiers $\forall$.
- To prove a there exists statement, simply give an example.
- To prove for all statements, you must prove for all values.
- To disprove a there exist statement, prove the opposite for all statements.
- To disprove a for-all statement, prove the opposite there exist statement.

Telescopic Cancelling

- Telescoping series is a series (sum of terms) where all terms cancel out except for the ______________ ______________________________________.
- The name comes from the visualisation as shown below.



- Proofs of telescoping series can be done via ____ ____.

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Question 14

a. Using partial fractions, find the values of a and b such that:

$$\frac{3}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}$$

b. Hence, prove that:

$$\frac{3}{1 \times 2} + \frac{3}{2 \times 3} + \cdots + \frac{3}{n \times (n+1)} = \frac{3n}{n+1}$$

Proof by Induction

- We will be following the given steps:" ______________".
  - **Step T:** Test the statement for its first possible value.
  - **Step A:** Assume that the statement is true for $n = k$.
  - **Step P:** Prove that if the statement holds true for $n = k$, It also holds true for $n = k + 1$.
    - Add "By Assumption".
  - **Step E:** "By the principle of mathematical induction, the statement is true for a set of values."

Question 15

Prove that for all $n \in N$, we have:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

NOTE: $1 + 2 + \cdots + n$ can be written as:

$$\sum_{i=1}^{n} i$$

Section B: Warmup (15 Marks)

Question 16 (4 marks)

Let $n \in N$. Consider the statement

$$\text{If } 3^n - 1 \text{ is prime, then } n \text{ is odd.}$$

a. Write down the contrapositive of the statement. (1 mark)

b. Prove that the contrapositive is true. (3 marks)

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Question 17 (3 marks)

Prove that if x is irrational then, $\sqrt{x-3}$ is irrational. Use the contradiction method in your proof.

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Question 18 (4 marks)

Prove that the following is true for all positive integers n: n is odd if and only if $n^2 + 4$ is odd.

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Question 19 (4 marks)

Prove that $\frac{1}{3\times5}+\frac{1}{5\times7}+\frac{1}{7\times9}+\cdots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{6n+9}$ for all $n\in N$.

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Section C: Exam Skills

Sub-Section: Solve Problems Using AM-GM Inequalities

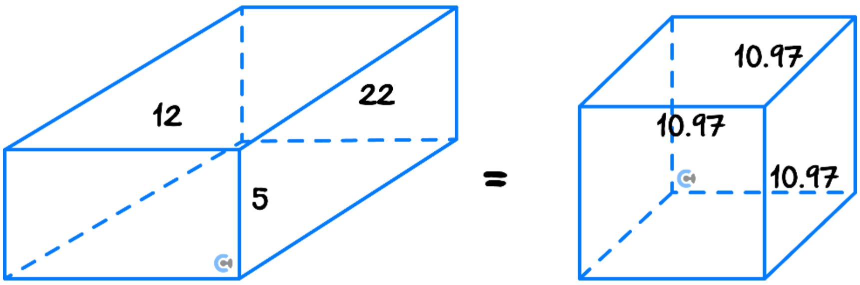
Context: Different Types of Means

- Normally, when we talk about a mean, we are talking about the 'arithmetic mean,' where we add the numbers and divide by the amount of numbers. In a sense, we do repeat addition (multiplication) and then undo that with division.
- There is also a geometric mean, where we multiply the different numbers, then undo repeated multiplication (exponentiation) by rooting the result.

Exploration: AM vs GM

- Consider the numbers $a_1, a_2, a_3 \ldots a_n$.
  - The arithmetic mean AM = ______________________________.
  - The geometric mean GM = ______________________________.

Discussion: Interpreting Geometric Mean



- Imagine that 3 numbers were the sides of a box. Then, imagine we squished the box into a cube, where the sides are the geometric mean. How are these two things related?

- Geometric mean = Average side length of a same-volume cube.

Question 20 Walkthrough.

Calculate the arithmetic and geometric mean of 2, 4 and 8.

Question 21

Evaluate the geometric mean of 2^3, 3^3 and 5^3.

Exploration: AM, GM Inequality

- For any collection of non-negative real numbers, AM $\geq$ GM.
- This can be proven with a special type of induction but is beyond the content of VCE!

Solve AM – GM Inequalities

- **Steps:**
  1. Rearrange the inequality into the form of an AM - GM inequality, noting your steps.
  2. Rewrite the steps backwards, starting from the AM - GM inequality and ending with the original inequality.

Question 22 Walkthrough.

Show that if $a, b > 0$ then $\frac{a}{b} + \frac{b}{a} \geq 2$.

Question 23

If $xyz = 27$, what is the minimum value of $x + y + z$?

Question 24 Additional Question.

If $wxyz = 16$, what is the minimum value of $w + x + y + z$?

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Sub-Section: Solve Arithmetic and Geometric Series Proofs

Context

- We saw how we can prove arithmetic series results using the general formula, but we can also prove these with results through induction!

Exploration: Proving the General Arithmetic Series Formula using Induction

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

- Test: ____.
- Assume: $S_k =$ ___________________________________.
- Prove: We need to ____.

  LHS =

- Explain: Hence, we have proven the arithmetic series formula by induction.

Now, let's look at particular examples!

Question 25 Walkthrough.

Using induction, prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \in N$.

Question 26

Prove by induction that $2 + 4 + 6 \ldots + 2n = n(n + 1)$ for all $n \ \in N$.

Question 27 Additional Question.

Prove by induction that $3 + 9 + 27 + \cdots + 3^n = \frac{3(3^n - 1)}{2}$ for all $n \in N$.

Sub-Section: Prove Divisibility with Induction

We've learnt to prove divisibility directly. What about through induction?

Proving that $a^n + b$ is Divisible by k using Induction

1. Let $a^n + b =$ ____________ where m is a natural number.

2. Rearrange so that $a^n =$ __________________.

3. Replace a^n with ____________________ when proving.

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Question 28 Walkthrough.

Prove that $7^n + 5$ is divisible by 6 for all $n \in N$.

Question 29

Prove that $11^n - 1$ is divisible by 10 for all $n \in N$.

Question 30 Additional Question.

Prove that $7^n - 3^n$ is divisible by 4 for all $n \in N$.

Section D: Exam 1 (24 Marks)

Question 31 (4 marks)

Prove that $3^{2n} - 1$ is divisible by 8 for all $n \in N$.

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Question 32 (3 marks)

If x and y are positive real numbers, determine the minimum value of:

$$(2x + 3y)\left(\frac{8}{x} + \frac{3}{y}\right)$$

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Question 33 (3 marks)

Prove by contradiction that if p and q are positive integers, then $\frac{2p}{q} + \frac{2q}{p} \geq 4$.

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Question 34 (5 marks)

Prove using induction that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

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Question 35 (4 marks)

Consider the sequence $t_n = \frac{3}{7}(8^n - 1), n \in N$.

Prove by induction that every term of this sequence is an integer.

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Question 36 (5 marks)

Prove using induction that $2^n > n^2$ for all integers $n \geq 5$.

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Section E: Exam 2 (5 Marks)

Question 37 (1 mark)

Which of the following best demonstrates a proof by contradiction?

A. Assume a statement is false and show that this leads to a logical inconsistency.

B. Use specific examples to verify the truth of a statement.

C. Start from a known fact and logically deduce the conclusion.

D. Test a hypothesis by substituting values into an equation.

Question 38 (1 mark)

A student claims: "The sum of any two odd numbers is even." To prove this, which of the following approaches would work?

A. Test the statement with one example, such as $3 + 5$.

B. Test the statement with multiple examples, such as $1 + 3$ and $7 + 9$.

C. Represent two odd numbers as $2n + 1$ and $2m + 1$, then show their sum is even.

D. Assume the statement is false and attempt to find a counter-example.

Question 39 (1 mark)

Which of the following is a valid proof for the statement: "If a number is divisible by 4, then it is even"?

A. Check if the statement is true for one number divisible by 4.

B. Write any number divisible by 4 as $4n$, and note that $4n$ is also divisible by 2.

C. Write the number as $4n + 1$ and verify it is divisible by 4.

D. None of the above.

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Question 40 (1 mark)

A rectangle has a fixed perimeter of 20 units. Which of the following values for the side lengths maximises its area?

A. Side lengths 4 and 6.

B. Side lengths 3 and 7.

C. Side lengths 5 and 5.

D. Side lengths 7 and 4.

Question 41 (1 mark)

Consider the statement: "If a number is divisible by 6, then it is divisible by 3." What is the contrapositive of this statement?

A. If a number is divisible by 3, then it is not divisible by 6.

B. If a number is not divisible by 3, then it is not divisible by 6.

C. If a number is not divisible by 6, then it is not divisible by 3.

D. If a number is divisible by 6, then it is not divisible by 3.

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Contour Check

Learning Objective: [2.3.1] - Solve Problems using AM-GM Inequalities

Key Takeaways

- [ ] The geometric mean of a_1, a_2, a_3 is ______________________________.
- [ ] When dealing with AM and GM, _________ ≥ ____________.

Learning Objective: [2.3.2] - Solve Arithmetic and Geometric Series Proofs

Key Takeaways

- [ ] Geometric and arithmetic series results can be proven using ______________________.

Learning Objective: [2.3.3] - Prove Divisibility with Induction

Key Takeaways

- [ ] To prove $a^n + b$ is divisible by something, we have to rearrange for ________.

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