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# VCE Specialist Mathematics ½
## Proofs Exam Skills [2.3]
## Homework Solutions

### Homework Outline:

## Section A: Compulsory Questions

### Sub-Section [2.3.1]: Solve Problems Using AM-GM Inequalities

**Question 1**

Show using the AM-GM inequality that for $x > 0$ we have:

$$3x + \frac{3}{x} \geq 6$$

By the AM-GM it must be that

$$\frac{3x + \frac{3}{x}}{2} \geq \sqrt{3x \cdot \frac{3}{x}}$$
$$\implies 3x + \frac{3}{x} \geq 2\sqrt{9}$$
$$3x + \frac{3}{x} \geq 6.$$

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**Question 2**

Find the maximum value of $2 - a - \frac{1}{2a}$ for all $a > 0$.

We write this as $2 - \left(a + \frac{1}{2a}\right)$. So our problem is to minimise $a + \frac{1}{2a}$.
By the AM-GM we have

$$\frac{a + \frac{1}{2a}}{2} \geq \sqrt{\frac{1}{2}}$$
$$\implies a + \frac{1}{2a} \geq \frac{2}{\sqrt{2}} = \sqrt{2}$$

Therefore the minimum value of $a + \frac{1}{2a}$ is $\sqrt{2}$.
So the maximum value of $2 - a - \frac{1}{2a}$ is $2 - \sqrt{2}$.

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**Question 3**

Let $a$ and $b$ be positive real numbers. Show that:

$$\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}$$

By AM-GM we have that

$$\frac{1/a+1/b}{2}\geq\sqrt{\frac{1}{a}\cdot\frac{1}{b}}=\frac{1}{\sqrt{ab}}$$

therefore,

$$\frac{1}{a}+\frac{1}{b}\geq\frac{2}{\sqrt{ab}}$$

another application of the AM-GM gives us that

$$\frac{a+b}{2}\geq\sqrt{ab}$$
$$\implies\frac{a+b}{4}\geq\frac{\sqrt{ab}}{2}$$
$$\implies\frac{4}{a+b}\leq\frac{2}{\sqrt{ab}}$$

Therefore we have that

$$\frac{4}{a+b}\leq\frac{2}{\sqrt{ab}}\leq\frac{1}{a}+\frac{1}{b}$$

which shows that $\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}$

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## Sub-Section [2.3.2]: Solve Arithmetic and Geometric Proofs

**Question 4**

Prove using induction that $5 + 8 + 11 + \cdots + (3n + 2) = \frac{n(3n+7)}{2}$.

Let $P(n)$ be the statement that $5 + 8 + 11 + \cdots + (3n + 2) = \frac{n(3n+7)}{2}$.

It is clear that $P(1)$ is true since $5 = \frac{1(3+7)}{2}$.

Now assume that $P(k)$ is true for some $k \in \mathbb{N}$ we then have that

$$\begin{aligned} 5 + 8 + 11 + \cdots + 3k + 2 + 3(k+1) + 2 &= \frac{k(3k+7)}{2} + 3(k+1) + 2 \\ &= \frac{3k^2 + 7k + 6k + 10}{2} \\ &= \frac{(k+1)(3k+10)}{2} \\ &= \frac{(k+1)(3(k+1)+7)}{2} \end{aligned}$$

therefore $P(k+1)$ is true. So by the principle of mathematical induction the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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**Question 5**

Prove using induction that $1 \cdot 4 + 2 \cdot 5 + \cdots + n(n+3) = \frac{n(n+1)(n+5)}{3}$.

Let $P(n)$ be the statement that $1 \cdot 4 + 2 \cdot 5 + \cdots + n(n+3) = \frac{n(n+1)(n+5)}{3}$.

It is clear that $P(1)$ is true since $4 = \frac{1(2)(6)}{3}$.

Now assume that $P(k)$ is true for some $k \in \mathbb{N}$ we then have that

$$\begin{aligned}
1 \cdot 4 + 2 \cdot 5 + \cdots + k(k+3) + (k+1)(k+4) &= \frac{k(k+1)(k+5)}{3} + (k+1)(k+4) \\
&= \frac{k(k+1)(k+5) + 3(k+1)(k+4)}{3} \\
&= \frac{(k+1)(k(k+5) + 3(k+4))}{3} \\
&= \frac{(k+1)(k^2 + 8k + 12)}{3} \\
&= \frac{(k+1)(k+2)(k+6)}{3}
\end{aligned}$$

therefore $P(k+1)$ is true. So by the principle of mathematical induction the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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**Question 6**

Prove using induction that $3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 3 \cdot 2^n = 6(2^n - 1)$.

Let $P(n)$ be the statement that $3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 3 \cdot 2^n = 6(2^n - 1)$.
It is clear that $P(1)$ is true since $6 = 6(2^1 - 1)$.
Now assume that $P(k)$ is true for some $k \in \mathbb{N}$ we then have that

$$\begin{aligned} 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 3 \cdot 2^n + 3 \cdot 2^{n+1} &= 6(2^n - 1) + 3 \cdot 2^{n+1} \\ &= 3 \cdot 2^{n+1} - 6 + 3 \cdot 2^{n+1} \\ &= 6 \cdot 2^{n+1} - 6 \\ &= 6(2^{n+1} - 1) \end{aligned}$$

therefore $P(k+1)$ is true. So by the principle of mathematical induction the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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## Sub-Section [2.3.3]: Prove Divisibility With Induction

**Question 7**

Prove using induction that $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$.

Let $f(n) = 5^n - 1$.
Base case: $f(1) = 5 - 1 = 4$ is divisible by 4.
Inductive step: Assume that $f(k)$ is divisible by 4 for some $k \in \mathbb{N}$. That is $f(k) = 4m$, $m \in \mathbb{N}$. Then we have

$$\begin{aligned} f(k+1) - f(k) &= 5^{k+1} - 1 - (5^k - 1) \\ &= 5^{k+1} - 5^k + 4m \\ &= 5^k(5-1) + 4m \\ &= 4(5^k + m) \\ &= 4p, \quad p \in \mathbb{N} \end{aligned}$$

thus $f(k+1)$ is divisible by 4 and so by the principle of mathematical induction $f(n)$ is divisible by 4 for all $n \in \mathbb{N}$.

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**Question 8**

Prove using induction that $4^n + 6n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

Let $f(n) = 4^n + 6n - 1$.
Base case: $f(1) = 4 + 6 - 1 = 9$ is divisible by 3.
Inductive step: Assume that $f(k)$ is divisible by 3 for some $k \in \mathbb{N}$. That is $f(k) = 3m$, $m \in \mathbb{N}$. Then we have

$$\begin{aligned} f(k+1) - f(k) &= 4^{k+1} + 6(k+1) - 1 - (4^k + 6k - 1) \\ &= 4^{k+1} - 4^k + 6 \\ \implies f(k+1) &= 4^k(4-1) + 6 + 3m \\ &= 3 \cdot 4^k + 3(2+m) \\ &= 3(4^k + 2 + m) \\ &= 3p, \quad p \in \mathbb{N} \end{aligned}$$

thus $f(k+1)$ is divisible by 3 and so by the principle of mathematical induction $f(n)$ is divisible by 3 for all $n \in \mathbb{N}$.

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**Question 9**

Prove using induction that $5^{2n} + 3n - 1$ is divisible by 9 for all $n \in \mathbb{N}$.

Let $f(n) = 5^{2n} + 3n - 1$.
Base case: $f(1) = 25 + 3 - 1 = 27$ is divisible by 9.
Inductive step: Assume that $f(k)$ is divisible by 9 for some $k \in \mathbb{N}$. That is $f(k) = 9m$, $m \in \mathbb{N}$. Then we have

$$\begin{aligned} f(k+1) &= 5^{2k+2} + 3k + 3 - 1 \\ &= 25(5^{2k}) + 3k + 2 \\ &= 25(9m - 3k + 1) + 3k + 2 \\ &= 25(9m) + 27 - 72k \\ &= 9(25m) + 9(3 - 8k) \\ &= 9(25m - 8k + 3) \\ &= 9p, \quad p \in \mathbb{N} \end{aligned}$$

thus $f(k+1)$ is divisible by 9 and so by the principle of mathematical induction $f(n)$ is divisible by 9 for all $n \in \mathbb{N}$.

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## Sub-Section: Exam 1 Questions

**Question 10**

Consider the statement below:

$$\text{If } a + b \geq 17, \text{ then } a \geq 8 \text{ or } b \geq 8.$$

**a.** Write down a statement to begin a proof by contradiction for the statement above.

If $a + b \geq 17$, then $a < 8$ and $b < 8$.

**b.** Hence, obtain a contradiction and prove the original statement.

If $a < 8$ and $b < 8$, then $a + b < 16$, which contradicts the fact that $a + b \geq 17$.

Therefore, we conclude that the original statement holds.

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**Question 11**

Use a direct proof to prove that $n^3 + 3n^2 + 2n$ is divisible by 6 for all $n \in \mathbb{N}$.

Let $f(n) = n^3 + 3n^2 + 2n$. We factorise $f(n)$.

$$\begin{aligned} f(n) &= n(n^2 + 3n + 2) \\ &= n(n+1)(n+2) \end{aligned}$$

Now $f(n)$ is the product of three consecutive integers and is therefore divisible by both 2 and 3.
Therefore $f(n)$ must be divisible by $2 \times 3 = 6$.

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**Question 12**

Prove using induction that $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}$, for all $n \in \mathbb{N}$.

Let $P(n)$ be the statement that $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}$, for all $n \in \mathbb{N}$.

Base case: $P(1)=\frac{1}{2}=1-\frac{1}{2}$ is true.

Inductive step: Suppose that $P(k)$ holds for some $k \in \mathbb{N}$ then it must be that

$$\begin{aligned}\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &= 1-\frac{1}{2^k}+\frac{1}{2^{k+1}}\\ &= \frac{2^{k+1}-2+1}{2^{k+1}}\\ &= \frac{2^{k+1}-1}{2^{k+1}}\\ &= 1-\frac{1}{2^{k+1}}\end{aligned}$$

therefore $P(k+1)$ is true and so by the POMI $P(n)$ must be true for al $n \in \mathbb{N}$.

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**Question 13**

Prove that for all $n \in \mathbb{N}$, $n^3$ is odd, if and only if, $n$ is odd.

**Hint**: Use the contrapositive for one of the directions.

($\Rightarrow$) For this direction we will prove the contrapositive:

$$\text{If } n \text{ is even then } n^3 \text{ is even.}$$

Let $n = 2k$ for $k \in \mathbb{Z}$, then

$$n^3 = (2k)^3 = 8k^3 = 2(4k^2)$$

which is even, since $4k^2 \in \mathbb{Z}$.

($\Leftarrow$) if $n$ is odd then $n^3$ is odd.
Let $n = 2k + 1$ for $k \in \mathbb{Z}$, then

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

which is odd, since $4k^3 + 6k^2 + 3k \in \mathbb{Z}$.
We have proved both directions so our proof is complete.

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**Question 14**

Prove by contradiction that $\sqrt{2}+\sqrt{5}>\sqrt{13}$.

Suppose for a contradiction that $\sqrt{2}+\sqrt{5}\leq\sqrt{13}$. Then we will have

$$(\sqrt{2}+\sqrt{5})^2 \leq 13$$
$$2+5+2\sqrt{10} \leq 13$$
$$2\sqrt{10} \leq 6$$
$$\sqrt{10} \leq 3$$

however this last statement is clearly false since $\sqrt{10} > \sqrt{}$

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**Question 15**

Prove using induction that $25^n + 75 \cdot 25^{n-1}$ is divisible by $100$ for all $n \in \mathbb{N}$.

Let $f(n) = 25^n + 75 \cdot 25^{n-1}$.
Base case: $f(1) = 25 + 75 = 100$ is divisible by 100.
Inductive step: Suppose that $f(k)$ is divisible by 100 for some $k \in \mathbb{N}$, then $f(k) = 100m$ for some $m \in \mathbb{N}$. This implies that $75 \cdot 25^{n-1} = 100m - 25^k$

$$f(k+1) = 25^{k+1} + 75 \cdot 25^k$$
$$= 25 \cdot 25^k + 75 \cdot 25 \cdot 25^{k-1} \quad = 25 \cdot 25^k + 25(100m - 25^k)$$
$$= 25(25^k + 100m - 25^k)$$
$$= 100(25m)$$
$$= 100p, \quad p \in \mathbb{N}$$

so $f(k+1)$ is divisible by 100. Thus by the priniple of mathematical induction $f(n)$ is divisible by 100 for all $n \in \mathbb{N}$.

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**Question 16**

Bernoulli's inequality states that if $a \in \mathbb{R}$, $a \geq -1$ and $n \in \mathbb{N}$, $n \geq 1$, then

$$(1 + a)^n \geq 1 + an.$$

Prove this inequality using induction.

Let $P(n)$ be the statement that if $a \in \mathbb{R}$, $a \geq -1$ and $n \in \mathbb{N}$, $n \geq 1$, then $(1+a)^n \geq 1 + an$
Base case: $P(1)$: $(1+a)^1 = 1 + 1 \cdot a$ so is true.
Inductive step: Suppose that $P(k)$ holds for some $k \in \mathbb{N}$ then it must be that

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k(1+a)\\ &\geq (1+ak)(1+a)\\ &= 1 + ak + a + ka^2\\ &= 1 + a(k+1) + ka^2\\ &\geq 1 + (k+1)a \quad \text{since } ka^2 \geq 0\end{aligned}$$

therefore $P(k+1)$ is true and so by the POMI $P(n)$ must be true for al $n \in \mathbb{N}$.

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## Sub-Section: Exam 2 Questions

### Question 17

A teacher claims: "To directly prove that the square of any odd number is odd, we can start by representing an odd number as $2n + 1$." What would the next step in the proof be?

**A.** Test this with specific odd numbers, such as 3 or 5.

**B.** Square $2n + 1$ and simplify to $4n^2 + 4n + 1$, then show this is odd.

**C.** Assume the square of an odd number is not odd and find a contradiction.

**D.** Conclude that odd numbers squared are always greater than even numbers.

### Question 18

Which of the following statements is true about mathematical proofs?

**A.** A proof provides a logical argument that guarantees a statement is true in all cases.

**B.** A proof can only be written for statements about numbers.

**C.** A proof is valid if it works for at least two examples.

**D.** A proof is the same as a hypothesis

### Question 19

The statement "If a triangle is equilateral, then it is also isosceles" is given. What is the converse of this statement?

**A.** If a triangle is equilateral, then it is not isosceles.

**B.** If a triangle is isosceles, then it is equilateral.

**C.** If a triangle is not equilateral, then it is not isosceles.

**D.** If a triangle is not isosceles, then it is not equilateral.

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**Question 20**

Which of the following statements is logically equivalent to "If it is snowing, then it is cold"?

**A.** If it is not snowing, then it is not cold.

**B.** If it is not cold, then it is not snowing.

**C.** If it is cold, then it is snowing.

**D.** If it is snowing, then it is not cold.

**Question 21**

Which of the following is an example of a biconditional statement?

**A.** If a number is even, then it is divisible by 2.

**B.** If a polygon is a square, then it has four sides.

**C.** A polygon is a square, if and only if, it has four equal sides and four right angles.

**D.** If it rains, then the grass will be wet.

**Question 22**

Consider the statement: "If a student studies, then they will pass the exam." Which of the following is true?

**A.** The contrapositive is "If a student does not pass the exam, then they did not study."

**B.** The converse is "If a student does not study, then they will not pass the exam."

**C.** The converse is "If a student passes the exam, then they did not study."

**D.** The contrapositive is "If a student passes the exam, then they studied."

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## Section B: Supplementary Questions

### Sub-Section [2.3.1]: Solve Problems Using AM-GM Inequalities

**Question 23**

Show using the AM-GM inequality that for $x > 0$ we have:

$$5x + \frac{5}{x} \geq 10$$

Recall that the AM-GM inequality states that $\frac{a+b}{2} \geq \sqrt{ab}$ where $a, b > 0$. Let $a = 5x$ and $b = 5/x$. Then, $\frac{5x+5/x}{2} \geq \sqrt{5x \cdot 5/x} = 5$. Thus, we obtain the inequality $5x + 5/x \geq 10$.

**Question 24**

Minimise $2x + \frac{2}{x}$ over $x > 0$ by applying the AM-GM inequality, and hence maximise $6 - 2x - \frac{2}{x}$.

Using the AM-GM inequality (similar to in the previous problem), we conclude that $2x + 2/x \geq 4$. Furthermore, we see that $2x + 2/x = 4$ when $x = 1$. Thus, $2x + 2/x$ attains a minimum of 4. Now, $6 - 2x - 2/x$ will be maximised as long as $2x + 2/x$ is minimised because in this situation, we would be subtracting the smallest possible number away from 6. Thus, the maximal value that $6 - 2x - 2/x$ can achieve is 2.

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**Question 25**

Find an expression for the area of a rectangle that has a perimeter of 4 units and a width of $x$ units, and hence use the AM-GM inequality to maximise the area of such a rectangle.

A rectangle with width $x$ must have length $2 - x$ so that the perimter is 4 units. Thus, the area of such a shape is $A(x) = x(2 - x)$. Now, by the AM-GM inequality with $a = x$ and $b = 2 - x$, we see that $ab \leq \left(\frac{a+b}{2}\right)^2 = \left(\frac{x+2-x}{2}\right)^2 = 1$. Note that this value is achieved by $x = 1$. Hence, the maximum area is 1.

**Question 26**

Let $x, y > 0$. Furthermore, suppose that $xy = 4$. Find the minimum value of $xy^3 + x^3y$.

Applying the AM-GM inequality with $a = xy^3$ and $b = x^3y$, we find that $\frac{xy^3+x^3y}{2} \geq \sqrt{xy^3 \cdot x^3y} = x^2y^2$. Therefore, $xy^3 + x^3y \geq 2x^2y^2 = 2 \cdot (4)^2 = 32$.

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## Sub-Section [2.3.2]: Solve Arithmetic and Geometric Proofs

**Question 27**

Prove using induction that $2 + 7 + 12 + \cdots + (5n - 3) = \frac{n(5n-1)}{2}$.

For $n = 1$, we see that RHS $= \frac{1\cdot(5\cdot 1-1)}{2} =$ LHS. Now, assume the statement holds for some $N \in \mathbb{N}$. Observe that

$$\begin{aligned} 2+7+12+\cdots+(5(n+1)-3) &= 2+7+12+\cdots+(5n-3)+(5n+2) \\ &= \frac{n(5n-1)}{2}+5n+2 \\ &= \frac{5n^2-n+10n+4}{2} \\ &= \frac{5n^2+9n+4}{2} \\ &= \frac{(n+1)(5n+4)}{2} \\ &= \frac{(n+1)(5(n+1)-1)}{2} \end{aligned}$$

Therefore, the statement holds for $n+1$ and by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$.

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**Question 28**

Prove using induction that $1 \cdot 7 + 2 \cdot 8 + \cdots + n(n+6) = \frac{n(n+1)(2n+19)}{6}$.

For $n = 1$, we see that RHS $= \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 19)}{6} = 7 = 1 \cdot 7 =$ LHS. Therefore, the statement holds for $n = 1$. Now, assume that the statement holds for some $n \in \mathbb{N}$. Observe that

$$\begin{aligned} 1 \cdot 7 + 2 \cdot 8 + \cdots (n+1)((n+1)+6) &= 1 \cdot 7 + 2 \cdot 8 + \cdots + n(n+6) + (n+1)(n+7) \\ &= \frac{n(n+1)(2n+19)}{6} + (n+1)(n+7) \\ &= \frac{n(n+1)(2n+19) + 6(n+1)(n+7)}{6} \\ &= \frac{(n+1)(2n^2 + 19n + 6n + 42)}{6} \\ &= \frac{(n+1)(2n^2 + 25n + 42)}{6} \\ &= \frac{(n+1)(n+2)(2n+21)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+19)}{6} \end{aligned}$$

Therefore, the statement is true for $n+1$ and by induction, the statement holds for all $n \in \mathbb{N}$.

**Question 29**

Prove using induction that $2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^n = 3^{n+1} - 3$.

For $n = 1$, we see that RHS $= 3^2 - 3 = 6 = 2 \cdot 3 =$ LHS. Hence, the base case is true. Now, assume that the statement holds for some $n \in \mathbb{N}$. Observe that

$$\begin{aligned} 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n+1} &= 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^n + 2 \cdot 3^{n+1} \\ &= 3^{n+1} - 3 + 2 \cdot 3^{n+1} \\ &= 3 \cdot 3^{n+1} - 3 \\ &= 3^{n+2} - 3 \\ &= 3^{(n+1)+1} - 3 \end{aligned}$$

Therefore, the statement holds for $n+1$ and by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$.

**Question 30**

**a.** Prove using induction that for all $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

For $n = 1$, we have RHS $= \frac{1^2 \cdot 2^2}{4} = 1 = 1^3 =$ LHS. Hence, we see that the statement holds for $n = 1$. Now, suppose that the statement holds for some $n \in \mathbb{N}$. Observe that

$$\begin{aligned}
1^3 + 2^3 + \cdots + (n+1)^3 &= 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \frac{(n+1)^2((n+1)+1)^2}{4}
\end{aligned}$$

Therefore, the statement holds for $n+1$ and by induction, the statement holds for all $n \in \mathbb{N}$.

**b.** Hence, write a rule for $2^3 + 4^3 + \cdots + (2n)^3$.

**Hint**: $2^3 + 4^3 + \cdots + (2n)^3$ is related to $1^3 + 2^3 + \cdots + n^3$ in a reasonably simple way.

Observe $2^3 + 4^3 + \cdots + (2n)^3 = (2 \cdot 1)^3 + (2 \cdot 2)^3 + \cdots (2n)^3 = 8 \cdot (1^3 + 2^3 + \cdots n^3) = 2n^2(n+1)^2$.

**c.** Now, deduce a rule for $1^3 + 3^3 + \cdots + (2n-1)^3$ using the rule you obtained above.

Observe that $1^3 + \cdots + (2n)^3 = (1^3 + 3^3 + \cdots + (2n-1)^3) + (2^3 + 4^3 + \cdots + (2n)^3)$. The sum of the first $2n$ cubes comes from the first formula using $2n$ instead of $n$. Therefore,

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2n-1)^3 &= \frac{(2n)^2(2n+1)^2}{4} - 2n^2(n+1)^2 \\
&= 2n^4 - n^2
\end{aligned}$$

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## Sub-Section [2.3.3]: Prove Divisibility With Induction

**Question 31**

Prove using induction that if $n \in \mathbb{N}$, then $8^n - 1$ is divisible by 7.

We proceed by induction. For $n = 1$, we have $8^n - 1 = 7$, which is divisible by 7 as $7 = 7 \cdot 1$. Therefore, the base case holds. Now, assume that $8^n - 1$ is divisible by 7 for some $n \in \mathbb{N}$. We want to show that $8^{n+1} - 1$ is divisible by 7. Notice that $8^{n+1} - 1 = 8 \cdot 8^n - 1 = 8(8^n - 1) + 7$. By assumption, $8^n - 1 = 7k$ for some $k \in \mathbb{Z}$. Therefore, $8^{n+1} - 1 = 7(8l + 1) = 7m$ where $m = 8l + 1 \in \mathbb{Z}$. Therefore, we conclude that $8^{n+1}$-7 is divisible by 7 and by the principle of mathematical induction, $8^n - 1$ is divisible by 7 for all $n \in \mathbb{N}$.

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**Question 32**

Prove using induction that if $n \in \mathbb{N}$, then $n^3 + 3n^2 + 2n$ is divisible by 3.

**Note:** If you want to make this question a bit harder, you can instead show that $n^3 + 3n^2 + 2n$ is also divisible by 6. You might need to use the fact that the product of two consecutive integers is always even.

We proceed by induction. For $n = 1$, we have $1^3 + 3 \cdot 1^2 + 2 \cdot 1 = 6 = 3 \cdot 2$, which is divisible by 3. Now, assume that $n^3 + 3n^2 + 2n$ is divisible by 3 for some $n \in \mathbb{N}$. Therefore, we can write $n^3 + 3n^2 + 2n = 3m$ for some $m \in \mathbb{Z}$. Notice that

$$\begin{aligned}
(n+1)^3 + 3(n+1)^2 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 3n^2 + 6n + 3 + 2n + 2 \\
&= (n^3 + 3n^2 + 2n) + (3n^2 + 3n + 1 + 6n + 3 + 2) \\
&= (n^3 + 3n^2 + 2n) + (3n^2 + 9n + 6) \\
&= 3m + 3(n^2 + 3n + 2) \\
&= 3(m + n^2 + 3n + 2) \\
&= 3p
\end{aligned}$$

where $p = m + n^2 + 3n + 2 \in \mathbb{Z}$. Therefore, $(n+1)^3 + 3(n+1)^2 + 2(n+1)$ is divisible by 3. Furthermore, using the principle of mathematical induction, $n^3 + 3n^2 + 2n$ is divisible by 3 for all numbers $n \in \mathbb{N}$.

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**Question 33**

Prove using induction that if $n \in \mathbb{N}$, then $10^{n+1} + 10^n + 1$ is divisible by 3.

**Note:** The statement says that 111, 1101, 11001, etc. are all divisible by 3.

For $n = 1$, we have $10^{n+1} + 10^n + 1 = 111 = 3 \cdot 37$. Therefore, the base case holds. Now assume that $10^{n+1} + 10^n + 1$ is divisible by 3 for some $n \in \mathbb{N}$. In particular, this means $10^{n+1} + 10^n + 1 = 3k$ for some $k \in \mathbb{Z}$. Furthermore,

$$\begin{aligned} 10^{n+2} + 10^{n+1} + 1 &= 10(10^{n+1} + 10^n) + 1 \\ &= 10(10^{n+1} + 10^n + 1) - 9 \\ &= 10 \cdot 3k - 9 \\ &= 3(10p - 3) \\ &= 3m, \end{aligned}$$

where $m = 10p - 3 \in \mathbb{Z}$. Therefore, we may conclude that $10^{n+2} + 10^{n+1} + 1$ is divisible by 3 and by the principle of mathematical induction $10^{n+1} + 10^n + 1$ is divisible by 3 for all. $n \in \mathbb{N}$.

**Question 34**

Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. For example, $3! = 1 \cdot 2 \cdot 3$. Prove using induction that if $n \in \mathbb{N}$, then $(2n)!$ is divisible by $2^n$.

For $n = 1$, we have $(2n)! = 2! = 2 = 2^1 \cdot 1$, which we see is divisible by $2^1$. Now, assume that $(2n)!$ is divisible by $2^n$ if $n$ is some natural number. Then, we may write $(2n)! = 2^n \cdot k$, where $k \in \mathbb{Z}$. Furthermore, we see that

$$(2(n+1))! = (2n+2)! = (2n)! \cdot (2n+1) \cdot (2n+2) = 2 \cdot 2^n \cdot k \cdot (2n+1)(n+1) = 2^{n+1}p,$$

where $p = k(2n+1)(n+1) \in \mathbb{Z}$. Therefore, $(2(n+1))!$ is divisible by $2^{n+1}$ and by the principle of mathematical induction, $(2n)!$ is divisible by $2^n$ for all $n \in \mathbb{N}$.

## Sub-Section: Exam 1 Questions

**Question 35**

Prove using induction that for all $n \in \mathbb{N}$, $n < 2^n$.

We begin by verfiying the base case: If $n = 1$ then LHS $= 1 < 2 = 2^n =$ RHS. Now, assume that $n < 2^n$ for some $n \in \mathbb{N}$. We see that $n + 1 < 2^n + 1 < 2^n + 2^n = 2^{n+1}$, where we have also used the fact that $1 < 2^n$ for all $n \in \mathbb{N}$. Therefore, we have shown that $n + 1 < 2^{n+1}$. Using the principle of mathematical induction, we conclude that $n < 2^n$ for all $n \in \mathbb{N}$.

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**Question 36**

Consider the statement below:

There cannot exist two integers $m$ and $n$ such that $5m + 10n = 3$.

**a.** Write down a statement to begin a proof by contradiction for the statement above.

There exists $m, n \in \mathbb{Z}$ such that $5m + 10n = 3$.

**b.** Hence, obtain a contradiction and prove the original statement.

If there are two integers $m, n \in \mathbb{Z}$ so that $5m + 10n = 3$, then the left-hand side is divisible by 5, but the right-hand side is not divisible by 5, which is a contradiction. Therefore, there cannot exist two integers $m$ and $n$ such that $5m + 10n = 3$.

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**Question 37**

Prove using induction that $6^n + 4$ is divisible by 5 for all $n \in \mathbb{N}$.

Let $f(n) = 6^n + 4$.
Base Case: $f(1) = 6 + 4 = 10$ is divisible by 5.
Inductive step: Suppose that $f(k)$ is divisible by 5 for any $k \in \mathbb{N}$. We then have $f(k) = 5m$ for some $m \in \mathbb{N}$. Now,

$$\begin{aligned} f(k+1) - f(k) &= 6^{k+1} + 4 - (6^k + 4) \\ \implies f(k+1) &= 6^{k+1} - 6^k + 5m \\ &= 6^k(6-1) + 5m \\ &= 5(6^k + m) \\ &= 5p, \quad p \in \mathbb{N} \end{aligned}$$

therefore $f(k+1)$ is divisible by 5 and thus by the priniple of mathematical induction $f(n)$ is divisible by 5 for all $n \in \mathbb{N}$.

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**Question 38**

Prove using induction that for all $n \in \mathbb{N}$, it holds that $\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$.

For the base case where $n=1$, we see that the left-hand side is $1+\frac{1}{1}=2=1+1$, which is equal to the right-hand side. Therefore, the base case has been verified. Now, suppose that $\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$ for some $n \in \mathbb{N}$. We use the induction hypothesis to conclude

$$\begin{aligned}\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right) &= (n+1)\left(1+\frac{1}{n+1}\right)\\ &= (n+1)+1\\ &= n+2\end{aligned}$$

Therefore, by the principle of mathematical induction, it holds that

$$\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$$

for all $n \in \mathbb{N}$.

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**Question 39**

Prove the following biconditional statement for $x, y \in \mathbb{Z}$:

$$x + y \text{ is even, if and only if, } x^2 + y^2 \text{ is even.}$$

($\Rightarrow$) Suppose that $x + y$ is even. Then $x + y = 2n$ for some $n \in \mathbb{Z}$. Recall $(x+y)^2 = x^2+2xy+y^2$. Therefore, $x^2+y^2 = (x+y)^2-2xy = 4n^2-2xy = 2(2n^2-xy) = 2k$, where $k = 2n^2 - xy \in \mathbb{Z}$. Therefore, $x^2 + y^2$ is even.

($\Leftarrow$) We shall prove the reverse direction by proving its contrapositive: If $x + y$ is odd, then $x^2 + y^2$ is odd. Thus, assume that $x + y = 2n + 1$ for some $n \in \mathbb{Z}$. Similar to above, $x^2+y^2 = (x+y)^2-2xy = (2n+1)^2-2xy = 4n^2+4n+1-2xy = 2(2n^2+2n-xy)+1 = 2k+1$, where $k = 2n^2 + 2n - xy \in \mathbb{Z}$. Therefore, $x^2 + y^2$ is odd.

Therefore, $x + y$ is even if and only if $x^2 + y^2$ is even.

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## Sub-Section: Exam 2 Questions

**Question 40**

The contrapositive to the statement, "If $n$ is even, then $n^2$ is even." is:

**A.** If $n^2$ is odd, then $n$ is odd.

**B.** If $n^2$ is even, then $n$ is even.

**C.** If $n$ is odd, then $n^2$ is even.

**D.** If $n$ is odd, then $n^2$ is odd.

**Question 41**

Consider the following:

$$\text{For all } k > K,\ 1.5^k < (k-1)!$$

What is the smallest value of $K \in \mathbb{N}$ such that the above holds:

**A.** 2

**B.** 3

**C.** 4

**D.** 5

**Question 42**

The negation of the statement, "All the cars in the carpark are black." is:

**A.** All the vans in the carpark are black.

**B.** There exists a car in the carpark that is not black.

**C.** There exists a bus in the carpark without a mirror.

**D.** All the cars in the carpark are yellow.

**Question 43**

Find the minimum value of $6x^2 + \frac{6}{x^2}$.

**A.** 6

**B.** 25

**C.** 15

**D.** 12

**Question 44**

Consider the following statement:

If a car in the carpark is black, then it costs a lot of money.

Which of the following is the converse of the above?

**A.** If a bus in the carpark costs a lot of money, then it is not black.

**B.** If a car costs a lot of money, then it is in the carpark.

**C.** If a car in the carpark is not black, then it costs a lot of money.

**D.** If a car in the carpark costs a lot of money, then it is black in colour.

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