CONTOUREDUCATION

VCE Specialist Mathematics ½
Proofs Exam Skills [2.3]
Homework

Homework Outline:

Compulsory Questions	Pg 2- Pg 19
Supplementary Questions	Pg 20- Pg 34

Section A: Compulsory Questions

Sub-Section [2.3.1]: Solve Problems Using AM-GM Inequalities

Question 1

Show using the AM-GM inequality that for $x > 0$ we have:

$$3x + \frac{3}{x} \geq 6$$

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 2

Find the maximum value of $2 - a - \frac{1}{2a}$ for all $a > 0$.

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 3

Let a and b be positive real numbers. Show that:

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$$

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Sub-Section [2.3.2]: Solve Arithmetic and Geometric Proofs

Question 4

Prove using induction that $5 + 8 + 11 + \cdots + (3n + 2) = \frac{n(3n+7)}{2}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 5

Prove using induction that $1 \cdot 4 + 2 \cdot 5 + \cdots + n(n + 3) = \frac{n(n+1)(n+5)}{3}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 6

Prove using induction that $3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 3 \cdot 2^n = 6(2^n - 1)$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Sub-Section [2.3.3]: Prove Divisibility With Induction

Question 7

Prove using induction that $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 8

Prove using induction that $4^n + 6n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 9

Prove using induction that $5^{2n} + 3n - 1$ is divisible by 9 for all $n \in \mathbb{N}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 10

Consider the statement below:

$$\text{If } a + b \geq 17, \text{ then } a \geq 8 \text{ or } b \geq 8.$$

a. Write down a statement to begin a proof by contradiction for the statement above.

_____

b. Hence, obtain a contradiction and prove the original statement.

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 11

Use a direct proof to prove that $n^3 + 3n^2 + 2n$ is divisible by 6 for all $n \in \mathbb{N}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 12

Prove using induction that $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$, for all $n \in \mathbb{N}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 13

Prove that for all $n \in \mathbb{N}$, n^3 is odd, if and only if, n is odd.

Hint: Use the contrapositive for one of the directions.

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 14

Prove by contradiction that $\sqrt{2} + \sqrt{5} > \sqrt{13}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 15

Prove using induction that $25^n + 75 \cdot 25^{n-1}$ is divisible by 100 for all $n \in \mathbb{N}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 16

Bernoulli's inequality states that if $a \in \mathbb{R}$, $a \geq -1$ and $n \in \mathbb{N}$, $n \geq 1$, then

$$(1 + a)^n \geq 1 + an.$$

Prove this inequality using induction.

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 17

A teacher claims: "To directly prove that the square of any odd number is odd, we can start by representing an odd number as $2n + 1$." What would the next step in the proof be?

A. Test this with specific odd numbers, such as 3 or 5.

B. Square $2n + 1$ and simplify to $4n^2 + 4n + 1$, then show this is odd.

C. Assume the square of an odd number is not odd and find a contradiction.

D. Conclude that odd numbers squared are always greater than even numbers.

Question 18

Which of the following statements is true about mathematical proofs?

A. A proof provides a logical argument that guarantees a statement is true in all cases.

B. A proof can only be written for statements about numbers.

C. A proof is valid if it works for at least two examples.

D. A proof is the same as a hypothesis

Question 19

The statement "If a triangle is equilateral, then it is also isosceles" is given. What is the converse of this statement?

A. If a triangle is equilateral, then it is not isosceles.

B. If a triangle is isosceles, then it is equilateral.

C. If a triangle is not equilateral, then it is not isosceles.

D. If a triangle is not isosceles, then it is not equilateral.

Space for Personal Notes

Question 20

Which of the following statements is logically equivalent to "If it is snowing, then it is cold"?

A. If it is not snowing, then it is not cold.

B. If it is not cold, then it is not snowing.

C. If it is cold, then it is snowing.

D. If it is snowing, then it is not cold.

Question 21

Which of the following is an example of a biconditional statement?

A. If a number is even, then it is divisible by 2.

B. If a polygon is a square, then it has four sides.

C. A polygon is a square, if and only if, it has four equal sides and four right angles.

D. If it rains, then the grass will be wet.

Question 22

Consider the statement: "If a student studies, then they will pass the exam." Which of the following is true?

A. The contrapositive is "If a student does not pass the exam, then they did not study."

B. The converse is "If a student does not study, then they will not pass the exam."

C. The converse is "If a student passes the exam, then they did not study."

D. The contrapositive is "If a student passes the exam, then they studied."

Space for Personal Notes

Section B: Supplementary Questions

Sub-Section [2.3.1]: Solve Problems Using AM-GM Inequalities

Question 23

Show using the AM-GM inequality that for $x > 0$ we have:

$$5x + \frac{5}{x} \geq 10$$

_____

_____

_____

_____

Question 24

Minimise $2x + \frac{2}{x}$ over $x > 0$ by applying the AM-GM inequality, and hence maximise $6 - 2x - \frac{2}{x}$.

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 25

Find an expression for the area of a rectangle that has a perimeter of 4 units and a width of x units, and hence use the AM-GM inequality to maximise the area of such a rectangle.

_____

_____

_____

_____

_____

_____

Question 26

Let $x, y > 0$. Furthermore, suppose that $xy = 4$. Find the minimum value of $xy^3 + x^3y$.

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Sub-Section [2.3.2]: Solve Arithmetic and Geometric Proofs

Question 27

Prove using induction that $2 + 7 + 12 + \cdots + (5n - 3) = \frac{n(5n-1)}{2}$.

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 28

Prove using induction that $1 \cdot 7 + 2 \cdot 8 + \cdots + n(n+6) = \frac{n(n+1)(2n+19)}{6}$.

_____

_____

_____

_____

_____

_____

_____

_____

Question 29

Prove using induction that $2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^n = 3^{n+1} - 3$.

_____

_____

_____

_____

_____

_____

_____

_____

Question 30

a. Prove using induction that for all $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

_____

_____

_____

_____

_____

_____

b. Hence, write a rule for $2^3 + 4^3 + \cdots + (2n)^3$.

Hint: $2^3 + 4^3 + \cdots + (2n)^3$ is related to $1^3 + 2^3 + \cdots + n^3$ in a reasonably simple way.

_____

_____

_____

_____

c. Now, deduce a rule for $1^3 + 3^3 + \cdots + (2n-1)^3$ using the rule you obtained above.

_____

_____

_____

_____

Space for Personal Notes

Question 31

Prove using induction that if $n \in \mathbb{N}$, then $8^n - 1$ is divisible by 7.

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 32

Prove using induction that if $n \in \mathbb{N}$, then $n^3 + 3n^2 + 2n$ is divisible by 3.

Note: If you want to make this question a bit harder, you can instead show that $n^3 + 3n^2 + 2n$ is also divisible by 6. You might need to use the fact that the product of two consecutive integers is always even.

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 33

Prove using induction that if $n \in \mathbb{N}$, then $10^{n+1} + 10^n + 1$ is divisible by 3.

Note: The statement says that 111, 1101, 11001, etc. are all divisible by 3.

_____

_____

_____

_____

_____

_____

_____

_____

Question 34

Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. For example, $3! = 1 \cdot 2 \cdot 3$. Prove using induction that if $n \in \mathbb{N}$, then $(2n)!$ is divisible by 2^n.

_____

_____

_____

_____

_____

_____

_____

Question 35

Prove using induction that for all $n \in \mathbb{N}, n < 2^n$.

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 36

Consider the statement below:

There cannot exist two integers m and n such that $5m + 10n = 3$.

a. Write down a statement to begin a proof by contradiction for the statement above.

_____

b. Hence, obtain a contradiction and prove the original statement.

_____

_____

_____

_____

Space for Personal Notes

Question 37

Prove using induction that $6^n + 4$ is divisible by 5 for all $n \in \mathbb{N}$.

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 38

Prove using induction that for all $n \in \mathbb{N}$, it holds that $\left(1 + \frac{1}{1}\right)\left(2 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{n}\right) = n + 1$.

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Question 39

Prove the following biconditional statement for $x, y \in \mathbb{Z}$:

$$x + y \text{ is even, if and only if, } x^2 + y^2 \text{ is even.}$$

_____

_____

_____

_____

_____

_____

_____

_____

_____

Space for Personal Notes

Sub-Section: Exam 2 Questions

Question 40

The contrapositive to the statement, "If n is even, then n^2 is even." is:

A. If n^2 is odd, then n is odd.

B. If n^2 is even, then n is even.

C. If n is odd, then n^2 is even.

D. If n is odd, then n^2 is odd.

Question 41

Consider the following:

$$\text{For all } k > K, 1.5^k < (k-1)!$$

What is the smallest value of $K \in \mathbb{N}$ such that the above holds:

A. 2

B. 3

C. 4

D. 5

Question 42

The negation of the statement, "All the cars in the carpark are black." is:

A. All the vans in the carpark are black.

B. There exists a car in the carpark that is not black.

C. There exists a bus in the carpark without a mirror.

D. All the cars in the carpark are yellow.

Question 43

Find the minimum value of $6x^2 + \frac{6}{x^2}$.

A. 6

B. 25

C. 15

D. 12

Question 44

Consider the following statement:

> If a car in the carpark is black, then it costs a lot of money.

Which of the following is the converse of the above?

A. If a bus in the carpark costs a lot of money, then it is not black.

B. If a car costs a lot of money, then it is in the carpark.

C. If a car in the carpark is not black, then it costs a lot of money.

D. If a car in the carpark costs a lot of money, then it is black in colour.

Space for Personal Notes

CONTOUREDUCATION

VCE Specialist Mathematics ½

Free 1-on-1 Consults

What Are 1-on-1 Consults?

- ➤ **Who Runs Them?** Experienced Contour tutors (45 + raw scores and 99 + ATARs).
- ➤ **Who Can Join?** Fully enrolled Contour students.
- ➤ **When Are They?** 30-minute 1-on-1 help sessions, after-school weekdays, and all-day weekends.
- ➤ **What To Do?** Join on time, ask questions, re-learn concepts, or extend yourself!
- ➤ **Price?** Completely free!
- ➤ **One Active Booking Per Subject:** Must attend your current consultation before scheduling the next :)

SAVE THE LINK, AND MAKE THE MOST OF THIS (FREE) SERVICE!

Booking Link

bit.ly/contour-specialist-consult-2025