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VCE Specialist Mathematics ½
Proofs II [2.2]
Workbook

Outline:

Proving Conditional Statements Pg 2-18
- Conditional Statements
- Methods of Proving Conditional Statements
- Direct Proof
- Proof by Contrapositive
- Proof by Contradiction

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- Universal Quantifier
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- Negation of Universal and Existence Statement
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Learning Objectives:

- ☐ **SM12 [2.2.1]** - Direct proofs, proofs by contrapositive and contradiction.
- ☐ **SM12 [2.2.2]** - Converse and the equivalent of a conditional statement.
- ☐ **SM12 [2.2.3]** - Proofs involving the universal and existence quantifiers.
- ☐ **SM12 [2.2.4]** - Proofs by induction and telescoping series.

Section A: Proving Conditional Statements

Sub-Section: Conditional Statements

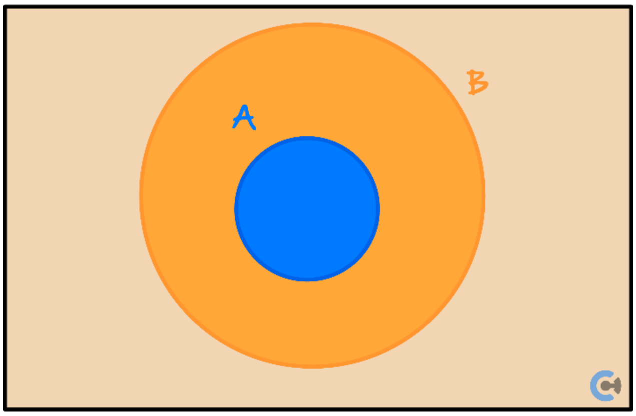
Context

- After this workbook, you will have the skills to tackle proof questions in Unit 3 and 4 level!

Exploration: Visualisation of Conditional Statements

$$A \Rightarrow B$$

Visualisation of $A \Rightarrow B$



If A, then B.

Conditional Statements

- Conditional Statement: "Hypothesis implies Conclusion"
- Note: Notation for "implies": $\Longrightarrow$

$$\text{"}Hypothesis \Longrightarrow Conclusion\text{"}$$

NOTE: Order Matters!

Let's look at some questions together!

Question 1 Walkthrough.

Write a conditional statement for the following:

A customer will receive a coupon if they spend $500.

If a customer spends $500 then they receive a coupon

Recall!

<u>Active Recall:</u> **Conditional statements**

- The relationship between the hypothesis and the conclusion is that the hypothesis $\Rightarrow$ conclusion
  implies

Your Turn!

Question 2

Write a conditional statement for the following:

Doing strength training grows your muscles.

If you do strength training then your muscles will grow.

Question 3 Additional Question.

Write a conditional statement for the following:

All books in the library's collection are about history.

If book in library's collection, it is about history.

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Sub-Section: Methods of Proving Conditional Statements

Okay, how do we prove them?

Proving Conditional Statements

1. Direct Proof
2. Proof by Contrapositive
3. Proof by Contradiction

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Sub-Section: Direct Proof

Context

- The proof question for the most recent VCAA Exam 1 was a direct proof!

REMINDER

- From last week, we found that an even number can be written in the form:

$$2k, \; k \in \mathbb{Z}$$

- An odd number can be written in the form:

$$2k+1, \; k \in \mathbb{Z}$$

- If a number is divisible by 3, then it can be written in the form:

$$3k, \; k \in \mathbb{Z}$$

- If a number is not divisible by 3, then it can be written in the form:

$$3k+1 \text{ or } 3k+2, \; k \in \mathbb{Z}$$

- For proofs involving divisibility, we often need to split into cases.

Method 1: Direct Proof

- The simplest method of proof. These are the proofs we have been working on in [2.1].
- Here we are not altering the statement we need to prove.

Question 4 Walkthrough.

Prove that if n is even, then n^3 is divisible by 8.

Let $n = 2k$, $k \in \mathbb{Z}$

Then $n^3 = (2k)^3$

$= 8k^3$

But $k^3 \in \mathbb{Z}$ $\therefore$ n^3 is divisible by 8 as req.

NOTE: We did not change the statement itself! Hence, a direct proof.

Recall!

Active Recall: Direct proofs

- In a direct proof, we do not change the original statement.

Your Turn!

Question 5

Prove that for all integers m and n, if m is divisible by 5 and n is divisible by 2, then $10m + 3n$ is even.

Let $m = 5k$ & $n = 2j$ $k, j \in \mathbb{Z}$

Then $10m + 3n = 10(5k) + 3(2j)$

$= 50k + 6j$

$= 2(25k + 3j)$

$= 2p$

Where $p = 25k + 3j \in \mathbb{Z}$

$\therefore$ is even, as req.

Question 6 Additional Question.

Prove that if the last two digits of n are divisible by 4, then n is divisible by 4.

Write $n = 100a + b$, $a, b \in \mathbb{N} \cup \{0\}$ and $0 \leq b < 100$

Then, if $b = 4k$, $k \in \mathbb{N}$

Then $n = 100a + 4k$

$= 4(25a + k)$

$= 4p$, $p = 25a + k \in \mathbb{N}$

$\therefore$ n is divisible by 4.

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Sub-Section: Proof by Contrapositive

Context

- Check out this question from the Sample Exam for Specialist Mathematics Units 3 and 4 Exam 2, which asks about contrapositive statements.

Question 1

Consider the following statement.

'For all integers n, if n^2 is even, then n is even.'

Which one of the following is the contrapositive of this statement?

A. For all integers n, if n^2 is odd, then n is odd.

B. There exists an integer n such that n^2 is even and n is odd.

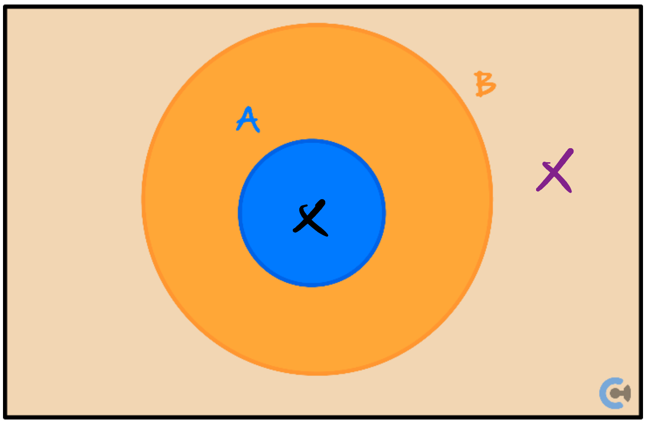
C. There exists an integer n such that n is even and n^2 is odd.

D. For all integers n, if n is odd, then n^2 is odd.

E. For all integers n, if n is even, then n^2 is even.

What is a contrapositive statement?

Contrapositive Statement



(Contrapositive Statement of $A \Rightarrow B$) is $\neg B \Rightarrow \neg A$

NOTE: Swap the order and negate the statements.

Question 7 Walkthrough.

Write down a contrapositive of the following statement:

If you starve for a day, then you are hungry.

If you are not hungry then you haven't starved for a day

NOTE: We negate and flip the order.

Recall!

Active Recall: Contrapositive statements

- The contrapositive of the statement $A \Rightarrow B$ is:

$$\neg B \Rightarrow \neg A$$

Your Turn!

Question 8

Write down the contrapositive of the following statement:

If all the sides of a triangle have equal length, then the triangle is equilateral.

If a triangle is not equilateral, not all of its sides have equal length.

Question 9 Additional Question.

Write down the contrapositive of the following statement:

If x is an irrational number, then $\sqrt{x}$ is irrational as well.

If $\sqrt{x}$ is rational then x is also rational

Discussion: What happens when you prove the contrapositive?

You prove O.G

Method 2: Proof by Contrapositive (Indirect Proof)

- Instead of proving $A \Rightarrow B$, we can prove its contrapositive $\neg B \Rightarrow \neg A$.

Prove a contrapositive statement instead.

- Considered to be an indirect proof, as the original statement is altered.
- **Steps**:
  1. Set up the contrapositive statement.
  2. Prove the contrapositive statement to be true.
  3. Conclude by saying "As contrapositive is true, the original statement is true."

Let's look at some questions together!

Question 10 Walkthrough.

Prove the following statement by contrapositive:

If $x^2 - 6x + 5$ is even, then x is odd.

It suffices to instead prove the contrapositive:

If x is even then $x^2 - 6x + 5$ is odd

Proof: Let $x = 2k,\ k \in \mathbb{Z}$

Then $x^2 - 6x + 5$

$= (2k)^2 - 6(2k) + 5$

$= 4k^2 - 12k + 5$

$= 2(2k^2 - 6k + 2) + 1$

$= 2p + 1,\ p = 2k^2 - 6k + 2 \in \mathbb{Z}$

$\therefore\ x^2 - 6x + 5$ is odd, as req.

<u>Discussion:</u> Imagine if we tried to do a direct proof for the previous question. Would it be easy?

Recall!

Active Recall: Proof by Contrapositive

- **Steps:**

1. Set up the contrapositive statement.
2. Prove the contrapositive statement to be true.
3. Conclude by saying "since contrapos. is true, the original statement is true"

Your Turn!

If n is rational
$n = p/q$, $p, q \in Z$ & $q \neq 0$

Question 11

Prove the following conditional statement using contrapositive:

Let $x \in R$. If x is irrational, then $\sqrt{x + \frac{1}{5}}$ is irrational.

① Contrapositive: If $\sqrt{x + \frac{1}{5}}$ is rational, then x is rational.

② Proof: Let $\sqrt{x + \frac{1}{5}} = p/q$, $p, q \in Z$ & $q \neq 0$

Then $x + \frac{1}{5} = \frac{p^2}{q^2}$

$x = \frac{p^2}{q^2} - \frac{1}{5}$

$x = \frac{5p^2 - q^2}{5q^2}$

But $5p^2 - q^2 \in Z$ & $5q^2 \in Z$

$\therefore$ x is rational.

③ Hence, since the contrapositive is true, so is the original statement.

Question 12 Additional Question.

Prove the following statement below by first writing down its contrapositive.:

$$\text{Let } a, b \in R^+. \text{ If } \sqrt{ab} \neq \frac{a+b}{2}, \text{ then } a \neq b.$$

Note: This means that if the geometric mean and arithmetic mean of two numbers a and b are not equal, then a and b are themselves not equal.

Prove instead the contrapositive, that is,

If $a = b$, then $\sqrt{ab} = \frac{a+b}{2}$

$$\begin{aligned} LHS = \sqrt{ab} &= \sqrt{a^2} \\ &= a \\ &= \frac{a+a}{2} \\ &= \frac{a+b}{2} = RHS \end{aligned}$$

$\therefore$ The original statement is true.

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Sub-Section: Proof by Contradiction

Context

- Check out this proof-by-contradiction question from the Specialist Mathematics Units 3 and 4 Sample Exam 2. Give it a try after completing this sub-section!

Question 4 (3 marks)
Use proof by contradiction to prove that if n is odd, where $n \in N$, then $n^3 + 1$ is even.

What is a contradicting statement?

Contradicting Statement

(Contradicting Statement of $A \Rightarrow B$) is $A \Rightarrow \neg B$

- Negate the conclusion.

Question 13

State the contradicting statement of the following:

If x is rational, then x^2 is rational.

If x is rational, then x^2 is irrational

Discussion: If we prove that the contradicting statement is false, then what are we also proving?

O.G is true

Method 3: Proof by Contradiction (Indirect Proof)

To Prove $A \Rightarrow B$

Assume $A \Rightarrow \neg B$ is true

And show that the assumption is FALSE.

- Steps:
  1. First, assume that the contradicting statement is true.
  2. Show that the assumption has a contradiction, and is hence false.
  3. Conclude by saying "Since the contradicting statement is false, the original statement is true."

odd = even
pos = neg
divisible by m = not divisible by m.

- Considered to be an indirect proof, as the original statement is altered.

Let's look at some questions together!

Question 14 Walkthrough.

Prove the simple statement below using contradiction:

$\log_5(9)$ is irrational

1 Assume instead, that $\log_5(9)$ is rational.

2 Let $\log_5(9) = p/q$, $p, q \in \mathbb{Z}$ & $q \neq 0$

$9 = 5^{p/q}$

$9^q = 5^p$

But LHS is not divisible by 5, & RHS is divisible by 5 $\therefore$ contradiction.

3 Since the negation is false, the original statement is true.

Active Recall: Steps for Proof by Contradictions

- Steps:

1. First, assume that the contradicting statement is true.
2. Show that the assumption has a contradiction, and is hence false.
3. Conclude by saying "since the negation is false, the original is true"

Active Recall: De Morgan's Law

$$\neg(A \wedge B) = \neg A \vee \neg B$$

$$\neg(A \vee B) = \neg A \wedge \neg B$$

Your Turn!

Question 15

Let $a, b \in R$. Prove that if $a + b > 150$, then $a > 75$ or $b > 75$.

Assume instead that $a+b > 150 \Rightarrow a \leq 75$ and $b \leq 75$

But $a \leq 75$ & $b \leq 75 \rightarrow a+b \leq 75+75$

$\therefore a+b \leq 150$

$a+b \leq 150$ & $a+b > 150$ is a contradiction.

$\therefore$ The original statement is true.

Question 16 Additional Question.

Use proof by contradiction to prove the following statement:

Let $x, y > 0$ and $x \neq$ y. Show that $\frac{x}{y} + \frac{y}{x} > 2$.

① Assume $\frac{x}{y} + \frac{y}{x} \leq 2$

Then $\frac{x^2+y^2}{yx} \leq 2$

$x^2+y^2 \leq 2xy$

$x^2-2xy+y^2 \leq 0$

$(x-y)^2 \leq 0$

This is a contradiction as $a^2 \geq 0$ for all $a \in R$

$\therefore$ The original statement is true.

__Key Takeaways__

- ☑ Direct proof involves proving without changing the conditional statement.
- ☑ The contrapositive of a statement $A \to B$ is given by $\neg B \to \neg A$.
- ☑ Proof by contrapositive involves proving the contrapositive of the conditional statement instead.
- ☑ Proof by contradiction requires first assuming that the contradicting statement is true, then showing contradiction.

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Section B: Converse Statements and Equivalent Statements

Sub-Section: Converse Statements

Context: Sam is being Converse

- Sam is being annoying and talks in the opposite direction of what Jacob says:

Jacob: "If you like chocolate, then you are Sam"

- What would Sam say?

Sam: "If you are Sam, then you like Chocolate"

- Here we call Sam the "converse" of Jacob.

Converse Statements

- **Definition**: Conditional statement that flows in the opposite direction.

(Converse Statement of $A \Rightarrow B$) is $B \Rightarrow A$

Question 17 Walkthrough.

For the following statement, write down the converse statement, and conclude whether the converse is true:

If a shape is a square, then it is a rectangle. ☑

If a shape is a rectangle, then it is a square ☒

This is false.

NOTE: This is NOT the same as negation! It's simply an opposite flow.

Recall!

Active Recall: Converse statements

- The converse of the statement $A \Rightarrow B$ is

$$B \Rightarrow A$$

Your Turn!

Question 18

For the following statement, write down the converse statement, and conclude whether the converse is true:

If x is divisible by 2 and 5, then it is divisible by 10.

If x is divisible by 10, then it is divisible by 2 & 5.

True ☑

Discussion: If your friend told you that x is divisible by 2 and 5, is that equivalent to x is divisible by 10?

Yes

What happens if the converse is also true?

Equivalent Statements (Biconditional)

- It is a conditional statement where if the original is proven to be true, its converse is ALWAYS true.

$$A \Rightarrow B \text{ and } B \Rightarrow A$$

$$A \Leftrightarrow B$$

$$A \text{ is true, if and only if } B$$

- In description, B is true if & only if A is true.
- To prove equivalent statements, we prove each direction separately.

Question 19 Walkthrough.

Let n be an integer. Prove that $\underbrace{n \text{ is even}}_{A}$, $\Leftrightarrow$ if and only if $\underbrace{3n+3 \text{ is odd}}_{B}$.

($\Rightarrow$) If n is even then $3n+3$ is odd

Proof: $n = 2k$, $k \in \mathbb{Z}$

Then $3n + 3 = 3(2k) + 3$

$= 6k + 3$

$= 2(3k+1) + 1$

Which is odd as $3k+1 \in \mathbb{Z}$

($\Leftarrow$) If $3n+3$ is odd, then n is even

Use contrapositive: If n is odd, then $3n+3$ is even

Let $n = 2k+1$, then $3n + 3 = 6k + 3 + 3$

$= 2(3k+3)$

Which is even.

NOTE: For if and only if (equivalent statement), we must prove both converse statements.

$\therefore$ The statement is proved.

Recall!

Active Recall: Proving biconditional statements

- To prove an "if and only if" statement $A \Leftrightarrow B$, we need to show both $A \Rightarrow B$ and $B \Rightarrow A$.

Your Turn!

Question 20

Let n be an integer. Prove that n is odd, if and only if n^2 is odd.

① n is odd $\Rightarrow$ n^2 is odd

Let $n = 2k+1$, $k \in \mathbb{Z}$

Then $n^2 = (2k+1)^2$

$= 4k^2+4k+1$

$= 2(2k^2+2k)+1$

But $2k^2+2k \in \mathbb{Z}$, then n^2 is odd as req.

② n^2 is odd $\Rightarrow$ n is odd

Prove by contrapositive.

Contrapositive: n is even $\Rightarrow$ n^2 is even

Let $n = 2k$, $k \in \mathbb{Z}$

Then $n^2 = (2k)^2$

$= 2(2k^2)$

But $2k^2 \in \mathbb{Z}$

$\therefore$ n^2 is even.

$\therefore$ n odd $\Leftrightarrow$ n^2 is odd, as req.

Question 21 Additional Question.

Let x be a real number. Show that $x^2 + y^2 = 0$, if and only if $x = 0$ and $y = 0$.

$x=0, y=0 \Rightarrow x^2+y^2=0$

$0^2+0^2 = 0+0 = 0$

$\therefore$ True

$x^2+y^2=0 \Rightarrow x=0$ & $y=0$

Prove instead the contraposition, that is if $x\neq 0$ or $y\neq 0$ then $x^2+y^2\neq 0$

If $x\neq 0$, then $x^2+y^2 \geq x^2 > 0$ since $x\neq 0$

$\therefore x^2+y^2\neq 0$

If $y\neq 0$, then $x^2+y^2\geq y^2>0$ since $y\neq 0$

$\therefore x^2+y^2\neq 0$

$\therefore$ The original statement is true since contrapos. is true.

$$\therefore x^2+y^2=0 \Leftrightarrow x=0 \text{ \& } y=0$$

Key Takeaways

- The converse statement of $A \to B$ is given by $B \to A$.
- Equivalent statement is when a statement and its converse both are proved to be true.
- "If and only if" stands for equivalent statement.

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Section C: Universal and Existential Quantifiers

Sub-Section: Universal Quantifier

How do we represent all members of a given set?

Universal Quantifiers

- Universal Quantifiers is a way to represent all members of a given set.

'For all real numbers x, x^2 is never negative.'

- Notation: ∀ (Universal Quantifier) 'for all'.

$$\forall x \in R, x^2 \geq 0$$

Discussion: Where does the symbol ∀ come from?

All → '∀'

Question 22 Walkthrough.

Convert the following conditional statement using the universal quantifier.

For all real numbers x, e^x is strictly positive.

$$\forall x \in R, e^x > 0$$

Recall!

Your Turn!

Question 23

Rewrite the following statement using the universal quantifier:

For all integers n, $n^2 - 4n$ is an integer.

$\forall n \in \mathbb{Z},\ n^2 - 4n \in \mathbb{Z}$

Question 24 Additional Question.

Rewrite the following statement using the universal quantifier:

For all natural numbers n, n is at most equal to n^2.

$\forall n \in \mathbb{N},\ n \leq n^2$

Sub-Section: Existence Quantifier

How do we represent certain members of a given set?

Existence Quantifiers

- Existence quantifier is a way to represent certain members of a given set.

There exists an integer such that $x^2 - x - 12 = 0$.

- Notation: ∃ (Existential Quantifier) 'There exists'.

$$\exists x \in \mathbb{Z}, x^2 - x - 12 = 0.$$

Discussion: How can we prove the existence statement?

Give an example

Question 25 Walkthrough.

Convert the following conditional statement using the existence quantifier.

There exist a real number x such that $x^2 + 4 = 16$.

$\exists x \in R, x^2 + 4 = 16$

Recall!

Active Recall: Existence quantifier

- The symbol for the existence quantifier $\exists$ represents Some members of the set.

Your Turn!

Question 26

Rewrite the following statement using the existence quantifier.

There exists a positive whole number n such that $2^n = 4$.

$$\exists n \in N, 2^n = 4$$

Question 27 Additional Question.

Rewrite the following statement using the existence quantifier.

There exists a positive real number x such that $x^2 < x$.

$$\exists x \in R^+, x^2 < x$$

Sub-Section: Negation of Universal and Existence Statement

Discussion: What would be the opposite of saying that all living humans breathe?

At least one human doesn't breath

Discussion: What would be the opposite of saying that some humans are taller than $190\ cm$?

All are 190cm or shorter.

Negation of Universal and Existence Statements

- They are opposites of each other (with opposite conclusions).

$$\neg\text{Universal Statement} = \text{Existence Statement with Opposite Conclusion}$$

- And vice versa.

Question 28 Walkthrough.

Write down the negation of the following statements.

a. For all natural numbers, $3n \geq 2n - 1$.

$\exists\, n \in N, 3n < 2n - 1$

b. There exists a real number x such that $x^2 = 4$.

$\forall\, n \in N,\ x^2 \neq 4$

Recall!

<u>Active Recall:</u> Negation involving the universal and existence quantifiers.

$\neg$ Universal statement = Existence statement with opposite concl.

$\neg$ Existence statement = Universal Statement with opp. concl.

Your Turn!

Question 29

Write down the negation for the following statements below.

a. If n is a natural number, then $n + 1 > n$.

$\exists\, n \in N,\ n+1 \leq n$

b. There exists an integer k such that $k^2 = k + 4$.

$\forall\, k \in \mathbb{Z},\ k^2 \neq k+4$

Question 30 Additional Question.

Write the negation for the following statements below.

a. If n is a natural number, then $6n^2 + 4n + 1$ is divisible by n.

$\exists\, n \in N$, $6n^2+4n+1$ is not divisible by n

b. There exist integers p and q such that $q \neq 0$ such that $\pi = \frac{p}{q}$.

$\forall\, p, q \in \mathbb{Z}$ & $q \neq 0$, $\pi \neq \frac{p}{q}$

Sub-Section: Disproving a Universal Statement

Context

- Consider the following questionable statement:

"All of the vehicles in the carpark are black."

- How can you disprove this statement?

Find a car that isn't black

- Basically, you have just provided a counter-example.

Disproving Universal Statement

- We prove the *opposite* (negation) existence statement.
- We call this proof by counter-example.
  - Giving a counter example will be proving an opposite existence statement.

Question 31 Walkthrough.

Disprove the following statement: **For all positive integers m, if m is prime then $m^2 + 4$ is also prime.**

Let $m = 2$

m is prime. But $2^2 + 4 = 8$

& 8 is not composite.

$\therefore$ The statement is false by counterexample.

NOTE: We simply give ourselves a counter example to disprove a universal statement.

Recall!

Active Recall: Disproving a universal statement.

- To show that a "for all ..." statement is not true, it suffices to provide a counter-example.

Your Turn!

Question 32

Disprove the following statement:

$$\forall a, b \in R, \text{ if } a^2 = b^2 \text{ then } a = b.$$

Take $a = -b$

$a \neq b$ but $a^2 = b^2$

$\therefore$ by counterexample, the statement is true.

Question 33 Additional Question.

Disprove the statement given below:

For all positive integers n, $2^n > n^2$.

Sub-Section: Disproving an Existence Statement

Disproving Existence Statements

- We prove the *opposite* (negation) universal statement.
(Direct or Indirect Proof)

Question 34 Walkthrough.

Disprove the following statement: There exists $n \in N$ such that $n^2 + 9n + 20$ is a prime number.

Negation: $\forall n \in N$, $n^2+9n+20$ is NOT prime.

$$n^2 + 9n + 20 = (n+5)(n+4)$$

$n+5 \neq \{0,1\}$ since $n \in N$

$n+4 \neq \{0,1\}$ since $n \in N$

$\therefore$ $n^2+9n+20$ has factors that are not just 1 & itself

$\therefore$ $n^2+9n+20$ is composite (Not prime)

$\therefore$ The original statement is false.

NOTE: To disprove their exist statement, you must show the opposite universal statement as true.

Recall!

Active Recall: Disproving an existence statement.

- To disprove a "there exists ..." statement, we need to show the opposite universal statement is true.

Your Turn!

Question 35

Disprove the following statements:

There exists a real number x, such that $10 + 3x^2 = 3 + x^2$.

Negation: $\forall x \in R$, $10+3x^2 \neq 3+x^2$

$2x^2 \neq -7$

$x^2 \neq -7/2$

Which is true since $x^2 \geq 0$ $\forall x \in R$.

$\therefore$ The original statement is false.

Question 36 Additional Question.

Disprove the following statement below:

There exists $n \in N$ so that $n^3 - 2n^2 + 5n + 4$ is divisible by $n + 2$.

Prove instead that "$\forall n \in N$, n^3-2n^2+5n+4 is not divisible by $n+2$"

Let $P(x) = x^3-2x^2+5x+4$

$P(-2) = -8-8-10+4 = -22 \neq 0$

$\therefore$ $n+2$ is not a factor $P(x)$ by factor theorem.

$\therefore$ The original statement is false

Key Takeaways

- [x] There exists a quantifier $\exists$.
- [x] For all quantifiers $\forall$.
- [x] To prove a there exists statement, simply give an example.
- [x] To prove for all statements, you must prove for all values.
- [x] To disprove a there exist statement, prove the opposite for all statements.
- [x] To disprove a for-all statement, prove the opposite there exist statement.

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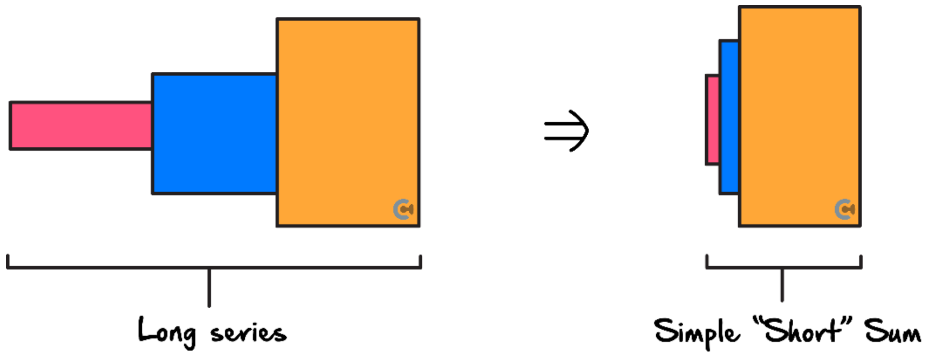
Section D: Other Proof Techniques

Sub-Section: Telescopic Cancelling

Telescopic Cancelling

- Telescoping series is a series (sum of terms) where all terms cancel out except for the first & last term.
- The name comes from the visualisation as shown below.



- Proofs of telescoping series can be done via cancelling terms in a sequence 'Telescoping cancelling'

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Question 37 Walkthrough.

a. Using partial fractions, find the values of a and b such that:

$$\frac{3}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}$$

$$\frac{(a+b)k + a}{k(k+1)} \equiv \frac{3}{k(k+1)}$$

$a = 3$

$a + b = 0$

$b = -3$

$a = 3, b = -3$

b. Hence, prove that:

$$\frac{3}{1 \times 2} + \frac{3}{2 \times 3} + \cdots + \frac{3}{n \times (n+1)} = \frac{3n}{n+1}$$

Use result: $\frac{3}{1(1+1)} + \frac{3}{2(2+1)} + \cdots \frac{3}{n(n+1)}$

$$= \left(\frac{3}{1} - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{4}}\right) \cdots + \left(\cancel{\frac{3}{n-1}} - \cancel{\frac{3}{n}}\right) + \left(\cancel{\frac{3}{n}} - \frac{3}{n+1}\right)$$

$$= 3 - \frac{3}{n+1}$$

$$= \frac{3n+3-3}{n+1} = \frac{3n}{n+1}, \blacksquare$$

Recall!

<u>Active Recall:</u> **Telescoping sums**

- The key idea for telescoping sums is to find cancellation among terms

Your Turn!

Question 38

a. Using partial fraction decomposition, find values of A and B such that:

$$\frac{1}{n^2+3n+2} \equiv \frac{A}{n+1} + \frac{B}{n+2}$$

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$
$$= \frac{(A+B)n + (2A+B)}{(n+1)(n+2)}$$

$\therefore A+B=0$
$2A+B=1$

$\therefore A=1, B=-1$

b. Hence, prove that:

$$\sum_{n=1}^{k} \frac{1}{n^2+3n+2} = \frac{k}{2(k+2)}$$

$$\sum_{n=1}^{k} \frac{1}{n^2+3n+2} = \sum_{n=1}^{k}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$
$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) \quad a_1$$
$$+ \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \quad a_2$$
$$+ \cdots \left(\cancel{\frac{1}{k}} - \cancel{\frac{1}{k+1}}\right) \quad a_{k-1}$$
$$+ \left(\cancel{\frac{1}{k+1}} - \frac{1}{k+2}\right) \quad a_k$$
$$= \frac{1}{2} - \frac{1}{k+2}$$
$$= \frac{k}{2(k+2)}, \text{ as req.}$$

Question 39 Additional Question.

By using the fact that $\frac{2(2k+1)}{k^2(k+1)^2} = \frac{2}{k^2} - \frac{2}{(k+1)^2}$, show that:

$$\frac{2\times 5}{2^2\times 3^2} + \cdots + \frac{2(2n+1)}{n^2(n+1)^2} = \frac{(n-1)(n+3)}{2(n+1)^2}$$

Starts at $k=2$

$$\left(\frac{2}{2^2} - \cancel{\frac{2}{3^2}}\right)$$
$$+ \left(\cancel{\frac{2}{3^2}} - \cancel{\frac{2}{4^2}}\right)$$
$$+ \left(\cancel{\frac{2}{4^2}} - \cancel{\frac{2}{5^2}}\right)$$
$$+ \cdots + \left(\cancel{\frac{2}{(n-1)^2}} - \cancel{\frac{2}{n^2}}\right)$$
$$+ \left(\cancel{\frac{2}{n^2}} - \frac{2}{(n+1)^2}\right)$$

$$= \frac{1}{2} - \frac{2}{(n+1)^2}$$
$$= \frac{n^2+2n+1-4}{2(n+1)^2}$$
$$= \frac{n^2+2n-3}{2(n+1)^2}$$
$$= \frac{(n-1)(n+3)}{2(n+1)^2}, \text{ as req.}$$

Sub-Section: Proof by Induction

Context

- Check out this question from the Specialist Mathematics Units 3 and 4 Sample Exam.

Question 3 (4 marks)
Prove by mathematical induction that the number $9^n - 5^n$ is divisible by 4 for all $n \in N$.

Proof by Induction

- We will be following the given steps:" TAPE ".
  - **Step T:** Test the statement for its first possible value.
  - **Step A:** Assume that the statement is true for $n = k$.
  - **Step P:** Prove that if the statement holds true for $n = k$, It also holds true for $n = k + 1$.
    - Add "By Assumption".
  - **Step E:** "By the principle of mathematical induction, the statement is true for a set of values."

Space for Personal Notes

Question 40 Walkthrough.

Prove that for $n \in N$, $1 + 3 + 5 \cdots (2n - 1) = n^2$.

$P(n)$: $1 + 3 + \ldots + (2n-1) = n^2$

(1) Base Case: $n = 1$

LHS $= 1$, RHS $= 1^2 = 1$

$\therefore$ Base Case True

(2) Let $k \in N$ be arbitrary, and assume $P(k)$ true.

That is $1 + 3 + \ldots + (2k-1) = k^2$

(3) LHS of $P(k+1)$

$= 1 + 3 + 5 + \ldots + (2k-1) + (2(k+1)-1)$

$= k^2 + 2k + 1$ (by assumption)

$= (k+1)^2$

$=$ RHS of $P(k+1)$

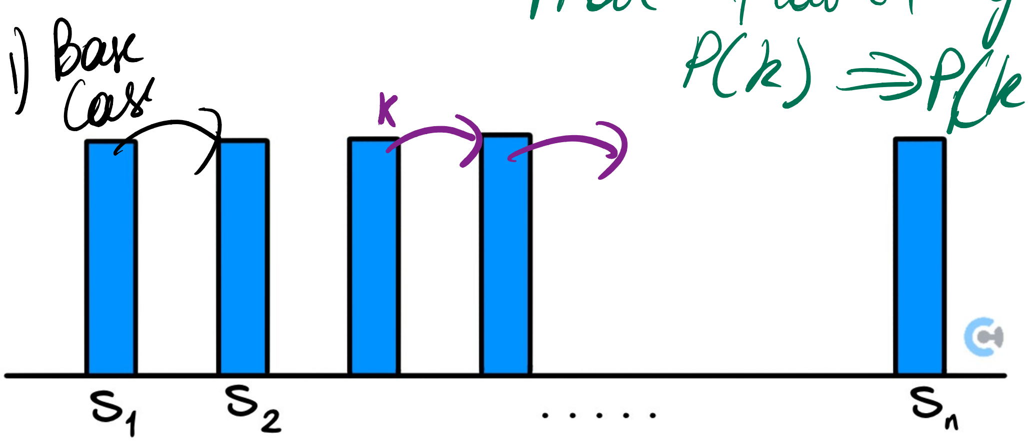
(4) Since $P(1)$ true, & $P(k) \Rightarrow P(k+1)$ then $P(n)$ true for all $n \in N$ by the principle of induction.

Exploration: How do we visualise the idea of proof by induction?

- Imagine we have dominos of statements we need to prove.
- Each domino is a statement with a specific value of n.

Dominos of Statement

Prove flow of logic $P(k) \Rightarrow P(k+1)$



- In Step 1: We test/prove the first statement.
- In Steps A and P: We prove the relationship between a "previous" domino block, and the "next" one.
- If one domino falls, the next one also falls.

  (If $P(k)$ true, then $P(k+1)$ also true)
- Since the first one falls, all of them fall!

Recall!

Active Recall: Proofs by induction

- When doing a proof-by-induction question, we have four key steps:

1. Test the Base Case holds.
2. Assume the statement holds for an $n =$ k in the required set of values for k.
3. Prove that the statement holds for $k+1$.
4. Explain using the principle of mathematical induction. that the statement holds for all k.

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Your Turn!

Question 41

Prove that for all $n \in N$, we have:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

NOTE: $1 + 2 + \cdots + n$ can be written as:

$$\sum_{i=1}^{n} i$$

RHS of $P(k+1) = \frac{(k+1)(k+2)}{2}$

$P(n)$: $1+2+\dots+n = \frac{n(n+1)}{2}$

Proof:

① Base Case: $n=1$

LHS $= 1$

RHS $= \frac{1(1+1)}{2} = 1$

$\therefore$ Base Case True

② Assume $k \in N$ be arbitrary & $P(k)$ true.

That is, $1+2+\dots+k = \frac{k(k+1)}{2}$

③ Then LHS of $P(k+1)$

$1+2+\dots+k+(k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ (by assumption)

$= \frac{k(k+1)+2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$ = RHS of $P(k+1)$

④ Since $P(1)$ True & $P(k) \Rightarrow P(k+1)$, then $P(n)$ is true for all $n \in N$ by the principles of induction.

Question 42 Additional Question.

Prove by induction that for any natural number $n \geq 1, n < 2^n$.

$P(n)$: $2^n > n$ for all $n \in \mathbb{N}$

① Test Base Case: $n=1$

$2^1 > 1$ $\therefore P(1)$ true

② Assume for $k \in \mathbb{N}$ that is arbitrary that $P(k)$ true, that is assume $2^k > k$

③ Prove $P(k+1)$

$LHS = 2^{k+1}$
$= 2(2^k)$
$= 2^k + 2^k$
$> k + k$ (by assumption)
$\geq k+1$ as $k \geq 1$
$= RHS$

$\therefore 2^k > k \Rightarrow 2^{k+1} > k+1$

④ $\therefore$ Since $P(1)$ True & $P(k) \Rightarrow P(k+1)$, $P(n)$ true $\forall n \in \mathbb{N}$ by induction.

Key Takeaways

- For telescoping series proof, always look for terms to cancel each other.
- For induction proof always use TAPE: Test for the first case, assume it's true for $n = k$, prove for $n = k + 1$ and explain at the end.

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Contour Check

Learning Objective: [2.2.1] – Direct proofs, proofs by contrapositive and contradiction.

Key Takeaways

- Direct proof is proving without changing the conditional statement.
- The contrapositive of a statement $A \rightarrow B$ is given by $\neg B \rightarrow \neg A$.
- Proof by contrapositive is proving the contrapositive instead.
- Proof by contradiction requires first assuming the negation is true and showing contradiction.

Learning Objective: [2.2.2] – Converse and the equivalent of a conditional statement.

$A \leftrightarrow B$

Key Takeaways

- Converse statement of $A \rightarrow B$ is given by $B \rightarrow A$.
- Equivalent statement is when a statement & its converse both have to be true.
- "If and only if" stands for equivalence.

Learning Objective: [2.2.3] - Proofs involving the universal and existence quantifiers.

Key Takeaways

- There exists a quantifier ∃.
- For all quantifiers ∀.
- To prove a there exists statement, simply give an example.
- To prove for all statements, you must prove for all values.
- To disprove a there exist statement, prove opposite universal statement.
- To disprove a for-all statement, prove opposite there exists statement.

Learning Objective: [2.2.4] - Proofs by induction and telescoping series.

Key Takeaways

- For telescoping series proof, always look for terms to cancel each other..
- For induction proof always use TAPE: Test for base case
  Assume true for $n=k$.
  Prove that true for $n=k+1$
  Explain at end.

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