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Email: hello@contoureducation.com.au

VCE Specialist Mathematics ½
Proofs II [2.2]
Workbook

Outline:

Learning Objectives:

- ☐ **SM12 [2.2.1]** - Direct proofs, proofs by contrapositive and contradiction.
- ☐ **SM12 [2.2.2]** - Converse and the equivalent of a conditional statement.
- ☐ **SM12 [2.2.3]** - Proofs involving the universal and existence quantifiers.
- ☐ **SM12 [2.2.4]** - Proofs by induction and telescoping series.

Section A: Proving Conditional Statements

Sub-Section: Conditional Statements

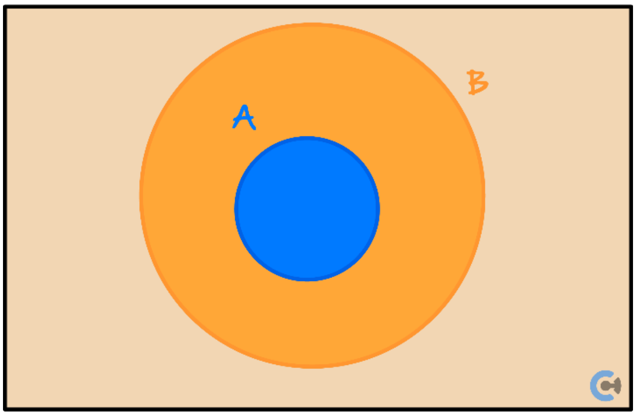
Context

- After this workbook, you will have the skills to tackle proof questions in Unit 3 and 4 level!

Exploration: Visualisation of Conditional Statements

$$A \Rightarrow B$$

Visualisation of $A \Rightarrow B$



If A, then B.

Conditional Statements

- Conditional Statement: ______________________________________.
- Note: Notation for "implies": $\Longrightarrow$

$$\text{"}Hypothesis \Longrightarrow Conclusion\text{"}$$

NOTE: Order Matters!

Let's look at some questions together!

Question 1 Walkthrough.

Write a conditional statement for the following:

A customer will receive a coupon if they spend $500.

Recall!

<u>Active Recall:</u> Conditional statements

- The relationship between the hypothesis and the conclusion is that ______________________________.

Your Turn!

Question 2

Write a conditional statement for the following:

Doing strength training grows your muscles.

Question 3 Additional Question.

Write a conditional statement for the following:

All books in the library's collection are about history.

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Sub-Section: Methods of Proving Conditional Statements

Okay, how do we prove them?

Proving Conditional Statements

1. Direct Proof
2. Proof by Contrapositive
3. Proof by Contradiction

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Sub-Section: Direct Proof

Context

- The proof question for the most recent VCAA Exam 1 was a direct proof!

REMINDER

- From last week, we found that an even number can be written in the form:

- An odd number can be written in the form:

- If a number is divisible by 3, then it can be written in the form:

- If a number is not divisible by 3, then it can be written in the form:

- For proofs involving divisibility, we often need to split into ________________.

Method 1: Direct Proof

- The simplest method of proof. These are the proofs we have been working on in [2.1].
- Here we are not altering the statement we need to prove.

Question 4 Walkthrough.

Prove that if n is even, then n^3 is divisible by 8.

NOTE: We did not change the statement itself! Hence, a direct proof.

Recall!

Active Recall: Direct proofs

- In a direct proof, we ____________________________ the original statement.

Your Turn!

Question 5

Prove that for all integers m and n, if m is divisible by 5 and n is divisible by 2, then $10m + 3n$ is even.

Question 6 Additional Question.

Prove that if the last two digits of n are divisible by 4, then n is divisible by 4.

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Sub-Section: Proof by Contrapositive

Context

- Check out this question from the Sample Exam for Specialist Mathematics Units 3 and 4 Exam 2, which asks about contrapositive statements.

Question 1

Consider the following statement.

'For all integers n, if n^2 is even, then n is even.'

Which one of the following is the contrapositive of this statement?

A. For all integers n, if n^2 is odd, then n is odd.

B. There exists an integer n such that n^2 is even and n is odd.

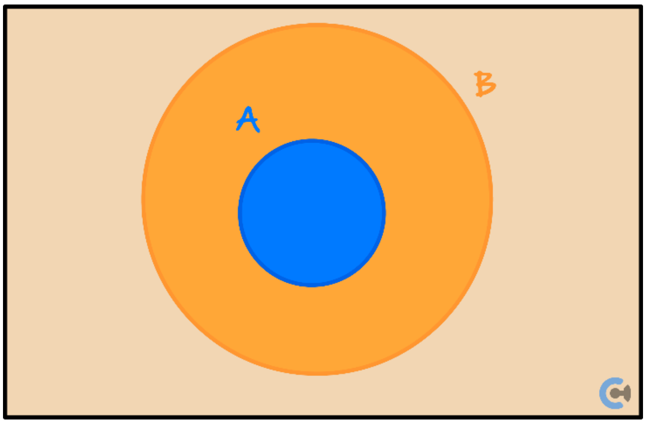
C. There exists an integer n such that n is even and n^2 is odd.

D. For all integers n, if n is odd, then n^2 is odd.

E. For all integers n, if n is even, then n^2 is even.

What is a contrapositive statement?

Contrapositive Statement



(Contrapositive Statement of $A \Rightarrow B$) is $\neg B \Rightarrow \neg A$

NOTE: Swap the order and negate the statements.

Question 7 Walkthrough.

Write down a contrapositive of the following statement:

If you starve for a day, then you are hungry.

NOTE: We negate and flip the order.

Recall!

Active Recall: Contrapositive statements

- The contrapositive of the statement $A \Rightarrow B$ is:

Your Turn!

Question 8

Write down the contrapositive of the following statement:

If all the sides of a triangle have equal length, then the triangle is equilateral.

Question 9 Additional Question.

Write down the contrapositive of the following statement:

If x is an irrational number, then $\sqrt{x}$ is irrational as well.

Discussion: What happens when you prove the contrapositive?

Method 2: Proof by Contrapositive (Indirect Proof)

- Instead of proving $A \Rightarrow B$, we can prove its contrapositive $\neg B \Rightarrow \neg A$.

Prove a contrapositive statement instead.

- Considered to be an ____________________, as the original statement is altered.
- **Steps**:
  1. Set up the contrapositive statement.
  2. Prove the contrapositive statement to be true.
  3. Conclude by saying "As contrapositive is true, the original statement is true."

Let's look at some questions together!

Question 10 Walkthrough.

Prove the following statement by contrapositive:

$$\text{If } x^2 - 6x + 5 \text{ is even, then } x \text{ is odd.}$$

<u>Discussion:</u> Imagine if we tried to do a direct proof for the previous question. Would it be easy?

Recall!

Active Recall: Proof by Contrapositive

- **Steps:**

1. Set up the ___________________________ statement.
2. Prove the contrapositive statement to be _______________.
3. Conclude by saying "____."

Your Turn!

Question 11

Prove the following conditional statement using contrapositive:

$$\text{Let } x \in R. \text{ If } x \text{ is irrational, then } \sqrt{x + \frac{1}{5}} \text{ is irrational.}$$

Question 12 Additional Question.

Prove the following statement below by first writing down its contrapositive.:

$$\text{Let } a, b \in R^+. \text{ If } \sqrt{ab} \neq \frac{a+b}{2}, \text{ then } a \neq b.$$

Note: This means that if the geometric mean and arithmetic mean of two numbers a and b are not equal, then a and b are themselves not equal.

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Sub-Section: Proof by Contradiction

Context

- Check out this proof-by-contradiction question from the Specialist Mathematics Units 3 and 4 Sample Exam 2. Give it a try after completing this sub-section!

Question 4 (3 marks)
Use proof by contradiction to prove that if n is odd, where $n \in N$, then $n^3 + 1$ is even.

What is a contradicting statement?

Contradicting Statement

$$\text{(Contradicting Statement of } A \Rightarrow B\text{) is } A \Rightarrow \neg B$$

- Negate the ______________________.

Question 13

State the contradicting statement of the following:

If x is rational, then x^2 is rational.

Discussion: If we prove that the contradicting statement is false, then what are we also proving?

Method 3: Proof by Contradiction (Indirect Proof)

To Prove $A \Rightarrow B$

Assume $A \Rightarrow \neg B$ is true

And show that the assumption is FALSE.

- Steps:
  1. First, assume that the contradicting statement is true.
  2. Show that the assumption has a contradiction, and is hence false.
  3. Conclude by saying "Since the contradicting statement is false, the original statement is true."
- Considered to be an ________________________, as the original statement is altered.

Let's look at some questions together!

Question 14 Walkthrough.

Prove the simple statement below using contradiction:

$$\log_5(9) \text{ is irrational}$$

Active Recall: Steps for Proof by Contradictions

- Steps:

1. First, assume that the contradicting statement is ____________.
2. Show that the assumption ______________________________, and is hence false.
3. Conclude by saying "____"

Active Recall: De Morgan's Law

$$\neg(A \wedge B) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\neg(A \vee B) = \_\_\_\_\_\_\_\_\_\_\_\_$$

Your Turn!

Question 15

Let $a, b \in R$. Prove that if $a + b > 150$, then $a > 75$ or $b > 75$.

Question 16 Additional Question.

Use proof by contradiction to prove the following statement:

$$\text{Let } x, y > 0 \text{ and } x \neq y. \text{ Show that } \frac{x}{y} + \frac{y}{x} > 2.$$

Key Takeaways

- ☑ Direct proof involves proving without changing the conditional statement.
- ☑ The contrapositive of a statement $A \to B$ is given by $\neg B \to \neg A$.
- ☑ Proof by contrapositive involves proving the contrapositive of the conditional statement instead.
- ☑ Proof by contradiction requires first assuming that the contradicting statement is true, then showing contradiction.

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Section B: Converse Statements and Equivalent Statements

Sub-Section: Converse Statements

Context: Sam is being Converse

- Sam is being annoying and talks in the opposite direction of what Jacob says:

Jacob: "If you like chocolate, then you are Sam"

- What would Sam say?

Sam:

- Here we call Sam the "_____________________" of Jacob.

Converse Statements

- **Definition**: Conditional statement that flows in the opposite direction.

(Converse Statement of $A \Rightarrow B$) is $B \Rightarrow A$

Question 17 Walkthrough.

For the following statement, write down the converse statement, and conclude whether the converse is true:

If a shape is a square, then it is a rectangle.

NOTE: This is NOT the same as negation! It's simply an opposite flow.

Recall!

Active Recall: Converse statements

- The converse of the statement $A \Rightarrow B$ is

Your Turn!

Question 18

For the following statement, write down the converse statement, and conclude whether the converse is true:

If x is divisible by 2 and 5, then it is divisible by 10.

Discussion: If your friend told you that x is divisible by 2 and 5, is that equivalent to x is divisible by 10?

What happens if the converse is also true?

Equivalent Statements (Biconditional)

- It is a conditional statement where if the original is ______________________, its converse is ______________________.

$$A \Rightarrow B \text{ and } B \Rightarrow A$$

$$A \Leftrightarrow B$$

$$A \text{ is true, if and only if } B$$

- In description, B is true ______________________ A is true.
- To prove equivalent statements, we prove each direction separately.

Question 19 Walkthrough.

Let n be an integer. Prove that n is even, if and only if $3n + 3$ is odd.

NOTE: For if and only if (equivalent statement), we must prove both converse statements.

Recall!

Active Recall: Proving biconditional statements

- To prove an "if and only if" statement $A \Leftrightarrow B$, we need to show both __________ and __________.

Your Turn!

Question 20

Let n be an integer. Prove that n is odd, if and only if n^2 is odd.

Question 21 Additional Question.

Let x be a real number. Show that $x^2 + y^2 = 0$, if and only if $x = 0$ and $y = 0$.

Key Takeaways

- The converse statement of $A \to B$ is given by $B \to A$.
- Equivalent statement is when a statement and its converse both are proved to be true.
- "If and only if" stands for equivalent statement.

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Section C: Universal and Existential Quantifiers

Sub-Section: Universal Quantifier

How do we represent all members of a given set?

Universal Quantifiers

- ________________________________ is a way to represent all members of a given set.

For all real numbers x, x^2 is never negative.

- Notation: ∀ (Universal Quantifier) _____________________.

$$\forall x \in R, x^2 \geq 0$$

Discussion: Where does the symbol ∀ come from?

Question 22 Walkthrough.

Convert the following conditional statement using the universal quantifier.

For all real numbers x, e^x is strictly positive.

Recall!

Your Turn!

Question 23

Rewrite the following statement using the universal quantifier:

For all integers n, $n^2 - 4n$ is an integer.

Question 24 Additional Question.

Rewrite the following statement using the universal quantifier:

For all natural numbers n, n is at most equal to n^2.

Sub-Section: Existence Quantifier

How do we represent certain members of a given set?

Existence Quantifiers

- ________________________ is a way to represent certain members of a given set.

There exists an integer such that $x^2 - x - 12 = 0$.

- Notation: ∃ (Existential Quantifier) ________________________.

$$\exists x \in \mathbb{Z}, x^2 - x - 12 = 0.$$

Discussion: How can we prove the existence statement?

Question 25 Walkthrough.

Convert the following conditional statement using the existence quantifier.

There exist a real number x such that $x^2 + 4 = 16$.

Recall!

Active Recall: Existence quantifier

- The symbol for the existence quantifier ∃ represents ____.

Your Turn!

Question 26

Rewrite the following statement using the existence quantifier.

There exists a positive whole number n such that $2^n = 4$.

Question 27 Additional Question.

Rewrite the following statement using the existence quantifier.

There exists a positive real number x such that $x^2 < x$.

Sub-Section: Negation of Universal and Existence Statement

Discussion: What would be the opposite of saying that all living humans breathe?

Discussion: What would be the opposite of saying that some humans are taller than $190\ cm$?

Negation of Universal and Existence Statements

- They are opposites of each other (with opposite conclusions).

$$\neg\text{Universal Statement} = \text{Existence Statement with Opposite Conclusion}$$

- And vice versa.

Question 28 Walkthrough.

Write down the negation of the following statements.

a. For all natural numbers, $3n \geq 2n - 1$.

b. There exists a real number x such that $x^2 = 4$.

Recall!

Active Recall: Negation involving the universal and existence quantifiers.

$\neg$ Universal statement =

$\neg$ Existence statement =

Your Turn!

Question 29

Write down the negation for the following statements below.

a. If n is a natural number, then $n + 1 > n$.

b. There exists an integer k such that $k^2 = k + 4$.

Question 30 Additional Question.

Write the negation for the following statements below.

a. If n is a natural number, then $6n^2 + 4n + 1$ is divisible by n.

b. There exist integers p and q such that $q \neq 0$ such that $\pi = \frac{p}{q}$.

Sub-Section: Disproving a Universal Statement

Context

- Consider the following questionable statement:

"All of the vehicles in the carpark are black."

- How can you disprove this statement?

- Basically, you have just provided a ______________________________.

Disproving Universal Statement

- We prove the *opposite* (negation) existence statement.
- We call this proof by _____________________________________.
  - Giving a counter example will be proving an opposite existence statement.

Question 31 Walkthrough.

Disprove the following statement: **For all positive integers m, if m is prime then $m^2 + 4$ is also prime.**

NOTE: We simply give ourselves a counter example to disprove a universal statement.

Recall!

Active Recall: Disproving a universal statement.

- To show that a "for all ..." statement is not true, it suffices to provide a ______________________.

Your Turn!

Question 32

Disprove the following statement:

$$\forall a, b \in R, \text{ if } a^2 = b^2 \text{ then } a = b.$$

Question 33 Additional Question.

Disprove the statement given below:

$$\text{For all positive integers } n,\ 2^n > n^2.$$

Sub-Section: Disproving an Existence Statement

Disproving Existence Statements

- We prove the *opposite* (negation) ______________________________.

Question 34 Walkthrough.

Disprove the following statement: There exists $n \in N$ such that $n^2 + 9n + 20$ is a prime number.

NOTE: To disprove their exist statement, you must show the opposite universal statement as true.

Recall!

Active Recall: Disproving an existence statement.

- To disprove a "there exists …" statement, we need to show ___________________________________.

Your Turn!

Question 35

Disprove the following statements:

There exists a real number x, such that $10 + 3x^2 = 3 + x^2$.

Question 36 Additional Question.

Disprove the following statement below:

There exists $n \in N$ so that $n^3 - 2n^2 + 5n + 4$ is divisible by $n + 2$.

Key Takeaways

- ☑ There exists a quantifier ∃.
- ☑ For all quantifiers ∀.
- ☑ To prove a there exists statement, simply give an example.
- ☑ To prove for all statements, you must prove for all values.
- ☑ To disprove a there exist statement, prove the opposite for all statements.
- ☑ To disprove a for-all statement, prove the opposite there exist statement.

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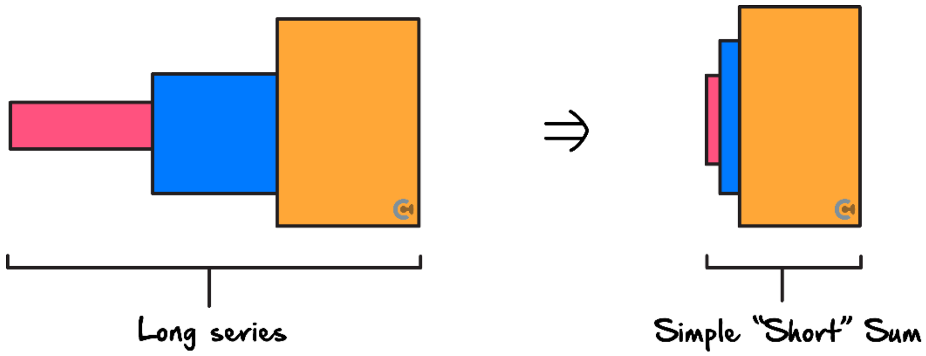
Section D: Other Proof Techniques

Sub-Section: Telescopic Cancelling

Telescopic Cancelling

- Telescoping series is a series (sum of terms) where all terms cancel out except for the ______________ ______________________________________.
- The name comes from the visualisation as shown below.



- Proofs of telescoping series can be done via ____ ____.

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Question 37 Walkthrough.

a. Using partial fractions, find the values of a and b such that:

$$\frac{3}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}$$

b. Hence, prove that:

$$\frac{3}{1 \times 2} + \frac{3}{2 \times 3} + \cdots + \frac{3}{n \times (n+1)} = \frac{3n}{n+1}$$

Recall!

Active Recall: Telescoping sums

- The key idea for telescoping sums is to ____.

Your Turn!

Question 38

a. Using partial fraction decomposition, find values of A and B such that:

$$\frac{1}{n^2+3n+2} \equiv \frac{A}{n+1} + \frac{B}{n+2}$$

b. Hence, prove that:

$$\sum_{n=1}^{k} \frac{1}{n^2+3n+2} = \frac{k}{2(k+2)}$$

Question 39 Additional Question.

By using the fact that $\frac{2(2k+1)}{k^2(k+1)^2} = \frac{2}{k^2} - \frac{2}{(k+1)^2}$, show that:

$$\frac{2\times 5}{2^2\times 3^2} + \cdots + \frac{2(2n+1)}{n^2(n+1)^2} = \frac{(n-1)(n+3)}{2(n+1)^2}$$

Sub-Section: Proof by Induction

Context

- Check out this question from the Specialist Mathematics Units 3 and 4 Sample Exam.

Question 3 (4 marks)
Prove by mathematical induction that the number $9^n - 5^n$ is divisible by 4 for all $n \in N$.

Proof by Induction

- We will be following the given steps:" ______________".
  - **Step T:** Test the statement for its first possible value.
  - **Step A:** Assume that the statement is true for $n = k$.
  - **Step P:** Prove that if the statement holds true for $n = k$, It also holds true for $n = k + 1$.
    - Add "By Assumption".
  - **Step E:** "By the principle of mathematical induction, the statement is true for a set of values."

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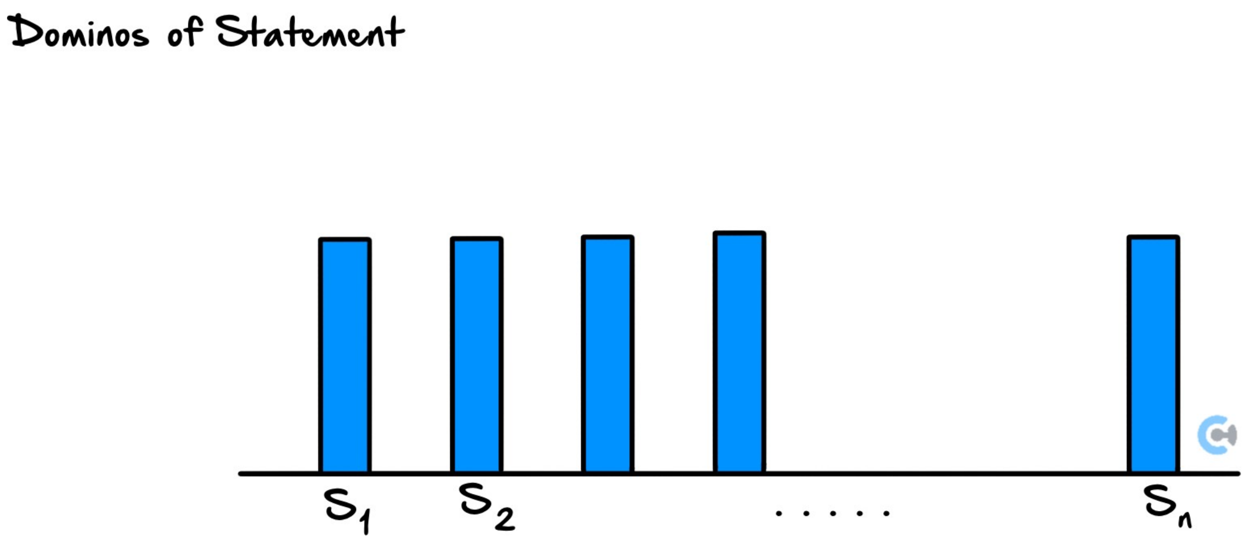
Question 40 Walkthrough.

Prove that for $n \in N$, $1 + 3 + 5 \cdots (2n - 1) = n^2$.

<u>Exploration</u>: **How do we visualise the idea of proof by induction?**

- Imagine we have ____________________ of statements we need to prove.
- Each domino is a statement with a specific value _________.



- In Step 1: We test/prove the first statement.
- In Steps A and P: We prove the relationship between a “previous” domino block, and the “next” one.
- If one domino falls, the next one also falls.
- Since the first one falls, all of them fall!

Recall!

Active Recall: Proofs by induction

- When doing a proof-by-induction question, we have four key steps:

1. ______________ the __________________________ holds.
2. ____________________ the statement holds for an $n =$ ______ in the required set of values for k.
3. ________________ that the statement holds for _______________.
4. ___________________ using the principle of ____ that the statement holds for all k.

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Your Turn!

Question 41

Prove that for all $n \in N$, we have:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

NOTE: $1 + 2 + \cdots + n$ can be written as:

$$\sum_{i=1}^{n} i$$

Question 42 Additional Question.

Prove by induction that for any natural number $n \geq 1$, $n < 2^n$.

Key Takeaways

- For telescoping series proof, always look for terms to cancel each other.
- For induction proof always use TAPE: Test for the first case, assume it's true for $n = k$, prove for $n = k + 1$ and explain at the end.

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Learning Objective: [2.2.1] – Direct proofs, proofs by contrapositive and contradiction.

Key Takeaways

- Direct proof is ____ the conditional statement.
- The contrapositive of a statement $A \to B$ is given by ___________________.
- Proof by contrapositive is proving the ____.
- Proof by contradiction requires first ____ and showing contradiction.

Learning Objective: [2.2.2] – Converse and the equivalent of a conditional statement.

Key Takeaways

- Converse statement of $A \to B$ is given by ________________.
- Equivalent statement is when _______________________________________ both have to be true.
- "If and only if" stands for _____________________________________.

Learning Objective: [2.2.3] – Proofs involving the universal and existence quantifiers.

Key Takeaways

- There exists a quantifier ∃.
- For all quantifiers ∀.
- To prove a there exists statement, simply ________________________________.
- To prove for all statements, you must prove for ____________________.
- To disprove a there exist statement, ____.
- To disprove a for-all statement, ____.

Learning Objective: [2.2.4] – Proofs by induction and telescoping series.

Key Takeaways

- For telescoping series proof, always look for terms to _________________________________.
- For induction proof always use TAPE: ____
____.

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