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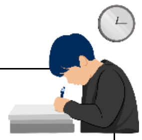
VCE Specialist Mathematics ½
Proofs II [2.2]
Homework Solutions

Homework Outline:



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Section A: Compulsory Questions

Sub-Section [2.2.1]: Direct and Indirect Proofs

Question 1

Prove the following statement using a direct proof: The sum of two even integers is always even.

Let the integers be m and n. Let $p, q \in \mathbb{Z}$, and since m and n are even we can write

$$\begin{aligned} m + n &= 2p + 2q \\ &= 2(p+q) \\ &= 2k \end{aligned}$$

which is even since $k \in \mathbb{Z}$.

Question 2

Prove the following statement using a proof by contrapositive: If n^3 is even, then n is even.

We will prove the contrapositive: n is odd $\implies$ n^3 is odd.
Let $n = 2k + 1$ where $k \in \mathbb{Z}$, then

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^2 + 6k^2 + 6) + 1 \\ &= 2m + 1 \end{aligned}$$

where $m \in \mathbb{Z}$ and so $n^3 = 2m + 1$ is odd.

Question 3

Prove the following statement using a proof by contradiction: $\sqrt{5} + \sqrt{3} < 4$.

Suppose for a contradiction that $\sqrt{5} + \sqrt{3} \geq 4$. Then

$$(\sqrt{5} + \sqrt{3})^2 \geq 16$$
$$5 + 3 + 2\sqrt{15} \geq 16$$
$$2\sqrt{15} \geq 8$$

but this statement is false since $2\sqrt{15} < 2\sqrt{16} = 8$. Thus we have a contradiction and therefore the assumption that $\sqrt{5} + \sqrt{3} \geq 4$ must be false and so $\sqrt{5} + \sqrt{3} < 4$.

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Sub-Section [2.2.2]: Proofs involving Converse and Equivalent Statements

Question 4

Write the converse of the following statements.

a. If it rains, the grass will be wet.

If the grass is wet, then it rains.

b. If a number is divisible by 2, then it is even.

If a number is even, then it is divisible by 2.

c. If a person is a teacher, then they enjoy teaching.

If a person enjoys teaching, then they are a teacher.

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Question 5

Prove the following statement: A number is odd, if and only if its square is odd.

($\Longrightarrow$) If a number n is odd, then $n = 2k + 1$ for some integer k.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is odd.
($\Longleftarrow$) If n^2 is odd, then it cannot be divisible by 2. This implies n cannot be divisible by 2 either, so n is odd.

Question 6

Prove the following statement: A four-digit number is divisible by 9, if and only if the sum of its digits is divisible by 9.

Let n have digits a, b, c, d from left to right.

$$\begin{aligned} n &= 1000a + 100b + 10c + d \\ &= a + b + c + d + 999a + 99b + 9c \\ &= a + b + c + d + 9(111a + 11b + c) \end{aligned}$$

Let $a + b + c + d = s$ and $111a + 11b + c = t$, then

$$n = s + 9t$$

($\Longrightarrow$) The second term is a multiple of 9 so for n to be a multiple of 9 we must also have s be a multiple of nine.
($\Longleftarrow$) s is a multiple of nine so

$$\begin{aligned} n &= 9u + 9t \\ &= 9(u + t) \end{aligned}$$

and so n is a multiple of nine.

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Sub-Section [2.2.3]: Proofs involving the Universal and Existence Quantifiers

Question 7

Write the following statements in terms of the universal (∀) and existential (∃) quantifiers.

a. All integers are even.

$\forall n \in \mathbb{Z}, n \text{ is even.}$

b. There exists a real number that is not a rational number.

$\exists x \in \mathbb{R}, x \notin \mathbb{Q}.$

c. For all real numbers x, if x is even, then x^2 is even.

$\forall x \in \mathbb{R}, x \text{ even} \implies x^2 \text{ is even}$

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Question 8

Negate the following statements involving universal and existential quantifiers.

a. $\forall n \in \mathbb{Z}, n^2 \geq 0$

$\nexists n \in \mathbb{Z}, n^2 < 0.$

b. $\exists x \in \mathbb{R}, x^2 = -1$

$\forall x \in \mathbb{R}, x^2 \neq -1.$

c. $\forall x \in \mathbb{R}, x + 1 > x$

$\exists x \in \mathbb{R}, x + 1 \leq x.$

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Question 9

Disprove the following statements by providing a counterexample.

a. Disprove that for all integers $n, n^2 + n + 1$ is always even.

Counterexample: Let $n = 1$. Then:

$$n^2 + n + 1 = 1^2 + 1 + 1 = 3,$$

which is odd. Hence, the statement is false.

b. Disprove that there exists an integer n such that, $n^2 = -1$.

The square of any integer n is non-negative, so $n^2 = -1$ is impossible. Hence, the statement is false.

c. Disprove that for all real numbers x, x^3 is odd.

Counterexample: $x = 2$ then $x^3 = 8$ is even.

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Sub-Section [2.2.4]: Telescoping Series and Proofs by Induction

Question 10

Simplify the following telescoping series using partial fraction decomposition and simplification.

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{n(n+1)}$$

Rewrite $\frac{1}{k(k+1)}$ using partial fractions:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Substitute into the series:

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} = \sum_{k=2}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

This is a telescoping series, so most terms cancel:

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Simplify to get:

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} = \frac{1}{2} - \frac{1}{n+1}.$$

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Question 11

Prove the following statement by induction:

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 1 \text{ for all integers } n \geq 1.$$

Base case $(n = 1)$:

$$1 = 2^1 - 1 = 1.$$

The base case holds.

Inductive step: Assume the formula is true for $n = m$, i.e.,

$$1 + 2 + 4 + \cdots + 2^{m-1} = 2^m - 1.$$

For $n = m + 1$, consider:

$$1 + 2 + 4 + \cdots + 2^{m-1} + 2^m.$$

Using the induction hypothesis:

$$\begin{aligned} 1 + 2 + 4 + \cdots + 2^{m-1} + 2^m &= (2^m - 1) + 2^m \\ &= 2 \cdot 2^m - 1 \\ &= 2^{m+1} - 1. \end{aligned}$$

Thus, the formula holds for $n = m + 1$.

Conclusion: By induction, the statement is true for all $n \geq 1$.

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Question 12

Prove the following statement by induction:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Base case $(n = 1)$:

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1.$$

The base case holds.

Inductive step: Assume the formula is true for $n = m$, i.e.,

$$1^3 + 2^3 + \cdots + m^3 = \left(\frac{m(m+1)}{2}\right)^2.$$

For $n = m + 1$, we now add $(m+1)^3$ to both sides:

$$\begin{aligned} 1^3 + 2^3 + \cdots + m^3 + (m+1)^3 &= \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3 \\ &= (m+1)^2\left(\frac{m^2}{4} + (m+1)\right) \\ &= (m+1)^2\left(\frac{1}{4}(m+2)^2\right) \\ &= \frac{(m+1)^2(m+2)^2}{4} \\ &= \left(\frac{(m+1)(m+2)}{2}\right)^2 \end{aligned}$$

Thus, the formula holds for $n = m + 1$.
By induction, the statement is true for all $n \geq 1$.

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Sub-Section: The 'Final Boss'

Question 13

a. Prove that $\sqrt{3}$ is irrational.

Proof by contradiction:
Assume $\sqrt{3}$ is rational. Then, it can be written as $\sqrt{3} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$. Squaring both sides:

$$3 = \frac{p^2}{q^2},$$
$$p^2 = 3q^2.$$

This shows p^2 is divisible by 3. By the properties of integers, this implies p is divisible by 3. Let $p = 3k$, where $k \in \mathbb{Z}$.
Substituting $p = 3k$ into $p^2 = 3q^2$:

$$(3k)^2 = 3q^2,$$
$$9k^2 = 3q^2,$$
$$q^2 = 3k^2.$$

This shows q^2 is divisible by 3, and hence q is divisible by 3.
Since both p and q are divisible by 3, this contradicts the assumption that $\gcd(p, q) = 1$.
Conclusion: The assumption that $\sqrt{3}$ is rational is false. Therefore, $\sqrt{3}$ is irrational.

b. Consider the statement:

$2^{3n} - 3^n$ is divisible by 5 for any integer n greater than or equal to 1.

Write the statement without any English words using the universal and existence quantifiers.

$$\forall n \in \mathbb{N}, \exists k \in \mathbb{Z},\ 2^{3n} - 3^n = 5k.$$

c. Prove the statement from **part b.** using mathematical induction.

Base case $(n = 1)$**:** For $n = 1$:

$$\begin{aligned} 2^{3(1)} - 3^1 &= 2^3 - 3, \\ &= 8 - 3, \\ &= 5. \end{aligned}$$

Since 5 is divisible by 5, the base case holds.
Inductive step: Assume the statement is true for $n = k$, i.e.,

$$2^{3k} - 3^k = 5m \quad \text{for some } m \in \mathbb{Z}.$$

We need to prove the statement holds for $n = k + 1$, i.e.,

$$2^{3(k+1)} - 3^{k+1} \text{ is divisible by 5.}$$

We have that

$$\begin{aligned} 2^{3(k+1)} - 3^{k+1} &= (2^{3k} \cdot 8) - (3^k \cdot 3) \\ &= 8(2^{3k} - 3^k) + 5 \cdot 3^k \\ &= 8(5m) + 5 \cdot 3^k \\ &= 5(8m + 1) \end{aligned}$$

which is divisble by 5.
Hence by the POMI $2^{3n} - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$.

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Section B: Supplementary Questions

Sub-Section [2.2.1]: Direct and Indirect Proofs

Question 14

Prove that all numbers of the form $n^3 - n$, where $n \in \mathbb{Z}$, are multiples of 6.

The expression is $n^3 - n$.

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1).$$

This represents the product of three consecutive integers.
One of these integers must be even, and another must be divisible by 3.
Therefore $n^3 - n$ is divisible by both 2 and 3 and thus it must be divisible by 6.

Question 15

Prove the following statement using a proof by contrapositive: If n^5 is odd, then n is odd.

We will prove the contrapositive: n is even $\implies$ n^5 is even.
Let $n = 2k$ where $k \in \mathbb{Z}$, then

$$\begin{aligned} n^5 &= (2k)^5 \\ &= 32k^5 \\ &= 2(16k^5) \\ &= 2m \end{aligned}$$

where $m \in \mathbb{Z}$ and so $n^5 = 2m$ is even.

Question 16

Prove the following statement using a proof by contradiction: $\sqrt{5}+\sqrt{7} < 5$.

Suppose for a contradiction that $\sqrt{5}+\sqrt{7} \geq 5$. Then

$$(\sqrt{5}+\sqrt{7})^2 \geq 25$$
$$5+7+2\sqrt{35} \geq 25$$
$$2\sqrt{35} \geq 13$$

but this statement is false since $2\sqrt{35} < 2\sqrt{36} = 12$. Thus we have a contradiction and therefore the assumption that $\sqrt{5}+\sqrt{7} \geq 5$ must be false and so $\sqrt{5}+\sqrt{7} < 5$.

Question 17

Prove that for $a, b > 0$, we have $a + b \geq \left(\frac{1}{a}+\frac{1}{b}\right)^{-1}$.

Assume the statement is false. That is, suppose:

$$a+b < \left(\frac{1}{a}+\frac{1}{b}\right)^{-1}.$$

Let:

$$H = \frac{1}{a}+\frac{1}{b}.$$

Then the assumption becomes:

$$a+b < \frac{1}{H}.$$

Multiply both sides by $H > 0$ (since $a, b > 0$):

$$H(a+b) < 1.$$

Now substituting $H = \frac{1}{a}+\frac{1}{b}$, we get:

$$\left(\frac{1}{a}+\frac{1}{b}\right)(a+b) < 1$$
$$2+\frac{b}{a}+\frac{a}{b} < 1$$
$$\frac{b}{a}+\frac{a}{b} < -1$$

However, for $a, b > 0$, $d\frac{b}{a}+\frac{a}{b} \geq 0$ which contradicts $\frac{b}{a}+\frac{a}{b} < -1$. We have a contradiction.

The assumption is false. Therefore, $a+b \geq \left(\frac{1}{a}+\frac{1}{b}\right)^{-1}$ is true.

Sub-Section [2.2.2]: Proofs involving Converse and Equivalent Statements

Question 18

Write the converse of the following statements.

a. If a person exercises regularly, they stay healthy.

If a person stays healthy, then they exercise regularly.

b. If a car is fuel-efficient, it saves money on gas.

If a car saves money on gas, then it is fuel-efficient.

c. If a student studies, they pass their exams.

If a student passes their exams, then they study.

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Question 19

Suppose $n \in \mathbb{Z}$. Prove that n is odd, if and only if $3n + 1$ is even.

($\Longrightarrow$) n odd then $3n+1$ is even.
Let $n = 2k+1$ for some $k \in \mathbb{Z}$, then

$$\begin{aligned} 3n+1 &= 3(2k+1)+1 \\ &= 6k+4 \\ &= 2(3k+2) \\ &= 2m \end{aligned}$$

where $m \in \mathbb{Z}$ and is therfore even.
($\Longleftarrow$) $3n+1$ even then n odd

$$\begin{aligned} 3n+1 &= 2k \\ 3n &= 2k-1 \\ n &= 2(k-n)-1 \\ &= 2m-1 \end{aligned}$$

where $m \in \mathbb{Z}$, therefore $3n+1$ is odd.

Question 20

Prove the following statement: $\frac{n(n+1)}{2}$ is a natural number, if and only if n is a natural number.

($\Longrightarrow$) $\dfrac{n(n+1)}{2} = k$ for some $k \in \mathbb{N}$. Then

$$n(n+1) = 2k$$

$n(n+1)$ is an even natural number and so n must be a natural number.
($\Longleftarrow$) If n is a natural number then $n(n+1)$ is the product of two consecutive natural numbers and is therefore even. So, for some $k \in \mathbb{Z}$,

$$\begin{aligned} n(n+1) &= 2k \\ \frac{n(n+1)}{2} &= k \end{aligned}$$

therefore, $\dfrac{n(n+1)}{2}$ is a natural number.

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Question 21

Prove the following statement: For any integer n, n is divisible by 3, if and only if the sum of its digits is divisible by 3.

Let n have k digits, from left to right these digits are $a_k, a_{k-1}, \ldots, a_2, a_1$. Then we can write

$$\begin{aligned} n &= a_1 + 10a_2 + 10^2 a_3 + \cdots + 10^{k-1} a_k \\ &= a_1 + a_2 + a_3 + \cdots + a_k + (9a_2 + 99a_3 + \cdots + (10^{k-1} - 1)a_k) \\ &= a_1 + a_2 + a_3 + \cdots + a_k + 3^2 \left(a_2 + 11a_3 + 111a_4 + \cdots + \frac{10^{k-1} - 1}{9} a_k \right) \end{aligned}$$

Let $a_1 + a_2 + a_3 + \cdots + a_k = d$ and $a_2 + 11a_3 + 111a_4 + \cdots + \dfrac{10^{k-1} - 1}{9} a_k = b$, then

$$n = d + 3^2 b$$

($\Longrightarrow$) The second term is a multiple of 3 so for n to be a multiple of 3 we must also have d be a multiple of three.

($\Longleftarrow$) d is a multiple of three so

$$n = 3c + 3^2 b = 3(c + 3b)$$

and so n is a multiple of three.

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Sub-Section [2.2.3]: Proofs involving the Universal and Existence Quantifiers

Question 22

Write the following statements in terms of the universal (∀) and existential (∃) quantifiers.

a. All positive integers are greater than zero.

$$\forall n \in \mathbb{Z}^+, n > 0.$$

b. There exists an integer that is a perfect square.

$$\exists n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n = k^2.$$

c. For all real numbers x, if $x > 0$, then $\frac{1}{x} > 0$.

$$\forall x \in \mathbb{R}, x > 0 \implies \frac{1}{x} > 0.$$

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Question 23

Negate the following statements involving universal and existential quantifiers.

a. $\forall n \in \mathbb{Z}, n + 0 = n$

$\exists n \in \mathbb{Z}, n + 0 \neq n.$

b. $\exists x \in \mathbb{R}, x^3 = 8$

$\forall x \in \mathbb{R}, x^3 \neq 8.$

c. $\forall x \in \mathbb{R}, x^2 \geq 0$

$\exists x \in \mathbb{R}, x^2 < 0.$

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Question 24

Disprove the following statements by providing a counterexample.

a. Disprove that for all integers $n, n^3 - n$ is always odd.

Counterexample: Let $n = 2$. Then:

$$n^3 - n = 2^3 - 2 = 8 - 2 = 6,$$

which is even. Hence, the statement is false.

b. Disprove that there exists an integer n such that, $2n + 1 = 0$.

The equation $2n+1 = 0$ implies $n = -\frac{1}{2}$, which is not an integer. Hence, the statement is false.

c. Disprove that for all real numbers $x, x^2 + x$ is greater than 1.

Counterexample: Let $x = -1$. Then:

$$x^2 + x = (-1)^2 + (-1) = 1 - 1 = 0,$$

which is not greater than 1. Hence, the statement is false.

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Question 25

Prove that:

$$\forall a, b \in \mathbb{R}^+ \cup \{0\}, \frac{a+b}{2} \geq \sqrt{ab}$$

Suppose that $\frac{a+b}{2} < \sqrt{ab}$ then

$$\frac{a^2+2ab+b^2}{4} < ab$$
$$\frac{a^2-2ab+b^2}{4} < 0$$
$$\left(\frac{a-b}{2}\right)^2 < 0$$

which is a contradiction since and real number squared is ≥ 0.

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Sub-Section [2.2.4]: Telescoping Series and Proofs by Induction

Question 26

Simplify the following telescoping series using partial fraction decomposition and simplification.

$$\sum_{k=1}^{n+1} \frac{1}{(k+1)(k+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)}$$

Rewrite $\frac{1}{(k+1)(k+2)}$ using partial fractions:

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}.$$

Substitute into the series:

$$\sum_{k=1}^{n+1} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{n+1} \left(\frac{1}{k+1} - \frac{1}{k+2} \right).$$

This is a telescoping series, so most terms cancel:

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right).$$

Simplify to get:

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} = \frac{1}{2} - \frac{1}{n+3}.$$

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Question 27

Prove the following statement by induction:

$$2 + 4 + 6 + \cdots + 2n = n(n+1) \text{ for all integers } n \geq 1.$$

Let $P(n)$ be the statement $2 + 4 + 6 + \cdots + 2n = n(n+1)$.
Base Case: $P(1) = 2 = 1(1+1) = 2$ holds.
Assume that $P(k)$ holds for some $k \in \mathbb{N}$. Then,

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2k + 2(k+1) &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \end{aligned}$$

which is the statement $P(k+1)$. Therefore by the POMI the statement $P(n)$ holds for all integers $n \geq 1$.

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Question 28

Prove the following statement by induction:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \text{ for all integers } n \geq 1.$$

Let $P(n)$ be the statement $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$.

Base Case: $P(1) = a = \dfrac{a(r-1)}{r-1} = a$ which is true.

Assume that $P(k)$ holds for some $k \in \mathbb{N}$. Then we have

$$\begin{aligned} a + ar + ar^2 + \cdots + ar^{k-1} + ar^k &= \frac{a(r^k - 1)}{r - 1} + ar^k \\ &= \frac{a(r^k - 1) + ar^k(r-1)}{r-1} \\ &= \frac{a(r^k - 1) + ar^{k+1} - ar^k}{r-1} \\ &= \frac{-a + ar^{k+1}}{r-1} \\ &= \frac{a(r^{k+1} - 1)}{r-1} \end{aligned}$$

so the statement $P(k+1)$ holds. Therefore, by the POMI the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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Question 29

Prove the following statement by induction:

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{n}{2}(2a + (n-1)d), \text{ for all integers } n \geq 1.$$

Let $P(n)$ be the statement $a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \frac{n}{2}(2a + (n-1)d)$.

Base Case: $P(1) = a = \frac{1}{2}(2a) = a$ which is true.

Assume that $P(k)$ hold for some $k \in \mathbb{N}$. Then we have

$$\begin{aligned} a + (a+d) + (a+2d) + \cdots + (a+(k-1)d) + (a+kd) &= \frac{k}{2}(2a + (k-1)d) + a + kd \\ &= ak + \frac{k}{2}(kd - d) + a + kd \\ &= (k+1)a + \frac{k}{2}(kd) + \frac{1}{2}kd \\ &= \frac{k+1}{2}(2a) + \frac{1}{2}kd(k+1) \\ &= \frac{k+1}{2}(2a + kd) \end{aligned}$$

which is equal to $P(k+1)$. Therefore by the POMI the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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