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VCE Specialist Mathematics ½

Proofs II [2.2]

Homework

Homework Outline:

Compulsory Questions	Pg 2-Pg 13
Supplementary Questions	Pg 14-Pg 26

Section A: Compulsory Questions

Sub-Section [2.2.1]: Direct and Indirect Proofs

Question 1

Prove the following statement using a direct proof: The sum of two even integers is always even.

Question 2

Prove the following statement using a proof by contrapositive: If n^3 is even, then n is even.

Question 3

Prove the following statement using a proof by contradiction: $\sqrt{5} + \sqrt{3} < 4$.

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Sub-Section [2.2.2]: Proofs involving Converse and Equivalent Statements

Question 4

Write the converse of the following statements.

a. If it rains, the grass will be wet.

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b. If a number is divisible by 2, then it is even.

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c. If a person is a teacher, then they enjoy teaching.

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Question 5

Prove the following statement: A number is odd, if and only if its square is odd.

Question 6

Prove the following statement: A four-digit number is divisible by 9, if and only if the sum of its digits is divisible by 9.

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Sub-Section [2.2.3]: Proofs involving the Universal and Existence Quantifiers

Question 7

Write the following statements in terms of the universal (∀) and existential (∃) quantifiers.

a. All integers are even.

b. There exists a real number that is not a rational number.

c. For all real numbers x, if x is even, then x^2 is even.

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Question 8

Negate the following statements involving universal and existential quantifiers.

a. $\forall n \in \mathbb{Z}, n^2 \geq 0$

b. $\exists x \in \mathbb{R}, x^2 = -1$

c. $\forall x \in \mathbb{R}, x + 1 > x$

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Question 9

Disprove the following statements by providing a counterexample.

a. Disprove that for all integers $n, n^2 + n + 1$ is always even.

b. Disprove that there exists an integer n such that, $n^2 = -1$.

c. Disprove that for all real numbers x, x^3 is odd.

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Sub-Section [2.2.4]: Telescoping Series and Proofs by Induction

Question 10

Simplify the following telescoping series using partial fraction decomposition and simplification.

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{n(n+1)}$$

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Question 11

Prove the following statement by induction:

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 1 \text{ for all integers } n \geq 1.$$

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Question 12

Prove the following statement by induction:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

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Sub-Section: The 'Final Boss'

Question 13

a. Prove that $\sqrt{3}$ is irrational.

b. Consider the statement:

$$2^{3n} - 3^n \text{ is divisible by } 5 \text{ for any integer } n \text{ greater than or equal to } 1.$$

Write the statement without any English words using the universal and existence quantifiers.

c. Prove the statement from **part b.** using mathematical induction.

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Section B: Supplementary Questions

Sub-Section [2.2.1]: Direct and Indirect Proofs

Question 14

Prove that all numbers of the form $n^3 - n$, where $n \in \mathbb{Z}$, are multiples of 6.

Question 15

Prove the following statement using a proof by contrapositive: If n^5 is odd, then n is odd.

Question 16

Prove the following statement using a proof by contradiction: $\sqrt{5} + \sqrt{7} < 5$.

Question 17

Prove that for $a, b > 0$, we have $a + b \geq \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}$.

Sub-Section [2.2.2]: Proofs involving Converse and Equivalent Statements

Question 18

Write the converse of the following statements.

a. If a person exercises regularly, they stay healthy.

b. If a car is fuel-efficient, it saves money on gas.

c. If a student studies, they pass their exams.

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Question 19

Suppose $n \in \mathbb{Z}$. Prove that n is odd, if and only if $3n + 1$ is even.

Question 20

Prove the following statement: $\frac{n(n+1)}{2}$ is a natural number, if and only if n is a natural number.

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Question 21

Prove the following statement: For any integer n, n is divisible by 3, if and only if the sum of its digits is divisible by 3.

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Sub-Section [2.2.3]: Proofs involving the Universal and Existence Quantifiers

Question 22

Write the following statements in terms of the universal (∀) and existential (∃) quantifiers.

a. All positive integers are greater than zero.

b. There exists an integer that is a perfect square.

c. For all real numbers x, if $x > 0$, then $\frac{1}{x} > 0$.

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Question 23

Negate the following statements involving universal and existential quantifiers.

a. $\forall n \in \mathbb{Z}, n + 0 = n$

b. $\exists x \in \mathbb{R}, x^3 = 8$

c. $\forall x \in \mathbb{R}, x^2 \geq 0$

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Question 24

Disprove the following statements by providing a counterexample.

a. Disprove that for all integers $n, n^3 - n$ is always odd.

b. Disprove that there exists an integer n such that, $2n + 1 = 0$.

c. Disprove that for all real numbers $x, x^2 + x$ is greater than 1.

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Question 25

Prove that:

$$\forall a, b \in \mathbb{R}^+ \cup \{0\}, \frac{a+b}{2} \geq \sqrt{ab}$$

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Sub-Section [2.2.4]: Telescoping Series and Proofs by Induction

Question 26

Simplify the following telescoping series using partial fraction decomposition and simplification.

$$\sum_{k=1}^{n+1} \frac{1}{(k+1)(k+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)}$$

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Question 27

Prove the following statement by induction:

$$2 + 4 + 6 + \cdots + 2n = n(n + 1) \text{ for all integers } n \geq 1.$$

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Question 28

Prove the following statement by induction:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1} \text{ for all integers } n \geq 1.$$

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Question 29

Prove the following statement by induction:

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{n}{2}(2a + (n - 1)d), \text{ for all integers } n \geq 1.$$

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