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VCE Specialist Mathematics ½
Proofs I [2.1]
Workbook

Outline:

Learning Objectives:

- ☐ SM12 [2.1.1] - Number sets
- ☐ SM12 [2.1.2] - Statements and operations
- ☐ SM12 [2.1.3] - Proving number sets

Section A: Sets and Notations

Context

- We will learn about proofs involving number sets later in this workbook.
- We need to first revise/preview the different number sets encountered in Specialist Mathematics.

What is a set?

Set

$$\{value_1, \quad value_2, \quad \ldots\}$$

- A set is simply a collection of multiple values.

Let's look at some questions together!

Question 1 Walkthrough.

Interpret the following set and simplify:

$$\{x | x^2 = 4\}$$

Your Turn!

Question 2

Interpret the following set and simplify:

$$\{x: x^2 - 4x + 3 = 0\}$$

Question 3 Additional Question.

Interpret the following set and simplify:

$$\{x: |x| = 8\}$$

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Definition

Interval Notation

- Between a and b.

$$(a, b)$$

- Between a and b inclusive.

$$[a, b]$$

- The interval of real numbers between a and b inclusive, but excluding c.

$$[a, b] \backslash \{c\}$$

Let's look at some questions together!

Question 4 Walkthrough.

Express each of the following subsets of R in interval notation.

a. $\left\{x: -\frac{\pi}{2} < x \leq 3\pi\right\}$

b. $\{x: x \neq -2\}$

c. $\{x: x < 3\} \cap \{x: x \neq -2\}$

Your Turn!

Question 5

Express each of the following subsets of R in interval notation.

a. $\{ x : x > 5 \}$

b. $\{x: -1 \leq x < 6\} \cap \{x \neq 5\}$

c. $\{x: x < 4\} \cap \{x: x \neq 0\}$

Question 6 Additional Question.

Express each of the following subsets of R in interval notation.

a. $\{x: x < 3\} \cap \{x: x \geq 2\}$

b. $\{x: x^2 \leq 4\} \cap \{x: x \neq 2\}$

How could we combine two sets?

Definition

<u>Operation of Sets</u>

- Negation: Everything **but**

$$\neg A \text{ or } A'$$

- Intersection

$$A \cap B$$

- Union

$$A \cup B$$

- Difference

$$A \backslash B$$

- Product

$$A \times B = \{(a, b), a \in A, b \in B\}$$

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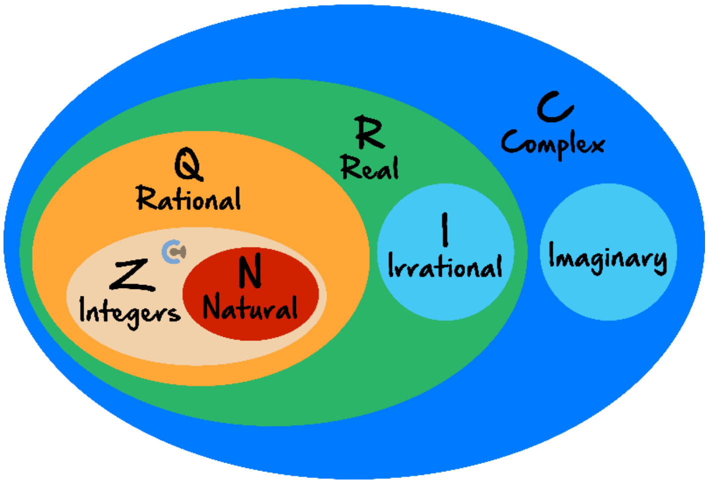
A quick recap of number sets!

<u>Number Sets</u>



$$N \subseteq Z \subseteq Q \subseteq R \subseteq C$$

- Naturals (N)

$$N =$$

- Integers (Z)

$$Z =$$

- Rationals (Q)

$$Q =$$

- Real (R)

$$R =$$

- Complex (C)

$$C =$$

Let's look at some questions together!

Question 7 Walkthrough.

For the following numbers, state all the number sets they are an element of.

a. $\sqrt{3}$

b. $\frac{1}{3}$

c. 3

d. $3 + i$

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Recall!

Active Recall: Determining Number Sets

- N is the set of ______________________________.
- Z is the set of ____.
- Q is the set of ________________________________.
- R is the set of _________________________.
- C is the set of ______________________________.

Your Turn!

Question 8

For the following numbers, state all the number sets they are an element of.

a. -4

b. $\frac{5}{12}$

c. $-4+i$

Question 9 Additional Question.

For the following numbers, state all the number sets they are an element of.

a. e

b. 10

c. i

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Sub-Section: Rationalising the Denominator

Context

- Take a look at the fraction below:

$$\frac{\sqrt{2}+3}{\sqrt{2}-1}$$

- How can we express this in the form of $\frac{a+\sqrt{b}}{c}$?
- VCAA loves these types of questions!
- EG: 2022 VCAA Specialist Exam 1.

The graph of $y = f(x)$ for $x \in \left[-\frac{\pi}{24}, \frac{\pi}{48}\right]$ is rotated about the x-axis to form a solid of revolution.

Find the volume of this solid. Give your answer in the form $\frac{\left(a-\sqrt{b}\right)\pi}{c}$, where $a, b, c \in R$. 3 marks

Rationalising the Denominator

- **Aim**: To remove surds in the denominator.

$$\frac{1}{a-\sqrt{b}} = \frac{1}{a-\sqrt{b}} \times \frac{a+\sqrt{b}}{a+\sqrt{b}}$$

- Multiply the top and bottom by the **conjugate of the denominator**.

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Discussion: How does this work? What does $(a - \sqrt{b}) \times (a + \sqrt{b})$ equal to?

Let's look at some questions together!

Question 10 Walkthrough.

Rationalise the denominator for the following:

$$\frac{1}{2 + \sqrt{7}}$$

Recall!

Active Recall

- To rationalise the denominator, we multiply the numerator and denominator by ______________________________ and expand.

Your Turn!

Question 11

Rationalise the denominator for the following.

a. $\frac{1+\sqrt{2}}{5-\sqrt{2}}$

b. $\frac{5-\sqrt{6}}{7+\sqrt{6}}$

Question 12 Additional Question.

Rationalise the denominator for the following. Write your answer in the form $\frac{a+b\sqrt{c}}{d}$.

$$\frac{3+\sqrt{2}}{6-\sqrt{2}}.$$

TIP: We will need this skill again when we look at complex numbers later!

Key Takeaways

- ☑ A set is a collection of multiple values.
- ☑ When using interval notation, square brackets are used to include the endpoints and round brackets are used to exclude the endpoints.
- ☑ Common number sets that are encountered in Specialist Mathematics are N (positive whole numbers), Z (integers), Q (rational numbers), R (real numbers) and C (complex numbers).
- ☑ To rationalise a fraction with a surd in the denominator, we can first multiply the numerator and denominator by the conjugate of the denominator and simplify by expanding.

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Section B: Statements

Sub-Section: Operation of Statements

What is a statement?

Context

- Before we start looking at proof questions, we need to learn about statements.

Statements and its Operations

- Statements can be very general.

$$A = \text{Doing SM12}$$

- Negation: Everything **but**

$$\neg A$$

- Intersection

$$A \wedge B$$

- Union

$$A \vee B$$

NOTE: We can use ∪ notation instead of ∨.

Space for Personal Notes

Let's look at some questions together!

Question 13 Walkthrough.

Consider the following statements:

$$A = \text{Doing SM12}$$

$$B = \text{Liking Maths}$$

Write down the statement for:

a. $\neg A$

b. $A \land B$

c. $A \lor B$

Recall!

Active Recall

- For two statements A and B, $\neg A$ is the statement ________, $A \wedge B$ is the statement ___________ and $A \vee B$ is the statement _____________.

Your Turn!

Question 14

Consider the following statements:

$$A = \text{Travelling to Argentina}$$

$$B = \text{Learning Spanish}$$

Write down the statement for:

a. $\neg A$

b. $A \wedge B$

c. $A \vee \neg B$

Question 15 Additional Question.

Consider the following statements:

$$A = \text{Taking Accounting } \frac{1}{2}$$

$$B = \text{Taking Accounting } \frac{3}{4}$$

Write down the statement for:

a. $\neg A$

b. $A \wedge \neg B$

c. $A \vee B$

Space for Personal Notes

Sub-Section: De Morgans's Law

Context

- We can do expanding and factorising involving the symbols $\neg$, $\wedge$ and $\vee$.

Exploration: De Morgan's Law

- Consider the statements:

$$A = \text{I Like Chocolate}$$

$$B = \text{I Like Strawberry}$$

- Consider $\neg(A \wedge B)$. Write it in words below!

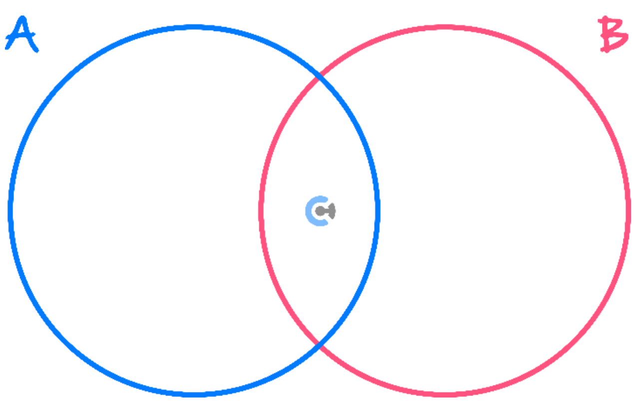
- Consider $\neg A \vee \neg B$. Write it in words below!

- Hence, we can say,

$$\neg(A \wedge B) = \neg A \vee \neg B$$

- We can also say,

$$\neg(A \vee B) = \neg A \wedge \neg B$$



NOTE: If you expand or factorise negation, the union and intersection flips!

De Morgan's Law

$$\neg(A \wedge B) = \neg A \vee \neg B$$

$$\neg(A \vee B) = \neg A \wedge \neg B$$

Let's look at some questions together!

Question 16 Walkthrough.

Using De Morgan's Law, write down the statement below in a different way.

Negation of "Sam doesn't like SM or MM."

Recall!

Active Recall

- $\neg(A \wedge B) =$ ____________________ and $\neg(A \vee B) =$ ____________________.

Your Turn!

Question 17

Using De Morgan's Law, write down the statement below in a different way.

Negation of "Bobby has an iPhone but doesn't have a Mac."

Question 18 Additional Question.

Using De Morgan's Law, write down the statement below in a different way.

Negation of "Bobby can speak Portuguese and English."

Key Takeaways

- We can do the following operations involving statements: $\neg A$ meaning "not A," $A \wedge B$ meaning "A or B," and $A \vee B$, meaning "A and B."
- De Morgan's Laws state that $\neg(A \wedge B) = \neg A \vee \neg B$ and $\neg(A \vee B) = \neg A \wedge \neg B$.
- We notice that in De Morgan's Laws, the intersections and unions flip.

Section C: Proving Number Sets

Sub-Section: Conditional Statements

Context

- Conditional statements will be the most important for us when doing a question involving proof.
- Check out this question from the sample Specialist Mathematics 3/4 exam. The "If ..., then ..." statement is an example of a conditional statement.

Question 1

Consider the following statement.

'For all integers n, if n^2 is even, then n is even.'

Which one of the following is the contrapositive of this statement?

A. For all integers n, if n^2 is odd, then n is odd.

B. There exists an integer n such that n^2 is even and n is odd.

C. There exists an integer n such that n is even and n^2 is odd.

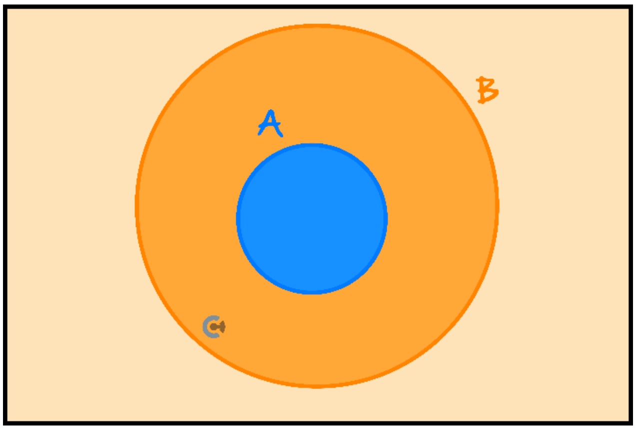
D. For all integers n, if n is odd, then n^2 is odd.

E. For all integers n, if n is even, then n^2 is even.

Discussion: If you are living, what must you be doing?

NOTE: Visually, imply is the same as "subset".

Conditional Statements



$$A \implies B$$

- Notation for "implies": $\implies$

$$\text{"}Hypothesis \implies Conclusion\text{"}$$

- A is a ____________ of B.

Let's look at some questions together!

Question 19 Walkthrough.

Write a conditional statement from each of the following:

All students at Contour Education are hard-working and diligent.

NOTE: Order Matters!

Recall!

Active Recall: What is the relationship between the hypothesis and the conclusion?

____

____

____

Your Turn!

Question 20

Write a conditional statement from each of the following:

Students doing SM12 are all brave.

Question 21 Additional Question.

Write a conditional statement from each of the following:

You must do well enough in English in order to study Law at University.

Sub-Section: Even and Odd Numbers

Context

- Exam 1 of Specialist Mathematics 3/4 usually has a proof question. Here is a question from the sample exam. After this workbook, you will be proving the statement!

Question 4 (3 marks)
Use proof by contradiction to prove that if n is odd, where $n \in N$, then $n^3 + 1$ is even.

Exploration: Even and Odd Numbers

- Write out a few **even** numbers:

- We can rewrite them in the following way:

- Do you notice a pattern?

$$Even =$$

- Now write out a few **odd** numbers:

- We can rewrite them in the following way:

- There is a similar pattern for odd numbers!

$$Odd =$$

Definition

Proofs involving Even and Odd numbers

- Simply show:

$$Even = 2k, k \in Z$$

$$Odd = 2k + 1, k \in Z$$

Let's look at some questions together!

Question 22 Walkthrough.

Prove that if n is even, then n^2 is also even.

NOTE: Start with the hypothesis by saying "let..."

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Recall!

<u>Active Recall:</u> Odd and Even Numbers

- An even number can be written in the form ______________________________.
- An odd number can be written in the form _______________________________.
- It is important to include $k \in Z$.

Your Turn!

Question 23

Prove that if n is odd, then n^2 is also odd.

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Question 24 Additional Question.

Prove that if two numbers a and b are odd, then their sum is even.

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Sub-Section: Divisibility

Context

- Even numbers are divisible by 2. How about divisibility by other numbers? Check out this divisibility question from the sample exam.

> **Question 3** (4 marks)
> Prove by mathematical induction that the number $9^n - 5^n$ is divisible by 4 for all $n \in N$.

Exploration:

- Write out a few numbers that are divisible by 3.

- Notice that these numbers can be written in the form:

- Therefore, any number that is divisible by 3 can be written in the form:

$$n =$$

Space for Personal Notes

Let's look at some questions together!

Question 25 Walkthrough.

Prove that if n is a multiple of 3, n^2 must be divisible by 9.

NOTE: For a number n to be divisible by a, n MUST be an integer multiple of a, that is you MUST be able to write $n = a \times \text{INTEGER}$.

Recall!

Active Recall

- If n is divisible by a, then n can be written as ____________________________.

Space for Personal Notes

Your Turn!

Question 26

Prove the following statement: If n is a multiple of 4, n^2 must be divisible by 8.

Question 27 Additional Question.

Prove the following statements:

a. If n is a multiple of 3, then $n^2 + 10n + 9$ is divisible by 3.

b. If n is a multiple of 3, then $n^2 - 1$ is not divisible by 3.

c. If n is divisible by 3 and m is divisible by 10, then $5n + 7m$ is divisible by 5.

Definition

Divisibility Proof - Splitting by Cases

- For divisibility of 2, we split the cases into ___________.

$$\textit{Case } 1: 2k, \qquad 2k+1$$

- For divisibility of n we split the cases into ____________.

$$\textit{Case } 1: nk, \qquad nk+1, \qquad nk+2 \quad \cdots \quad nk+(n-1)$$

Let's look at some questions together!

Question 28 Walkthrough.

Prove the following statement:

For any integer n, $(n+1)(n+2)$ is an even number (divisible by 2).

NOTE: Split the case into two because we are checking the divisibility of 2.

Recall!

<u>Active Recall:</u> Splitting by cases

- For a proof question where we want to show something is divisible by 3, we can split it into three cases where ____.

Your Turn!

Question 29

Prove the following statement: For any integer n, $(n+1)(n+2)n$ is divisible by 3.

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Question 30 Additional Question.

Prove the following statement below:

Prove that if $n = m^2$ for some integer m, then $n = 4k$ or $n = 4k + 1$ for some integer k.

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Sub-Section: Rational Numbers

Proving Rationals ($\mathbb{Q}$)

- Write out a few rational numbers:

- Write out a few numbers that are not rational numbers!

- Not all fractions are rational numbers! What is the general form for a rational number?

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Let's look at some questions together!

Question 31 Walkthrough.

Prove the following conditional statement:

If x is a rational number, $\frac{x+2}{x+3}$ is also a rational number.

Recall!

<u>Active Recall:</u> **Rational numbers**

- A rational number can be written in the form:

Question 32

Prove the following conditional statement:

If x and y are rational numbers, $\frac{xy}{x+y}$ is also a rational number.

Question 33 Additional Question.

Prove the following statement below:

If x and y are rational numbers and y is not equal to zero, then x/y is also a rational number.

Key Takeaways

- Adding, subtracting and multiplying integers result in integers.
- An even number can be expressed as $2k$ where $k \in Z$ and an odd number can be expressed as $2k+1$ where $k \in Z$.
- If n is divisible by a, then $n = a \times k$ where $k \in Z$, meaning $n = a \times \text{INTEGER}$.
- Where helpful, we can approach a proof involving divisibility by considering different cases.

Contour Check

Learning Objective: [2.1.1] – Number sets

Key Takeaways

- A set is a collection of multiple values.
- When using interval notation, square brackets are used to include the endpoints and round brackets are used to exclude the endpoints.
- Common number sets that are encountered in Specialist Mathematics are N (positive whole numbers), Z (integers), Q (rational numbers), R (real numbers) and C (complex numbers).
- To rationalise a fraction with a surd in the denominator, we can first multiply the numerator and denominator by the conjugate of the denominator and simplify by expanding.

Learning Objective: [2.1.2] – Statements and operations

Key Takeaways

- We can do the following operations involving statements: $\neg A$ meaning "not A", $A \wedge B$ meaning "A or B", and $A \vee B$, meaning "A and B".
- De Morgan's Laws state that $\neg(A \wedge B) = \neg A \vee \neg B$ and $\neg(A \vee B) = \neg A \wedge \neg B$.
- We notice that in De Morgan's Laws, the intersection/union flips.

Learning Objective: [2.1.3] – Proving number sets

Key Takeaways

- Adding, subtracting and multiplying integers result in integers.
- An even number can be expressed as $2k$ where $k \in Z$ and an odd number can be expressed as $2k+1$ where $k \in Z$.
- If n is divisible by a, then $n = a \times k$ where $k \in Z$, meaning $n = a \times \text{INTEGER}$.
- Where helpful, we can approach a proof involving divisibility by considering different cases.

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