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VCE Specialist Mathematics ½
Proofs I [2.1]
Homework Solutions

Homework Outline:

Compulsory Questions	Pg 2-Pg 15
Supplementary Questions	Pg 16-Pg 31

Section A: Compulsory Questions

Sub-Section [2.1.1]: Number Sets

Question 1

State all the number sets that the following are an element of:

a. $\sqrt{5}$

$\mathbb{R}, \mathbb{C}$

b. 5

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

c. $\pi + i$

$\mathbb{C}$

d. $-\frac{3}{7}$

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$

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Question 2

Express each of the following subsets of $\mathbb{R}$ in interval notation:

a. $\{x: x > -4\}$

$(-4, \infty)$

b. $\{x: x \neq 1\} \cap \{x: x \geq -3\}$

$[-3, \infty)\backslash\{1\}$

c. $\{x: x \neq 0\} \cup \{x: x \leq 2\}$

$(-\infty, 2\}\backslash\{0\}$

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Question 3

Rationalise the denominator, then simplify the following expressions:

a. $\frac{5}{\sqrt{3}}$

$$\frac{5\sqrt{3}}{3}$$

b. $\frac{2}{1+\sqrt{2}}$

$$2\sqrt{2}-2$$

c. $\frac{\sqrt{8}+3}{2+\sqrt{5}}$

$$(2\sqrt{2}+3)(\sqrt{5}-2) = -4\sqrt{2}+3\sqrt{5}+2\sqrt{10}-6$$

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Sub-Section [2.1.2]: Operations on Statements

Question 4

Consider the following statements:

$$A = \text{It is raining.}$$
$$B = \text{I go out running.}$$

Write down the following:

a. $A \land B$

It is raining and I go out running.

b. $\neg A$

It is not raining.

c. $\neg A \lor B$

It is not raining or I don't go running.

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Question 5

Use De Morgan's Law to write down the negation of the following statements:

a. The cake is delicious and the coffee is hot.

The cake is not delicious or the coffee is cold.

b. It is raining or the sun is shining.

It is not raining and the sun is not shining.

c. The computer is cheap and slow.

The computer is expensive, or it is fast.

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Question 6

Write the following as conditional statements:

a. Customers that spend over $500 get a voucher.

If a customer spends more than $500, then they get a voucher.

b. All students who study hard pass their exams.

If a student studies hard, then they will pass their exams.

c. People who go to the gym grow their muscles.

If someone goes to the gym, then their muscles will grow.

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Sub-Section [2.1.3]: Proofs Involving Even and Odd Numbers

Question 7

For an integer n, show that if n is odd then n^2 is odd.

n is odd and so $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2m + 1 \end{aligned}$$

where $m \in \mathbb{Z}$, and so $n^2 = 2m + 1$ is odd.

Question 8

Show that $(2n + 5)^2 - (2n - 1)$ is always even for any $n \in \mathbb{Z}$.

$$\begin{aligned} (2n+5)^2 - (2n-1) &= 4n^2 + 20n + 25 - 2n + 1 \\ &= 4n^2 + 18n + 24 \\ &= 2(2n^2 + 9n + 12) \\ &= 2k \end{aligned}$$

where $k \in \mathbb{Z}$, and so $(2n+5)^2 - (2n-1)$ is even for any $n \in \mathbb{Z}$.

Question 9

Show that $n^2 + 5n + 6$ is even for all $n \in \mathbb{N}$.

Observe that

$$n^2 + 5n + 6 = (n+2)(n+3)$$

one of $n+2$ and $n+3$ is even and the other is odd.
We know that even $\times$ odd is always even.
Therefore, $n^2 + 4n + 3$ is even for all $n \in \mathbb{N}$

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Sub-Section [2.1.4]: Proofs Involving Divisibility

Question 10

Show that if n is divisible by 5, then n^2 is divisible by 5 for any $n \in \mathbb{N}$.

We can write $n = 5k$ for some $k \in \mathbb{N}$ and so

$$\begin{aligned} n^2 &= (5k)^2 \\ &= 25k^2 \\ &= 5(5k^2) \\ &= 5m \end{aligned}$$

where $m \in \mathbb{N}$ and so n^2 is divisible by 5.

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Question 11

Show that if n is divisible by 2 and m is divisible by 3, then $5n + 10m$ is even for all $n, m \in \mathbb{N}$.

Let $n = 2k$ where $k \in \mathbb{N}$ and $m = 3\ell$ where $\ell \in \mathbb{N}$. Then

$$\begin{aligned} 5n + 10m &= 10k + 30\ell \\ &= 2(5k + 15\ell) \\ &= 2r \end{aligned}$$

where $r \in \mathbb{N}$, and so $5n + 10m$ is even.

Question 12

Show by cases that for $n \in \mathbb{Z}$, if n is not divisible by 3, then n^2 is not divisible by 9.

If n is not divisible by 3, then n must leave a remainder of either 1 or 2 when divided by 3. Thus, we analyze two cases:

Case 1: $n = 3k + 1$ for some $k \in \mathbb{Z}$ In this case:

$$\begin{aligned} n^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(k^2 + 2k) + 1 \\ &= 3m + 1 \end{aligned}$$

where $m \in \mathbb{Z}$, which is not divisible by 3 and therefore not divisible by 9.

Case 2: $n = 3k + 2$ for some $k \in \mathbb{Z}$ In this case:

$$\begin{aligned} n^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \\ &= 3m + 1 \end{aligned}$$

where $m \in \mathbb{Z}$, which is not divisible by 3 and therefore not divisible by 9.

Conclusion: In both cases, n^2 is not divisible by 9 if n is not divisible by 3.

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Sub-Section [2.1.5]: Proofs Involving Rational Numbers

Question 13

Show that if $\sqrt{x}$ is rational, then x is rational for any $x \in \mathbb{R}$.

$\sqrt{x} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$x = \frac{p^2}{q^2} = \frac{m}{n}$$

where $m, n \in \mathbb{Z}$ and $n \neq 0$, and so $x \in \mathbb{Q}$.

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Question 14

Show that if both x and y are rational, then $x^2 - y^3$ is rational.

Assume $x = \frac{p}{q}$ and $y = \frac{r}{s}$, where p, q, r, s are integers, and $q, s \neq 0$.

$$x^2 - y^3 = \left(\frac{p}{q}\right)^2 - \left(\frac{r}{s}\right)^3 = \frac{p^2}{q^2} - \frac{r^3}{s^3}$$
$$= \frac{p^2 s^3 - r^3 q^2}{q^2 s^3}.$$

Since $p^2 s^3 - r^3 q^2$ is an integer and $q^2 s^3 \neq 0$, the result is rational.

Question 15

Find possible values for $x, y \in \mathbb{R}$ such that x and y are irrational, but x^y is rational.

Note that $(\sqrt{2})^2 = 2$ which we can write as $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$

So we can have $x = (\sqrt{2})^{\sqrt{2}}$ and $y = \sqrt{2}$

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Sub-Section: The 'Final Boss'

Question 16

Consider the statements:

$$A : n \text{ is an even integer.}$$
$$B : n < 0$$

a. Write the statement $A \wedge \neg B$ in words.

n is an even integer that is greater than or equal to zero.

b. Prove that if $A \wedge \neg B$ is true, then $n^2 + 3n + 1$ is odd.

Let $n = 2k$ where $k \in \mathbb{Z}_{\geq 0}$. Then,

$$\begin{aligned} n^2 + 3n + 1 &= 4k^2 + 6k + 1 \\ &= 2(2k^2 + 3k) + 1 \\ &= 2p + 1 \end{aligned}$$

where $p \in \mathbb{Z}_{\geq 0}$, and so $n^2 + 3n + 1$ is odd.

c. Prove that the product of any two odd integers is odd.

Let $m = 2k + 1$ and $n = 2j + 1$ be odd integers, where $k, j \in \mathbb{Z}$. Compute the product:

$$mn = (2k + 1)(2j + 1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1.$$

Since $2kj + k + j \in \mathbb{Z}$, the product is of the form $2p + 1$, where $p \in \mathbb{Z}$. Hence, mn is odd.

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Section B: Supplementary Questions

Sub-Section [2.1.1]: Number Sets

Question 17

State all the number sets that the following are an element of:

a. $\sqrt{5}$

$\mathbb{R}, \mathbb{C}$

b. 5

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

c. $\pi + i$

$\mathbb{C}$

d. $-\frac{3}{7}$

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$

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Question 18

Express each of the following subsets of $\mathbb{R}$ in interval notation.

a. $\{x: x > 3\}$

$(3, \infty)$

b. $\{x: -8 < x < 1\} \cap \{x: x \geq -3\}$

$[-3,1)$

c. $\{x: x \neq 1\} \cup \{x: x \leq 5\}$

$(-\infty, 5]\backslash\{1\}$

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Question 19

Rationalise the denominator and then simplify the following expressions:

a. $\frac{2}{\sqrt{3}}$

$$\frac{2\sqrt{3}}{3}$$

b. $\frac{5}{3+\sqrt{2}}$

$$\frac{15-5\sqrt{2}}{7}$$

c. $\frac{\sqrt{30}+7}{4+\sqrt{7}}$

$$\frac{28-7\sqrt{7}+4\sqrt{30}-\sqrt{210}}{9}$$

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Question 20

Rationalise the denominator of the following expression and simplify:

$$\frac{x+\sqrt{y}}{\sqrt{a}+\sqrt{b}},$$

where $x, y, a, b > 0$ and $a \neq b$.

$$\frac{(x+\sqrt{y})(\sqrt{a}-\sqrt{b})}{a^2-b^2} = \frac{(\sqrt{a}-\sqrt{b})x-\sqrt{ay}-\sqrt{by}}{a^2-b^2}$$

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Sub-Section [2.1.2]: Operations on Statements

Question 21

Consider the following statements:

$$A = \text{It is hot outside.}$$
$$B = \text{I go to the beach.}$$

Write down the following:

a. $A \wedge B$

It is hot outside and I go to the beach.

b. $\neg B$

I do not go to the beach.

c. $\neg A \vee \neg B$

It is not hot outside, or I do not go to the beach.

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Question 22

Use De Morgan's Law to write down the negation of the following statements:

a. The movie is entertaining and the popcorn is tasty.

The movie is not entertaining or the popcorn is not tasty.

b. The traffic is light or the weather is clear.

The traffic is not light and the weather is not clear.

c. The phone is affordable and has a good camera.

The phone is not affordable or it does not have a good camera.

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Question 23

Write the following as conditional statements:

a. People who recycle help the environment.

If a person recycles, then they help the environment.

b. Employees who work overtime earn extra pay.

If an employee works overtime, then they earn extra pay.

c. Athletes who practice regularly improve their performance.

If an athlete practices regularly, then they improve their performance.

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Question 24

Simplify the following logical expression using De Morgan's Laws:

$$\neg((P \wedge Q) \vee (\neg R \wedge S)).$$

Give your answer in the form

$$(A \vee B) \wedge (C \vee D)$$

Using De Morgan's Law:

$$\neg\big((P \wedge Q) \vee (\neg R \wedge S)\big) = \neg(P \wedge Q) \wedge \neg(\neg R \wedge S).$$

Next, apply De Morgan's Laws to each part:

$$\neg(P \wedge Q) = \neg P \vee \neg Q,$$

and:

$$\neg(\neg R \wedge S) = \neg(\neg R) \vee \neg S = R \vee \neg S.$$

Substitute these into the expression:

$$\neg\big((P \wedge Q) \vee (\neg R \wedge S)\big) = (\neg P \vee \neg Q) \wedge (R \vee \neg S).$$

So our expression is:

$$(\neg P \vee \neg Q) \wedge (R \vee \neg S).$$

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Sub-Section [2.1.3]: Proofs Involving Even and Odd Numbers

Question 25

For an integer n, show that if n is even then n^3 is even.

n is even and so $n = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3) \\ &= 2m \end{aligned}$$

where $m \in \mathbb{Z}$, and so $n^3 = 2m$ is even.

Question 26

Show that, $(4n + 2)^2 - (2n - 1)$ is always odd for any $n \in \mathbb{Z}$.

$$\begin{aligned} (4n+2)^2 - (2n-1) &= 16n^2 + 16n + 4 - 2n + 1 \\ &= 2(16n^2 + 7n + 2) + 1 \\ &= 2k + 1 \end{aligned}$$

where $k \in \mathbb{Z}$, and so $(4n+2)^2 - (2n-1)$ is odd for any $n \in \mathbb{Z}$.

Question 27

Show that $n^2 + 7n + 10$ is even for all $n \in \mathbb{N}$.

Observe that

$$n^2 + 7n + 10 = (n+2)(n+5)$$

one of $n+2$ and $n+5$ is even and the other is odd.
We know that even $\times$ odd is always even.
Therefore, $n^2 + 7n + 10$ is even for all $n \in \mathbb{N}$

Question 28

Prove that the product of any two odd integers minus the sum of the same two integers is always even.

Let the two odd integers be $a = 2m+1$ and $b = 2n+1$, where $m, n \in \mathbb{Z}$.
We have

$$\begin{aligned} a \cdot b &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

so ab is odd.
Next, compute the sum of a and b:

$$\begin{aligned} a + b &= (2m+1) + (2n+1) \\ &= 2m + 2n + 2 \\ &= 2(m + n + 1) \end{aligned}$$

so $a + b$ is even.
Therefore, it must be that for $k, \ell \in \mathbb{Z}$

$$\begin{aligned} ab - (a+b) &= (2k+1) - 2\ell \\ &= 2(k - \ell) + 1 \end{aligned}$$

which is odd.

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Sub-Section [2.1.4]: Proofs Involving Divisibility

Question 29

Show that if n is divisible by 7, then n^2 is divisible by 7 for any $n \in \mathbb{N}$.

We can write $n = 7k$ for some $k \in \mathbb{N}$ and so

$$\begin{aligned} n^2 &= (7k)^2 \\ &= 49k^2 \\ &= 7(7k^2) \\ &= 7m \end{aligned}$$

where $m \in \mathbb{N}$ and so n^2 is divisible by 7.

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Question 30

Show that if n is divisible by 2 and m is divisible by 3, then $3n + 4m$ is divisible by 3 for all $n, m \in \mathbb{N}$.

Let $n = 2k$ where $k \in \mathbb{N}$ and $m = 3\ell$ where $\ell \in \mathbb{N}$. Then

$$\begin{aligned} 3n + 4m &= 6k + 12\ell \\ &= 3(2k + 4\ell) \\ &= 3r \end{aligned}$$

where $r \in \mathbb{N}$, and so $3n + 4m$ is divisible by 3.

Question 31

Prove that if m and n are even integers, then $m^2 + n^2$ and $m^2 - n^2$ are both divisible by 4.

Let $m = 2a$ and $n = 2b$ for $a, b \in \mathbb{Z}$. Then we have

$$\begin{aligned} m^2 + n^2 &= 4a^2 + 4b^2 \\ &= 4(a^2 + b^2) \end{aligned}$$

and so is divisible by four.
We also have

$$\begin{aligned} m^2 - n^2 &= 4a^2 - 4b^2 \\ &= 4(a^2 - b^2) \end{aligned}$$

and so is divisible by four.

Question 32

Prove that the sum of any two consecutive odd numbers is divisible by 4.

Let the two odd numbers be $a = 2n + 1$ and $b = 2n + 3$ for some $n \in \mathbb{Z}$. Then

$$\begin{aligned} a + b &= 2n + 1 + 2n + 3 \\ &= 4n + 4 \\ &= 4(n + 1) \end{aligned}$$

and so must be divisible by four.

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Sub-Section [2.1.5]: Proofs Involving Rational Numbers

Question 33

Show that if $\sqrt[3]{x}$ is rational, then x is rational for any $x \in \mathbb{R}$.

$\sqrt[3]{x} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$\begin{aligned} x &= \frac{p^3}{q^3} \\ &= \frac{m}{n} \end{aligned}$$

where $m, n \in \mathbb{Z}$ and $n \neq 0$, and so $x \in \mathbb{Q}$.

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Question 34

Show that if both x and y are rational, then $x^2 + y^2$ is rational.

Assume $x = \frac{p}{q}$ and $y = \frac{r}{s}$, where p, q, r, s are integers, and $q, s \neq 0$.

$$x^2 + y^2 = \frac{p^2}{q^2} + \frac{r^2}{s^2} = \frac{p^2s^2 + r^2q^2}{q^2s^2}$$

Since $p^2s^2 + r^2q^2$ is an integer and $q^2s^2 \neq 0$, the result is rational.

Question 35

Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is also rational.

Assume x is rational and $x \neq 0$. By the definition of rational numbers, we can write:

$$x = \frac{p}{q},$$

where $p, q \in \mathbb{Z}$ and $q \neq 0$.
Now,

$$\frac{1}{x} = \frac{1}{\frac{p}{q}} = \frac{q}{p}.$$

Since $p \neq 0$ (as $x \neq 0$) and both p and q are integers, $\frac{q}{p}$ is a fraction with an integer numerator and nonzero integer denominator. Thus, $\frac{1}{x} = \frac{q}{p}$ is rational.

Question 36

Prove that if x and y are rational and $x, y \neq 0$ then

$$\frac{(x-2y)^5 + x^2 + 3y}{x^2 + 2y^2}$$

is rational.

Let $a = x - 2y \in \mathbb{Q}$ then $a^5 = b \in \mathbb{Q}$, since the sum and product of rational numbers is rational.
Also $c = x^2 + 3y \in \mathbb{Q}$ and $d = x^2 + 2y^2 \in \mathbb{Q}$, since the sum and product of rational numbers is rational.
Therefore,

$$\frac{(x-2y)^5 + x^2 + 3y}{x^2 + 2y^2} = \frac{b+c}{d} \in \mathbb{Q}$$

since $b, c, d \in \mathbb{Q}$ and $d \neq 0$.

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