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## VCE Specialist Mathematics ½

### Proofs I [2.1]

### Homework Solutions

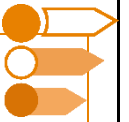
#### Homework Outline:

Compulsory Questions	Pg 2-Pg 15
Supplementary Questions	Pg 16-Pg 31



## Section A: Compulsory Questions

### Sub-Section [2.1.1]: Number Sets



#### Question 1



State all the number sets that the following are an element of:

a.  $\sqrt{5}$

\_\_\_\_\_  $\mathbb{R}, \mathbb{C}$  \_\_\_\_\_

b. 5

\_\_\_\_\_  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  \_\_\_\_\_

c.  $\pi + i$

\_\_\_\_\_  $\mathbb{C}$  \_\_\_\_\_

d.  $-\frac{3}{7}$

\_\_\_\_\_  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$  \_\_\_\_\_

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**Question 2**

Express each of the following subsets of  $\mathbb{R}$  in interval notation:

a.  $\{x: x > -4\}$

$$(-4, \infty)$$

b.  $\{x: x \neq 1\} \cap \{x: x \geq -3\}$

$$[-3, \infty) \setminus \{1\}$$

c.  $\{x: x \neq 0\} \cup \{x: x \leq 2\}$

$$(-\infty, 2] \setminus \{0\}$$

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**Question 3**

Rationalise the denominator, then simplify the following expressions:

a.  $\frac{5}{\sqrt{3}}$

$$\frac{5\sqrt{3}}{3}$$

b.  $\frac{2}{1+\sqrt{2}}$

$$2\sqrt{2} - 2$$

c.  $\frac{\sqrt{8}+3}{2+\sqrt{5}}$

$$(2\sqrt{2} + 3)(\sqrt{5} - 2) = -4\sqrt{2} + 3\sqrt{5} + 2\sqrt{10} - 6$$

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## Sub-Section [2.1.2]: Operations on Statements

### Question 4



Consider the following statements:

$A$  = It is raining.  
 $B$  = I go out running.

Write down the following:

a.  $A \wedge B$

It is raining and I go out running.

b.  $\neg A$

It is not raining.

c.  $\neg A \vee B$

It is not raining or I don't go running.

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**Question 5**

Use De Morgan's Law to write down the negation of the following statements:

- a. The cake is delicious and the coffee is hot.

\_\_\_\_\_

\_\_\_\_\_ The cake is not delicious or the coffee is cold. \_\_\_\_\_

\_\_\_\_\_

- b. It is raining or the sun is shining.

\_\_\_\_\_

\_\_\_\_\_ It is not raining and the sun is not shining. \_\_\_\_\_

\_\_\_\_\_

- c. The computer is cheap and slow.

\_\_\_\_\_

\_\_\_\_\_ The computer is expensive, or it is fast. \_\_\_\_\_

\_\_\_\_\_

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**Question 6**

Write the following as conditional statements:

- a. Customers that spend over \$500 get a voucher.

If a customer spends more than \$500, then they get a voucher.

- b. All students who study hard pass their exams.

If a student studies hard, then they will pass their exams.

- c. People who go to the gym grow their muscles.

If someone goes to the gym, then their muscles will grow.

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## Sub-Section [2.1.3]: Proofs Involving Even and Odd Numbers

### Question 7



For an integer  $n$ , show that if  $n$  is odd then  $n^2$  is odd.

$n$  is odd and so  $n = 2k + 1$  for some  $k \in \mathbb{Z}$ . Therefore,

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2m + 1 \end{aligned}$$

where  $m \in \mathbb{Z}$ , and so  $n^2 = 2m + 1$  is odd.

### Question 8



Show that  $(2n + 5)^2 - (2n - 1)$  is always even for any  $n \in \mathbb{Z}$ .

$$\begin{aligned} (2n + 5)^2 - (2n - 1) &= 4n^2 + 20n + 25 - 2n + 1 \\ &= 4n^2 + 18n + 24 \\ &= 2(2n^2 + 9n + 12) \\ &= 2k \end{aligned}$$

where  $k \in \mathbb{Z}$ , and so  $(2n + 5)^2 - (2n - 1)$  is even for any  $n \in \mathbb{Z}$ .





Question 9

Show that  $n^2 + 5n + 6$  is even for all  $n \in \mathbb{N}$ .

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Observe that

$$n^2 + 5n + 6 = (n + 2)(n + 3)$$

one of  $n + 2$  and  $n + 3$  is even and the other is odd.

We know that even  $\times$  odd is always even.

Therefore,  $n^2 + 5n + 6$  is even for all  $n \in \mathbb{N}$

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### Sub-Section [2.1.4]: Proofs Involving Divisibility



### Question 10

Show that if  $n$  is divisible by 5, then  $n^2$  is divisible by 5 for any  $n \in \mathbb{N}$ .

We can write  $n = 5k$  for some  $k \in \mathbb{N}$  and so

$$\begin{aligned} n^2 &= (5k)^2 \\ &= 25k^2 \\ &= 5(5k^2) \\ &= 5m \end{aligned}$$

where  $m \in \mathbb{N}$  and so  $n^2$  is divisible by 5.

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### Question 11

Show that if  $n$  is divisible by 2 and  $m$  is divisible by 3, then  $5n + 10m$  is even for all  $n, m \in \mathbb{N}$ .

Let  $n = 2k$  where  $k \in \mathbb{N}$  and  $m = 3\ell$  where  $\ell \in \mathbb{N}$ . Then

$$\begin{aligned} 5n + 10m &= 10k + 30\ell \\ &= 2(5k + 15\ell) \\ &= 2r \end{aligned}$$

where  $r \in \mathbb{N}$ , and so  $5n + 10m$  is even.

### Question 12



Show by cases that for  $n \in \mathbb{Z}$ , if  $n$  is not divisible by 3, then  $n^2$  is not divisible by 9.

If  $n$  is not divisible by 3, then  $n$  must leave a remainder of either 1 or 2 when divided by 3. Thus, we analyze two cases:

**Case 1:**  $n = 3k + 1$  for some  $k \in \mathbb{Z}$  In this case:

$$\begin{aligned} n^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(k^2 + 2k) + 1 \\ &= 3m + 1 \end{aligned}$$

where  $m \in \mathbb{Z}$ , which is not divisible by 3 and therefore not divisible by 9.

**Case 2:**  $n = 3k + 2$  for some  $k \in \mathbb{Z}$  In this case:

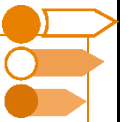
$$\begin{aligned} n^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \\ &= 3m + 1 \end{aligned}$$

where  $m \in \mathbb{Z}$ , which is not divisible by 3 and therefore not divisible by 9.

**Conclusion:** In both cases,  $n^2$  is not divisible by 9 if  $n$  is not divisible by 3.

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Sub-Section [2.1.5]: Proofs Involving Rational Numbers



Question 13



Show that if  $\sqrt{x}$  is rational, then  $x$  is rational for any  $x \in \mathbb{R}$ .

$\sqrt{x} = \frac{p}{q}$  for  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . Then

$$x = \frac{p^2}{q^2} = \frac{m}{n}$$

where  $m, n \in \mathbb{Z}$  and  $n \neq 0$ , and so  $x \in \mathbb{Q}$ .

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**Question 14**


Show that if both  $x$  and  $y$  are rational, then  $x^2 - y^3$  is rational.

Assume  $x = \frac{p}{q}$  and  $y = \frac{r}{s}$ , where  $p, q, r, s$  are integers, and  $q, s \neq 0$ .

$$\begin{aligned} x^2 - y^3 &= \left(\frac{p}{q}\right)^2 - \left(\frac{r}{s}\right)^3 = \frac{p^2}{q^2} - \frac{r^3}{s^3} \\ &= \frac{p^2 s^3 - r^3 q^2}{q^2 s^3}. \end{aligned}$$

Since  $p^2 s^3 - r^3 q^2$  is an integer and  $q^2 s^3 \neq 0$ , the result is rational.

**Question 15**


Find possible values for  $x, y \in \mathbb{R}$  such that  $x$  and  $y$  are irrational, but  $x^y$  is rational.

Note that  $(\sqrt{2})^2 = 2$  which we can write as  $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$   
So we can have  $x = (\sqrt{2})^{\sqrt{2}}$  and  $y = \sqrt{2}$

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## Sub-Section: The 'Final Boss'

### Question 16

Consider the statements:

$A : n$  is an even integer.

$B : n < 0$

- a. Write the statement  $A \wedge \neg B$  in words.

$n$  is an even integer that is greater than or equal to zero.

- b. Prove that if  $A \wedge \neg B$  is true, then  $n^2 + 3n + 1$  is odd.

Let  $n = 2k$  where  $k \in \mathbb{Z}_{\geq 0}$ . Then,

$$\begin{aligned} n^2 + 3n + 1 &= 4k^2 + 6k + 1 \\ &= 2(2k^2 + 3k) + 1 \\ &= 2p + 1 \end{aligned}$$

where  $p \in \mathbb{Z}_{\geq 0}$ , and so  $n^2 + 3n + 1$  is odd.

- c. Prove that the product of any two odd integers is odd.

Let  $m = 2k + 1$  and  $n = 2j + 1$  be odd integers, where  $k, j \in \mathbb{Z}$ . Compute the product:

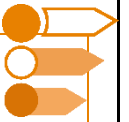
$$mn = (2k + 1)(2j + 1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1.$$

Since  $2kj + k + j \in \mathbb{Z}$ , the product is of the form  $2p + 1$ , where  $p \in \mathbb{Z}$ . Hence,  $mn$  is odd.

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## Section B: Supplementary Questions

### Sub-Section [2.1.1]: Number Sets



#### Question 17



State all the number sets that the following are an element of:

a.  $\sqrt{5}$

\_\_\_\_\_  $\mathbb{R}, \mathbb{C}$  \_\_\_\_\_

b. 5

\_\_\_\_\_  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  \_\_\_\_\_

c.  $\pi + i$

\_\_\_\_\_  $\mathbb{C}$  \_\_\_\_\_

d.  $-\frac{3}{7}$

\_\_\_\_\_  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$  \_\_\_\_\_

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**Question 18**

Express each of the following subsets of  $\mathbb{R}$  in interval notation.

a.  $\{x: x > 3\}$

$$(3, \infty)$$

b.  $\{x: -8 < x < 1\} \cap \{x: x \geq -3\}$

$$[-3, 1)$$

c.  $\{x: x \neq 1\} \cup \{x: x \leq 5\}$

$$(-\infty, 5] \setminus \{1\}$$

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**Question 19**

Rationalise the denominator and then simplify the following expressions:

a.  $\frac{2}{\sqrt{3}}$

$$\frac{2\sqrt{3}}{3}$$

b.  $\frac{5}{3+\sqrt{2}}$

$$\frac{15 - 5\sqrt{2}}{7}$$

c.  $\frac{\sqrt{30}+7}{4+\sqrt{7}}$

$$\frac{28 - 7\sqrt{7} + 4\sqrt{30} - \sqrt{210}}{9}$$

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Question 20

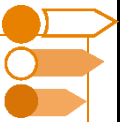
Rationalise the denominator of the following expression and simplify:

$$\frac{x+\sqrt{y}}{\sqrt{a}+\sqrt{b}}$$

where  $x, y, a, b > 0$  and  $a \neq b$ .

$$\frac{(x + \sqrt{y})(\sqrt{a} - \sqrt{b})}{a^2 - b^2} = \frac{(\sqrt{a} - \sqrt{b})x - \sqrt{ay} - \sqrt{by}}{a^2 - b^2}$$

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## Sub-Section [2.1.2]: Operations on Statements

### Question 21



Consider the following statements:

$A$  = It is hot outside.  
 $B$  = I go to the beach.

Write down the following:

a.  $A \wedge B$

It is hot outside and I go to the beach.

b.  $\neg B$

I do not go to the beach.

c.  $\neg A \vee \neg B$

It is not hot outside, or I do not go to the beach.

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**Question 22**

Use De Morgan's Law to write down the negation of the following statements:

- a. The movie is entertaining and the popcorn is tasty.

\_\_\_\_\_

\_\_\_\_\_ The movie is not entertaining or the popcorn is not tasty. \_\_\_\_\_

\_\_\_\_\_

- b. The traffic is light or the weather is clear.

\_\_\_\_\_

\_\_\_\_\_ The traffic is not light and the weather is not clear. \_\_\_\_\_

\_\_\_\_\_

- c. The phone is affordable and has a good camera.

\_\_\_\_\_

\_\_\_\_\_ The phone is not affordable or it does not have a good camera. \_\_\_\_\_

\_\_\_\_\_

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**Question 23**

Write the following as conditional statements:

- a. People who recycle help the environment.

\_\_\_\_\_

\_\_\_\_\_ If a person recycles, then they help the environment. \_\_\_\_\_

\_\_\_\_\_

- b. Employees who work overtime earn extra pay.

\_\_\_\_\_

\_\_\_\_\_ If an employee works overtime, then they earn extra pay. \_\_\_\_\_

\_\_\_\_\_

- c. Athletes who practice regularly improve their performance.

\_\_\_\_\_

\_\_\_\_\_ If an athlete practices regularly, then they improve their performance. \_\_\_\_\_

\_\_\_\_\_

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**Question 24**

Simplify the following logical expression using De Morgan's Laws:

$$\neg((P \wedge Q) \vee (\neg R \wedge S)).$$

Give your answer in the form

$$(A \vee B) \wedge (C \vee D)$$

Using De Morgan's Law:

$$\neg((P \wedge Q) \vee (\neg R \wedge S)) = \neg(P \wedge Q) \wedge \neg(\neg R \wedge S).$$

Next, apply De Morgan's Laws to each part:

$$\neg(P \wedge Q) = \neg P \vee \neg Q,$$

and:

$$\neg(\neg R \wedge S) = \neg(\neg R) \vee \neg S = R \vee \neg S.$$

Substitute these into the expression:

$$\neg((P \wedge Q) \vee (\neg R \wedge S)) = (\neg P \vee \neg Q) \wedge (R \vee \neg S).$$

So our expression is:

$$(\neg P \vee \neg Q) \wedge (R \vee \neg S).$$

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### Sub-Section [2.1.3]: Proofs Involving Even and Odd Numbers

#### Question 25



For an integer  $n$ , show that if  $n$  is even then  $n^3$  is even.

$n$  is even and so  $n = 2k$  for some  $k \in \mathbb{Z}$ . Therefore,

$$\begin{aligned} n^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3) \\ &= 2m \end{aligned}$$

where  $m \in \mathbb{Z}$ , and so  $n^3 = 2m$  is even.

#### Question 26



Show that,  $(4n + 2)^2 - (2n - 1)$  is always odd for any  $n \in \mathbb{Z}$ .

$$\begin{aligned} (4n + 2)^2 - (2n - 1) &= 16n^2 + 16n + 4 - 2n + 1 \\ &= 2(16n^2 + 7n + 2) + 1 \\ &= 2k + 1 \end{aligned}$$

where  $k \in \mathbb{Z}$ , and so  $(4n + 2)^2 - (2n - 1)$  is odd for any  $n \in \mathbb{Z}$ .



**Question 27**


Show that  $n^2 + 7n + 10$  is even for all  $n \in \mathbb{N}$ .

Observe that

$$n^2 + 7n + 10 = (n + 2)(n + 5)$$

one of  $n + 2$  and  $n + 5$  is even and the other is odd.

We know that even  $\times$  odd is always even.

Therefore,  $n^2 + 7n + 10$  is even for all  $n \in \mathbb{N}$

**Question 28**


Prove that the product of any two odd integers minus the sum of the same two integers is always even.

Let the two odd integers be  $a = 2m + 1$  and  $b = 2n + 1$ , where  $m, n \in \mathbb{Z}$ .

We have

$$\begin{aligned} a \cdot b &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

so  $ab$  is odd.

Next, compute the sum of  $a$  and  $b$ :

$$\begin{aligned} a + b &= (2m + 1) + (2n + 1) \\ &= 2m + 2n + 2 \\ &= 2(m + n + 1) \end{aligned}$$

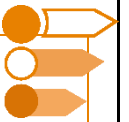
so  $a + b$  is even.

Therefore, it must be that for  $k, \ell \in \mathbb{Z}$

$$\begin{aligned} ab - (a + b) &= (2k + 1) - 2\ell \\ &= 2(k - \ell) + 1 \end{aligned}$$

which is odd.

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## Sub-Section [2.1.4]: Proofs Involving Divisibility

### Question 29



Show that if  $n$  is divisible by 7, then  $n^2$  is divisible by 7 for any  $n \in \mathbb{N}$ .

We can write  $n = 7k$  for some  $k \in \mathbb{N}$  and so

$$\begin{aligned} n^2 &= (7k)^2 \\ &= 49k^2 \\ &= 7(7k^2) \\ &= 7m \end{aligned}$$

where  $m \in \mathbb{N}$  and so  $n^2$  is divisible by 7.

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Question 30



Show that if  $n$  is divisible by 2 and  $m$  is divisible by 3, then  $3n + 4m$  is divisible by 3 for all  $n, m \in \mathbb{N}$ .

Let  $n = 2k$  where  $k \in \mathbb{N}$  and  $m = 3\ell$  where  $\ell \in \mathbb{N}$ . Then

$$\begin{aligned} 3n + 4m &= 6k + 12\ell \\ &= 3(2k + 4\ell) \\ &= 3r \end{aligned}$$

where  $r \in \mathbb{N}$ , and so  $3n + 4m$  is divisible by 3.

Question 31



Prove that if  $m$  and  $n$  are even integers, then  $m^2 + n^2$  and  $m^2 - n^2$  are both divisible by 4.

Let  $m = 2a$  and  $n = 2b$  for  $a, b \in \mathbb{Z}$ . Then we have

$$\begin{aligned} m^2 + n^2 &= 4a^2 + 4b^2 \\ &= 4(a^2 + b^2) \end{aligned}$$

and so is divisible by four.

We also have

$$\begin{aligned} m^2 - n^2 &= 4a^2 - 4b^2 \\ &= 4(a^2 - b^2) \end{aligned}$$

and so is divisible by four.


**Question 32**

Prove that the sum of any two consecutive odd numbers is divisible by 4.

Let the two odd numbers be  $a = 2n + 1$  and  $b = 2n + 3$  for some  $n \in \mathbb{Z}$ . Then

$$a + b = 2n + 1 + 2n + 3$$

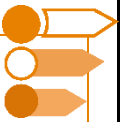
$$= 4n + 4$$

$$= 4(n + 1)$$

and so must be divisible by four.

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Sub-Section [2.1.5]: Proofs Involving Rational Numbers



Question 33



Show that if  $\sqrt[3]{x}$  is rational, then  $x$  is rational for any  $x \in \mathbb{R}$ .

$\sqrt[3]{x} = \frac{p}{q}$  for  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . Then

$$x = \frac{p^3}{q^3} = \frac{m}{n}$$

where  $m, n \in \mathbb{Z}$  and  $n \neq 0$ , and so  $x \in \mathbb{Q}$ .

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**Question 34**

Show that if both  $x$  and  $y$  are rational, then  $x^2 + y^2$  is rational.

Assume  $x = \frac{p}{q}$  and  $y = \frac{r}{s}$ , where  $p, q, r, s$  are integers, and  $q, s \neq 0$ .

$$\begin{aligned} x^2 + y^2 &= \frac{p^2}{q^2} + \frac{r^2}{s^2} \\ &= \frac{p^2 s^2 + r^2 q^2}{q^2 s^2} \end{aligned}$$

Since  $p^2 s^2 + r^2 q^2$  is an integer and  $q^2 s^2 \neq 0$ , the result is rational.


**Question 35**

Prove that if  $x$  is rational and  $x \neq 0$ , then  $\frac{1}{x}$  is also rational.

Assume  $x$  is rational and  $x \neq 0$ . By the definition of rational numbers, we can write:

$$x = \frac{p}{q},$$

where  $p, q \in \mathbb{Z}$  and  $q \neq 0$ .

Now,

$$\frac{1}{x} = \frac{1}{\frac{p}{q}} = \frac{q}{p}.$$

Since  $p \neq 0$  (as  $x \neq 0$ ) and both  $p$  and  $q$  are integers,  $\frac{q}{p}$  is a fraction with an integer numerator and nonzero integer denominator. Thus,  $\frac{1}{x} = \frac{q}{p}$  is rational.


**Question 36**

Prove that if  $x$  and  $y$  are rational and  $x, y \neq 0$  then

$$\frac{(x - 2y)^5 + x^2 + 3y}{x^2 + 2y^2}$$

is rational.

Let  $a = x - 2y \in \mathbb{Q}$  then  $a^5 = b \in \mathbb{Q}$ , since the sum and product of rational numbers is rational.

Also  $c = x^2 + 3y \in \mathbb{Q}$  and  $d = x^2 + 2y^2 \in \mathbb{Q}$ , since the sum and product of rational numbers is rational.

Therefore,

$$\frac{(x - 2y)^5 + x^2 + 3y}{x^2 + 2y^2} = \frac{b + c}{d} \in \mathbb{Q}$$

since  $b, c, d \in \mathbb{Q}$  and  $d \neq 0$ .

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