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VCE Specialist Mathematics ½
AOS 2 Revision [2.0]
Contour Check Solutions

Contour Check

Section A: [2.1] – Proofs I (Checkpoints)

Sub-Section [2.1.1]: Number Sets

Question 1

State all the number sets that the following are an element of:

a. $\sqrt{5}$

$\mathbb{R}, \mathbb{C}$

b. 5

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

c. $\pi + i$

$\mathbb{C}$

d. $-\frac{3}{7}$

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$

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Question 2

Express each of the following subsets of $\mathbb{R}$ in interval notation.

a. $\{x: x > 3\}$

$(3, \infty)$

b. $\{x: -8 < x < 1\} \cap \{x: x \geq -3\}$

$[-3,1)$

c. $\{x: x \neq 1\} \cup \{x: x \leq 5\}$

$(-\infty, 5]\backslash\{1\}$

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Question 3

Rationalise the denominator and then simplify the following expressions.

a. $\frac{2}{\sqrt{3}}$

$$\frac{2\sqrt{3}}{3}$$

b. $\frac{5}{3+\sqrt{2}}$

$$\frac{15-5\sqrt{2}}{7}$$

c. $\frac{\sqrt{30}+7}{4+\sqrt{7}}$

$$\frac{28-7\sqrt{7}+4\sqrt{30}-\sqrt{210}}{9}$$

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Question 4

Rationalise the denominator of the following expression and simplify

$$\frac{x+\sqrt{y}}{\sqrt{a}+\sqrt{b}},$$

Where $x, y, a, b > 0$ and $a \neq b$.

$$\frac{(x+\sqrt{y})(\sqrt{a}-\sqrt{b})}{a^2-b^2} = \frac{(\sqrt{a}-\sqrt{b})x-\sqrt{ay}-\sqrt{by}}{a^2-b^2}$$

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Sub-Section [2.1.2]: Operations on Statements

Question 5

Consider the following statements:

$$A = \text{It is hot outside.}$$
$$B = \text{I go to the beach.}$$

Write down the following:

a. $A \wedge B$.

It is hot outside and I go to the beach.

b. $\neg B$.

I do not go to the beach.

c. $\neg A \vee \neg B$.

It is not hot outside, or I do not go to the beach.

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Question 6

Use De Morgan's Law to write down the negation of the following statements:

a. The movie is entertaining and the popcorn is tasty.

The movie is not entertaining or the popcorn is not tasty.

b. The traffic is light or the weather is clear.

The traffic is not light and the weather is not clear.

c. The phone is affordable and has a good camera.

The phone is not affordable or it does not have a good camera.

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Question 7

Write the following as conditional statements.

a. People who recycle help the environment.

If a person recycles, then they help the environment.

b. Employees who work overtime earn extra pay.

If an employee works overtime, then they earn extra pay.

c. Athletes who practice regularly improve their performance.

If an athlete practices regularly, then they improve their performance.

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Question 8

Simplify the following logical expression using De Morgan's Laws:

$$\neg((P \wedge Q) \vee (\neg R \wedge S)).$$

Give your answer in the form:

$$(A \vee B) \wedge (C \vee D).$$

Using De Morgan's Law:

$$\neg((P \wedge Q) \vee (\neg R \wedge S)) = \neg(P \wedge Q) \wedge \neg(\neg R \wedge S).$$

Next, apply De Morgan's Laws to each part:

$$\neg(P \wedge Q) = \neg P \vee \neg Q,$$

and:

$$\neg(\neg R \wedge S) = \neg(\neg R) \vee \neg S = R \vee \neg S.$$

Substitute these into the expression:

$$\neg((P \wedge Q) \vee (\neg R \wedge S)) = (\neg P \vee \neg Q) \wedge (R \vee \neg S).$$

So our expression is:

$$(\neg P \vee \neg Q) \wedge (R \vee \neg S).$$

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Sub-Section [2.1.3]: Proofs Involving Even and Odd Numbers

Question 9

For an integer n, show that if n is even then n^3 is even.

n is even and so $n = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3) \\ &= 2m \end{aligned}$$

where $m \in \mathbb{Z}$, and so $n^3 = 2m$ is even.

Question 10

Show that $(4n + 2)^2 - (2n - 1)$ is always odd for any $n \in \mathbb{Z}$.

$$\begin{aligned} (4n+2)^2 - (2n-1) &= 16n^2 + 16n + 4 - 2n + 1 \\ &= 2(16n^2 + 7n + 2) + 1 \\ &= 2k + 1 \end{aligned}$$

where $k \in \mathbb{Z}$, and so $(4n+2)^2 - (2n-1)$ is odd for any $n \in \mathbb{Z}$.

Question 11

Show that $n^2 + 7n + 10$ is even for all $n \in \mathbb{N}$.

Observe that

$$n^2 + 7n + 10 = (n+2)(n+5)$$

one of $n+2$ and $n+5$ is even and the other is odd.
We know that even $\times$ odd is always even.
Therefore, $n^2 + 7n + 10$ is even for all $n \in \mathbb{N}$

Question 12

Prove that the product of any two odd integers minus the sum of the same two integers is always even.

Let the two odd integers be $a = 2m+1$ and $b = 2n+1$, where $m, n \in \mathbb{Z}$.
We have

$$\begin{aligned} a \cdot b &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

so ab is odd.
Next, compute the sum of a and b:

$$\begin{aligned} a + b &= (2m+1) + (2n+1) \\ &= 2m + 2n + 2 \\ &= 2(m + n + 1) \end{aligned}$$

so $a+b$ is even.
Therefore, it must be that for $k, \ell \in \mathbb{Z}$

$$\begin{aligned} ab - (a+b) &= (2k+1) - 2\ell \\ &= 2(k - \ell) + 1 \end{aligned}$$

which is odd.

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Sub-Section [2.1.4]: Proofs Involving Divisibility

Question 13

Show that if n is divisible by 7, then n^2 is also divisible by 7 for any $n \in \mathbb{N}$.

We can write $n = 7k$ for some $k \in \mathbb{N}$ and so

$$\begin{aligned} n^2 &= (7k)^2 \\ &= 49k^2 \\ &= 7(7k^2) \\ &= 7m \end{aligned}$$

where $m \in \mathbb{N}$ and so n^2 is divisible by 7.

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Question 14

Show that if n is divisible by 2 and m is divisible by 3, then $3n + 4m$ is divisible by 3 for all $n, m \in \mathbb{N}$.

Let $n = 2k$ where $k \in \mathbb{N}$ and $m = 3\ell$ where $\ell \in \mathbb{N}$. Then

$$\begin{aligned} 3n + 4m &= 6k + 12\ell \\ &= 3(2k + 4\ell) \\ &= 3r \end{aligned}$$

where $r \in \mathbb{N}$, and so $3n + 4m$ is divisible by 3.

Question 15

Prove that if m and n are even integers, then $m^2 + n^2$ and $m^2 - n^2$ are both divisible by 4.

Let $m = 2a$ and $n = 2b$ for $a, b \in \mathbb{Z}$. Then we have

$$\begin{aligned} m^2 + n^2 &= 4a^2 + 4b^2 \\ &= 4(a^2 + b^2) \end{aligned}$$

and so is divisible by four.
We also have

$$\begin{aligned} m^2 - n^2 &= 4a^2 - 4b^2 \\ &= 4(a^2 - b^2) \end{aligned}$$

and so is divisible by four.

Question 16

Prove that the sum of any two consecutive odd numbers is divisible by 4.

Let the two odd numbers be $a = 2n + 1$ and $b = 2n + 3$ for some $n \in \mathbb{Z}$. Then

$$\begin{aligned} a + b &= 2n + 1 + 2n + 3 \\ &= 4n + 4 \\ &= 4(n + 1) \end{aligned}$$

and so must be divisible by four.

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Sub-Section [2.1.5]: Proofs Involving Rational Numbers

Question 17

Show that if $\sqrt[3]{x}$ is rational, then x is rational for any $x \in \mathbb{R}$.

$\sqrt[3]{x} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$x = \frac{p^3}{q^3} = \frac{m}{n}$$

where $m, n \in \mathbb{Z}$ and $n \neq 0$, and so $x \in \mathbb{Q}$.

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Question 18

Show that if both x and y are rational, then $x^2 + y^2$ is rational.

Assume $x = \frac{p}{q}$ and $y = \frac{r}{s}$, where p, q, r, s are integers, and $q, s \neq 0$.

$$x^2 + y^2 = \frac{p^2}{q^2} + \frac{r^2}{s^2}$$
$$= \frac{p^2 s^2 + r^2 q^2}{q^2 s^2}$$

Since $p^2 s^2 + r^2 q^2$ is an integer and $q^2 s^2 \neq 0$, the result is rational.

Question 19

Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is also rational.

Assume x is rational and $x \neq 0$. By the definition of rational numbers, we can write:

$$x = \frac{p}{q},$$

where $p, q \in \mathbb{Z}$ and $q \neq 0$.
Now,

$$\frac{1}{x} = \frac{1}{\frac{p}{q}} = \frac{q}{p}.$$

Since $p \neq 0$ (as $x \neq 0$) and both p and q are integers, $\frac{q}{p}$ is a fraction with an integer numerator and nonzero integer denominator. Thus, $\frac{1}{x} = \frac{q}{p}$ is rational.

Question 20

Prove that if x and y are rational and $x, y \neq 0$ then,

$$\frac{(x-2y)^5 + x^2 + 3y}{x^2 + 2y^2}$$

is rational.

Let $a = x - 2y \in \mathbb{Q}$ then $a^5 = b \in \mathbb{Q}$, since the sum and product of rational numbers is rational.
Also $c = x^2 + 3y \in \mathbb{Q}$ and $d = x^2 + 2y^2 \in \mathbb{Q}$, since the sum and product of rational numbers is rational.
Therefore,

$$\frac{(x-2y)^5 + x^2 + 3y}{x^2 + 2y^2} = \frac{b+c}{d} \in \mathbb{Q}$$

since $b, c, d \in \mathbb{Q}$ and $d \neq 0$.

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Section B: [2.2] - Proofs II (Checkpoints)

Sub-Section [2.2.1]: Direct and Indirect Proofs

Question 21

Prove that all numbers of the form $n^3 - n$, where $n \in \mathbb{Z}$, are multiples of 6.

The expression is $n^3 - n$.

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1).$$

This represents the product of three consecutive integers.
One of these integers must be even, and another must be divisible by 3.
Therefore $n^3 - n$ is divisible by both 2 and 3 and thus it must be divisible by 6.

Question 22

Prove the following statement using a proof by contrapositive: If n^5 is odd, then n is odd.

We will prove the contrapositive: n is even $\implies$ n^5 is even.
Let $n = 2k$ where $k \in \mathbb{Z}$, then

$$\begin{aligned} n^5 &= (2k)^5 \\ &= 32k^5 \\ &= 2(16k^5) \\ &= 2m \end{aligned}$$

where $m \in \mathbb{Z}$ and so $n^5 = 2m$ is even.

Question 23

Prove the following statement using a proof by contradiction: $\sqrt{5} + \sqrt{7} < 5$.

Suppose for a contradiction that $\sqrt{5} + \sqrt{7} \geq 5$. Then

$$(\sqrt{5} + \sqrt{7})^2 \geq 25$$
$$5 + 7 + 2\sqrt{35} \geq 25$$
$$2\sqrt{35} \geq 13$$

but this statement is false since $2\sqrt{35} < 2\sqrt{36} = 12$. Thus we have a contradiction and therefore the assumption that $\sqrt{5} + \sqrt{7} \geq 5$ must be false and so $\sqrt{5} + \sqrt{7} < 5$.

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Question 24

Prove that for $a, b > 0$, we have $a + b \geq \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}$.

Assume the statement is false. That is, suppose:

$$a + b < \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}.$$

Let:

$$H = \frac{1}{a} + \frac{1}{b}.$$

Then the assumption becomes:

$$a + b < \frac{1}{H}.$$

Multiply both sides by $H > 0$ (since $a, b > 0$):

$$H(a + b) < 1.$$

Now substituting $H = \frac{1}{a} + \frac{1}{b}$, we get:

$$\left(\frac{1}{a} + \frac{1}{b}\right)(a + b) < 1$$
$$2 + \frac{b}{a} + \frac{a}{b} < 1$$
$$\frac{b}{a} + \frac{a}{b} < -1$$

However, for $a, b > 0$, $d\frac{b}{a} + \frac{a}{b} \geq 0$ which contradicts $\frac{b}{a} + \frac{a}{b} < -1$. We have a contradiction.

The assumption is false. Therefore, $a + b \geq \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}$ is true.

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Sub-Section [2.2.2]: Proofs involving Converse and Equivalent Statements

Question 25

Write the converse of the following statements.

a. If a person exercises regularly, they stay healthy.

If a person stays healthy, then they exercise regularly.

b. If a car is fuel-efficient, it saves money on gas.

If a car saves money on gas, then it is fuel-efficient.

c. If a student studies, they pass their exams.

If a student passes their exams, then they study.

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Question 26

Suppose $n \in \mathbb{Z}$. Prove that n is odd, if and only if $3n + 1$ is even.

($\Longrightarrow$) n odd then $3n+1$ is even.
Let $n = 2k+1$ for some $k \in \mathbb{Z}$, then

$$\begin{aligned} 3n+1 &= 3(2k+1)+1 \\ &= 6k+4 \\ &= 2(3k+2) \\ &= 2m \end{aligned}$$

where $m \in \mathbb{Z}$ and is therfore even.
($\Longleftarrow$) $3n+1$ even then n odd

$$\begin{aligned} 3n+1 &= 2k \\ 3n &= 2k-1 \\ n &= 2(k-n)-1 \\ &= 2m-1 \end{aligned}$$

where $m \in \mathbb{Z}$, therefore $3n+1$ is odd.

Question 27

Prove the following statement: $\frac{n(n+1)}{2}$ is a natural number, if and only if n is a natural number.

($\Longrightarrow$) $\dfrac{n(n+1)}{2} = k$ for some $k \in \mathbb{N}$. Then

$$n(n+1) = 2k$$

$n(n+1)$ is an even natural number and so n must be a natural number.
($\Longleftarrow$) If n is a natural number then $n(n+1)$ is the product of two consecutive natural numbers and is therefore even. So, for some $k \in \mathbb{Z}$,

$$\begin{aligned} n(n+1) &= 2k \\ \frac{n(n+1)}{2} &= k \end{aligned}$$

therefore, $\dfrac{n(n+1)}{2}$ is a natural number.

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Question 28

Prove the following statement: For any integer n, n is divisible by 3, if and only if the sum of its digits is divisible by 3.

Let n have k digits, from left to right these digits are $a_k, a_{k-1}, \ldots, a_2, a_1$. Then we can write

$$\begin{aligned} n &= a_1 + 10a_2 + 10^2 a_3 + \cdots + 10^{k-1} a_k \\ &= a_1 + a_2 + a_3 + \cdots + a_k + (9a_2 + 99a_3 + \cdots + (10^{k-1} - 1)a_k) \\ &= a_1 + a_2 + a_3 + \cdots + a_k + 3^2 \left(a_2 + 11a_3 + 111a_4 + \cdots + \frac{10^{k-1} - 1}{9} a_k \right) \end{aligned}$$

Let $a_1 + a_2 + a_3 + \cdots + a_k = d$ and $a_2 + 11a_3 + 111a_4 + \cdots + \frac{10^{k-1} - 1}{9} a_k = b$, then

$$n = d + 3^2 b$$

($\Longrightarrow$) The second term is a multiple of 3 so for n to be a multiple of 3 we must also have d be a multiple of three.
($\Longleftarrow$) d is a multiple of three so

$$n = 3c + 3^2 b = 3(c + 3b)$$

and so n is a multiple of three.

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Sub-Section [2.2.3]: Proofs involving the Universal and Existence Quantifiers

Question 29

Write the following statements in terms of the universal (∀) and existential (∃) quantifiers.

a. All positive integers are greater than zero.

$\forall n \in \mathbb{Z}^+, n > 0.$

b. There exists an integer that is a perfect square.

$\exists n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n = k^2.$

c. For all real numbers x, if $x > 0$, then $\frac{1}{x} > 0$.

$\forall x \in \mathbb{R}, x > 0 \implies \frac{1}{x} > 0.$

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Question 30

Negate the following statements involving universal and existential quantifiers.

a. $\forall n \in \mathbb{Z}, n + 0 = n$

$\exists n \in \mathbb{Z}, n + 0 \neq n.$

b. $\exists x \in \mathbb{R}, x^3 = 8$

$\forall x \in \mathbb{R}, x^3 \neq 8.$

c. $\forall x \in \mathbb{R}, x^2 \geq 0$

$\exists x \in \mathbb{R}, x^2 < 0.$

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Question 31

Disprove the following statements by providing a counterexample.

a. Disprove that for all integers $n, n^3 - n$ is always odd.

Counterexample: Let $n = 2$. Then:

$$n^3 - n = 2^3 - 2 = 8 - 2 = 6,$$

which is even. Hence, the statement is false.

b. Disprove that there exists an integer n such that, $2n + 1 = 0$.

The equation $2n+1 = 0$ implies $n = -\frac{1}{2}$, which is not an integer. Hence, the statement is false.

c. Disprove that for all real numbers $x, x^2 + x$ is greater than 1.

Counterexample: Let $x = -1$. Then:

$$x^2 + x = (-1)^2 + (-1) = 1 - 1 = 0,$$

which is not greater than 1. Hence, the statement is false.

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Question 32

Prove that:

$$\forall a, b \in \mathbb{R}^+ \cup \{0\}, \frac{a+b}{2} \geq \sqrt{ab}$$

Suppose that $\frac{a+b}{2} < \sqrt{ab}$ then

$$\frac{a^2+2ab+b^2}{4} < ab$$
$$\frac{a^2-2ab+b^2}{4} < 0$$
$$\left(\frac{a-b}{2}\right)^2 < 0$$

which is a contradiction since and real number squared is ≥ 0.

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Sub-Section [2.2.4]: Telescoping Series and Proofs by Induction

Question 33

Simplify the following telescoping series using partial fraction decomposition and simplification.

$$\sum_{k=1}^{n+1} \frac{1}{(k+1)(k+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)}$$

Rewrite $\frac{1}{(k+1)(k+2)}$ using partial fractions:

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}.$$

Substitute into the series:

$$\sum_{k=1}^{n+1} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{n+1} \left(\frac{1}{k+1} - \frac{1}{k+2} \right).$$

This is a telescoping series, so most terms cancel:

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right).$$

Simplify to get:

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} = \frac{1}{2} - \frac{1}{n+3}.$$

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Question 34

Prove the following statement by induction:

$$2 + 4 + 6 + \cdots + 2n = n(n + 1) \text{ for all integers } n \geq 1.$$

Let $P(n)$ be the statement $2+4+6+\cdots+2n = n(n+1)$.
Base Case: $P(1) = 2 = 1(1+1) = 2$ holds.
Assume that $P(k)$ holds for some $k \in \mathbb{N}$. Then,

$$\begin{aligned} 2+4+6+\cdots+2k+2(k+1) &= k(k+1)+2(k+1) \\ &= (k+1)(k+2) \end{aligned}$$

which is the statement $P(k+1)$. Therefore by the POMI the statement $P(n)$ holds for all integers $n \geq 1$.

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Question 35

Prove the following statement by induction:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \text{ for all integers } n \geq 1.$$

Let $P(n)$ be the statement $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$.

Base Case: $P(1) = a = \dfrac{a(r-1)}{r-1} = a$ which is true.

Assume that $P(k)$ holds for some $k \in \mathbb{N}$. Then we have

$$\begin{aligned} a + ar + ar^2 + \cdots + ar^{k-1} + ar^k &= \frac{a(r^k - 1)}{r - 1} + ar^k \\ &= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1} \\ &= \frac{a(r^k - 1) + ar^{k+1} - ar^k}{r - 1} \\ &= \frac{-a + ar^{k+1}}{r - 1} \\ &= \frac{a(r^{k+1} - 1)}{r - 1} \end{aligned}$$

so the statement $P(k+1)$ holds. Therefore, by the POMI the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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Question 36

Prove the following statement by induction:

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{n}{2}(2a + (n - 1)d), \text{ for all integers } n \geq 1.$$

Let $P(n)$ be the statement $a + (a+d) + (a+2d) + \cdots + (a + (n-1)d) = \frac{n}{2}(2a + (n-1)d)$.

Base Case: $P(1) = a = \frac{1}{2}(2a) = a$ which is true.

Assume that $P(k)$ hold for some $k \in \mathbb{N}$. Then we have

$$\begin{aligned} a + (a+d) + (a+2d) + \cdots + (a + (k-1)d) + (a + kd) &= \frac{k}{2}(2a + (k-1)d) + a + kd \\ &= ak + \frac{k}{2}(kd - d) + a + kd \\ &= (k+1)a + \frac{k}{2}(kd) + \frac{1}{2}kd \\ &= \frac{k+1}{2}(2a) + \frac{1}{2}kd(k+1) \\ &= \frac{k+1}{2}(2a + kd) \end{aligned}$$

which is equal to $P(k+1)$. Therefore by the POMI the statement $P(n)$ is true for all $n \in \mathbb{N}$.

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Section C: [2.3] - Proofs Exam Skills (Checkpoints)

Sub-Section [2.3.1]: Solve Problems Using AM-GM Inequalities

Question 37

Show using the AM-GM inequality that for $x > 0$ we have:

$$5x + \frac{5}{x} \geq 10$$

Recall that the AM-GM inequality states that $\frac{a+b}{2} \geq \sqrt{ab}$ where $a, b > 0$. Let $a = 5x$ and $b = 5/x$. Then, $\frac{5x+5/x}{2} \geq \sqrt{5x \cdot 5/x} = 5$. Thus, we obtain the inequality $5x + 5/x \geq 10$.

Question 38

Minimise $2x + \frac{2}{x}$ over $x > 0$ by applying the AM-GM inequality, and hence maximise $6 - 2x - \frac{2}{x}$.

Using the AM-GM inequality (similar to in the previous problem), we conclude that $2x + 2/x \geq 4$. Furthermore, we see that $2x + 2/x = 4$ when $x = 1$. Thus, $2x + 2/x$ attains a minimum of 4. Now, $6 - 2x - 2/x$ will be maximised as long as $2x + 2/x$ is minimised because in this situation, we would be subtracting the smallest possible number away from 6. Thus, the maximal value that $6 - 2x - 2/x$ can achieve is 2.

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Question 39

Find an expression for the area of a rectangle that has a perimeter of 4 units and a width of x units, and hence use the AM-GM inequality to maximise the area of such a rectangle.

A rectangle with width x must have length $2 - x$ so that the perimter is 4 units. Thus, the area of such a shape is $A(x) = x(2 - x)$. Now, by the AM-GM inequality with $a = x$ and $b = 2 - x$, we see that $ab \leq \left(\frac{a+b}{2}\right)^2 = \left(\frac{x+2-x}{2}\right)^2 = 1$. Note that this value is achieved by $x = 1$. Hence, the maximum area is 1.

Question 40

Let $x, y > 0$. Furthermore, suppose that $xy = 4$. Find the minimum value of $xy^3 + x^3y$.

Applying the AM-GM inequality with $a = xy^3$ and $b = x^3y$, we find that $\frac{xy^3+x^3y}{2} \geq \sqrt{xy^3 \cdot x^3y} = x^2y^2$. Therefore, $xy^3 + x^3y \geq 2x^2y^2 = 2 \cdot (4)^2 = 32$.

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Sub-Section [2.3.2]: Solve Arithmetic and Geometric Proofs

Question 41

Prove using induction that $2+7+12+\cdots+(5n-3)=\frac{n(5n-1)}{2}$.

For $n=1$, we see that RHS $=\frac{1\cdot(5\cdot1-1)}{2}=$ LHS. Now, assume the statement holds for some $N\in\mathbb{N}$. Observe that

$$\begin{aligned}2+7+12+\cdots+(5(n+1)-3) &= 2+7+12+\cdots+(5n-3)+(5n+2)\\ &= \frac{n(5n-1)}{2}+5n+2\\ &= \frac{5n^2-n+10n+4}{2}\\ &= \frac{5n^2+9n+4}{2}\\ &= \frac{(n+1)(5n+4)}{2}\\ &= \frac{(n+1)(5(n+1)-1)}{2}\end{aligned}$$

Therefore, the statement holds for $n+1$ and by the principle of mathematical induction, the statement holds for all $n\in\mathbb{N}$.

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Question 42

Prove using induction that $1 \cdot 7 + 2 \cdot 8 + \cdots + n(n+6) = \frac{n(n+1)(2n+19)}{6}$.

For $n = 1$, we see that RHS $= \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 19)}{6} = 7 = 1 \cdot 7 =$ LHS. Therefore, the statement holds for $n = 1$. Now, assume that the statement holds for some $n \in \mathbb{N}$. Observe that

$$\begin{aligned} 1 \cdot 7 + 2 \cdot 8 + \cdots (n+1)((n+1)+6) &= 1 \cdot 7 + 2 \cdot 8 + \cdots + n(n+6) + (n+1)(n+7) \\ &= \frac{n(n+1)(2n+19)}{6} + (n+1)(n+7) \\ &= \frac{n(n+1)(2n+19) + 6(n+1)(n+7)}{6} \\ &= \frac{(n+1)(2n^2 + 19n + 6n + 42)}{6} \\ &= \frac{(n+1)(2n^2 + 25n + 42)}{6} \\ &= \frac{(n+1)(n+2)(2n+21)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+19)}{6} \end{aligned}$$

Therefore, the statement is true for $n + 1$ and by induction, the statement holds for all $n \in \mathbb{N}$.

Question 43

Prove using induction that $2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^n = 3^{n+1} - 3$.

For $n = 1$, we see that RHS $= 3^2 - 3 = 6 = 2 \cdot 3 =$ LHS. Hence, the base case is true. Now, assume that the statement holds for some $n \in \mathbb{N}$. Observe that

$$\begin{aligned} 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n+1} &= 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^n + 2 \cdot 3^{n+1} \\ &= 3^{n+1} - 3 + 2 \cdot 3^{n+1} \\ &= 3 \cdot 3^{n+1} - 3 \\ &= 3^{n+2} - 3 \\ &= 3^{(n+1)+1} - 3 \end{aligned}$$

Therefore, the statement holds for $n + 1$ and by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$.

Question 44

a. Prove using induction that for all $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

For $n = 1$, we have RHS $= \frac{1^2 \cdot 2^2}{4} = 1 = 1^3 =$ LHS. Hence, we see that the statement holds for $n = 1$. Now, suppose that the statement holds for some $n \in \mathbb{N}$. Observe that

$$\begin{aligned} 1^3 + 2^3 + \cdots + (n+1)^3 &= 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \\ &= \frac{(n+1)^2((n+1)+1)^2}{4} \end{aligned}$$

Therefore, the statement holds for $n+1$ and by induction, the statement holds for all $n \in \mathbb{N}$.

b. Hence, write a rule for $2^3 + 4^3 + \cdots + (2n)^3$.

Hint: $2^3 + 4^3 + \cdots + (2n)^3$ is related to $1^3 + 2^3 + \cdots + n^3$ in a reasonably simple way.

Observe $2^3 + 4^3 + \cdots + (2n)^3 = (2 \cdot 1)^3 + (2 \cdot 2)^3 + \cdots (2n)^3 = 8 \cdot (1^3 + 2^3 + \cdots n^3) = 2n^2(n+1)^2$.

c. Now, deduce a rule for $1^3 + 3^3 + \cdots + (2n-1)^3$ using the rule you obtained above.

Observe that $1^3 + \cdots + (2n)^3 = (1^3 + 3^3 + \cdots + (2n-1)^3) + (2^3 + 4^3 + \cdots + (2n)^3)$. The sum of the first $2n$ cubes comes from the first formula using $2n$ instead of n. Therefore,

$$\begin{aligned} 1^3 + 3^3 + \cdots + (2n-1)^3 &= \frac{(2n)^2(2n+1)^2}{4} - 2n^2(n+1)^2 \\ &= 2n^4 - n^2 \end{aligned}$$

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Sub-Section [2.3.3]: Prove Divisibility With Induction

Question 45

Prove using induction that if $n \in \mathbb{N}$, then $8^n - 1$ is divisible by 7.

We proceed by induction. For $n = 1$, we have $8^n - 1 = 7$, which is divisible by 7 as $7 = 7 \cdot 1$. Therefore, the base case holds. Now, assume that $8^n - 1$ is divisible by 7 for some $n \in \mathbb{N}$. We want to show that $8^{n+1} - 1$ is divisible by 7. Notice that $8^{n+1} - 1 = 8 \cdot 8^n - 1 = 8(8^n - 1) + 7$. By assumption, $8^n - 1 = 7k$ for some $k \in \mathbb{Z}$. Therefore, $8^{n+1} - 1 = 7(8l + 1) = 7m$ where $m = 8l + 1 \in \mathbb{Z}$. Therefore, we conclude that 8^{n+1}-7 is divisible by 7 and by the principle of mathematical induction, $8^n - 1$ is divisible by 7 for all $n \in \mathbb{N}$.

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Question 46

Prove using induction that if $n \in \mathbb{N}$, then $n^3 + 3n^2 + 2n$ is divisible by 3.

Note: If you want to make this question a bit harder, you can instead show that $n^3 + 3n^2 + 2n$ is also divisible by 6. You might need to use the fact that the product of two consecutive integers is always even.

We proceed by induction. For $n = 1$, we have $1^3 + 3 \cdot 1^2 + 2 \cdot 1 = 6 = 3 \cdot 2$, which is divisible by 3. Now, assume that $n^3 + 3n^2 + 2n$ is divisible by 3 for some $n \in \mathbb{N}$. Therefore, we can write $n^3 + 3n^2 + 2n = 3m$ for some $m \in \mathbb{Z}$. Notice that

$$\begin{aligned}
(n+1)^3 + 3(n+1)^2 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 3n^2 + 6n + 3 + 2n + 2 \\
&= (n^3 + 3n^2 + 2n) + (3n^2 + 3n + 1 + 6n + 3 + 2) \\
&= (n^3 + 3n^2 + 2n) + (3n^2 + 9n + 6) \\
&= 3m + 3(n^2 + 3n + 2) \\
&= 3(m + n^2 + 3n + 2) \\
&= 3p
\end{aligned}$$

where $p = m + n^2 + 3n + 2 \in \mathbb{Z}$. Therefore, $(n+1)^3 + 3(n+1)^2 + 2(n+1)$ is divisible by 3. Furthermore, using the principle of mathematical induction, $n^3 + 3n^2 + 2n$ is divisible by 3 for all numbers $n \in \mathbb{N}$.

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Question 47

Prove using induction that if $n \in \mathbb{N}$, then $10^{n+1} + 10^n + 1$ is divisible by 3.

Note: The statement says that 111, 1101, 11001, etc., are all divisible by 3.

For $n = 1$, we have $10^{n+1} + 10^n + 1 = 111 = 3 \cdot 37$. Therefore, the base case holds. Now assume that $10^{n+1} + 10^n + 1$ is divisible by 3 for some $n \in \mathbb{N}$. In particular, this means $10^{n+1} + 10^n + 1 = 3k$ for some $k \in \mathbb{Z}$. Furthermore,

$$\begin{aligned} 10^{n+2} + 10^{n+1} + 1 &= 10(10^{n+1} + 10^n) + 1 \\ &= 10(10^{n+1} + 10^n + 1) - 9 \\ &= 10 \cdot 3k - 9 \\ &= 3(10p - 3) \\ &= 3m, \end{aligned}$$

where $m = 10p - 3 \in \mathbb{Z}$. Therefore, we may conclude that $10^{n+2} + 10^{n+1} + 1$ is divisible by 3 and by the principle of mathematical induction $10^{n+1} + 10^n + 1$ is divisible by 3 for all. $n \in \mathbb{N}$.

Question 48

Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. For example, $3! = 1 \cdot 2 \cdot 3$. Prove using induction that if $n \in \mathbb{N}$, then $(2n)!$ is divisible by 2^n.

For $n = 1$, we have $(2n)! = 2! = 2 = 2^1 \cdot 1$, which we see is divisible by 2^1. Now, assume that $(2n)!$ is divisible by 2^n if n is some natural number. Then, we may write $(2n)! = 2^n \cdot k$, where $k \in \mathbb{Z}$. Furthermore, we see that

$$(2(n+1))! = (2n+2)! = (2n)! \cdot (2n+1) \cdot (2n+2) = 2 \cdot 2^n \cdot k \cdot (2n+1)(n+1) = 2^{n+1}p,$$

where $p = k(2n+1)(n+1) \in \mathbb{Z}$. Therefore, $(2(n+1))!$ is divisible by 2^{n+1} and by the principle of mathematical induction, $(2n)!$ is divisible by 2^n for all $n \in \mathbb{N}$.

Section D: [2.4] – Logic & Algorithms I (Checkpoints)

Sub-Section [2.4.1]: Write and Understand Basic Algorithms

Question 49

Construct an algorithm that multiplies any input given by 10.

Step 1. Input A,

Step 2. $A \leftarrow 10A$

Step 3: Print A.

Question 50

Construct an algorithm that adds any input given by 5.

Step 1. Input A,

Step 2. $A \leftarrow A+5$

Step 3: Print A.

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Question 51

Construct an algorithm that subtracts any input given by 5 and multiplies by 2.

Step *1*. Input A,

Step *2*. *A*←*1/2A-5*

Step *3*: Print *A*.

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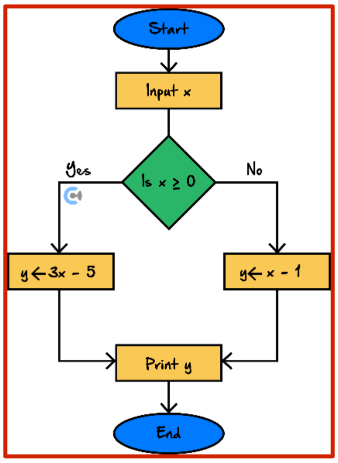
Sub-Section [2.4.2]: Understanding and Evaluating Algorithms That Have Conditional Statements and Represent Hybrid Functions as Algorithms

Question 52

Using a flowchart, describe an algorithm of the following hybrid function.

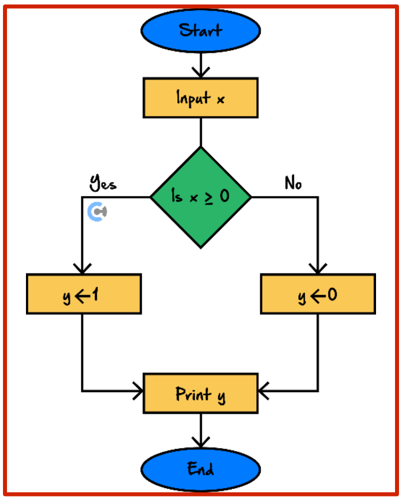
$$f(x) = \begin{cases} 3x - 5 & x \geq 0 \\ x - 1 & x < 0 \end{cases}$$



Question 53

Using a flowchart, describe an algorithm of the following hybrid function.

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Question 54

Turn the following function into an algorithm.

$$f(x) = \begin{cases} x^2 & x \geq 1 \\ -2x + 1 & x < 1 \end{cases}$$

Step 1: Input x

Step 2: If x>=1, then $y \leftarrow x$^2

else $y \leftarrow$-2x+1

Step 3: Print y.

Question 55

Turn the following function into an algorithm.

$$f(x) = max\{n \in \mathbb{R} | n \leq x\}$$

Step 1: Input x.
Step 2: $y \leftarrow max\{n \in \mathbb{R} | n \leq x\}$
Step 3: Print y.

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Sub-Section [2.4.3]: Understand and Evaluate Algorithms with Loops

Question 56

Check whether the following algorithm has any problems. If there is a problem, state the problem; if there is no problem, give the final output of the algorithm.

Step 1: $A \leftarrow 30$
Step 2: $A \leftarrow 3A - 20$
Step 3: Repeat 2 while $A > 65$.

It goes on infinitely. The condition of the loop is ALWAYS met.

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Question 57

Evaluate the following algorithm:

For a from 1 to 10
  if $a =$ even, then
    print "yes"
else
    print "no"
end for.

no yes no yes no yes no yes no yes.

Question 58

Check whether the following algorithm has any problems. If there is a problem, state the problem; if there is no problem, give the final output of the algorithm.

Step 1: $A \leftarrow 60$
Step 2: $A \leftarrow 2A - 50$
Step 3: Repeat 2 while $A \leq 130$.

$A = 210$

Question 59

Evaluate the following output:

$a \leftarrow 5$
$b \leftarrow 10$
if $a - b < 5$
  $a \leftarrow a - 5$
  $b \leftarrow b - 10$
end if
print a, b.

$a = 0, b = 0.$

Space for Personal Notes

Sub-Section [2.4.4]: Write and Evaluate Functions using Pseudocode

Question 60

$A \leftarrow [\ ]$
for n from 1 to 5
  append n to A.
  if $n = 1$, then
   return
else
  $A = \sqrt{n^2 + A[n-1]}$
  if $A =$ integer
   print "$A[n-1], n, A$ is a perfect triangle."
end for.

$3, 4, 5$ is a perfect triangle.

Space for Personal Notes

Question 61

a. Roger decided to invest \$1000 at an interest rate of 10% compounded monthly. Construct an algorithm that computes the number of years needed for Roger's investment to double.

Step 1: $\tau \leftarrow 1000$, rate $\leftarrow 10$, $T \leftarrow 0$

Step 2: $\tau \leftarrow \tau \times \left(1 + \frac{0.1}{12}\right)^{12}$, $T \leftarrow T + 1$

Step 3: Repeat 2 while $\tau \leq 2000$.

Step 4: Print T.

b. Jacob decided to invest \$500 at an interest rate of 15% compounded annually. Construct an algorithm that computes the number of years needed for Jacob's investment to increase by 50%.

Step 1: $\tau \leftarrow 500$, rate $\leftarrow 0.15$, $T \leftarrow 0$

Step 2: $\tau \leftarrow \tau \times (1 + \text{rate})^1$, $T \leftarrow T + 1$

Step 3: Repeat 2 while $\tau \leq 750$.

Step 4: Print T.

Space for Personal Notes

Question 62

Using pseudocode, write an algorithm to find all the primes less or equal to 100.

```
Prime 1st ← [1]        plistless= 1
    For number from 2 to 100
        check ← 0
        For index from 1 to plistless
            if number / prime list [index] = int,
                check ← check +1
            else
                return
        end for.

        if check = 0
            append number to prime list.
            plistless ← plistless +1
end for.
print "primelist"
```

Question 63

Using pseudocode, construct an algorithm for the following:

Find the shortest distance between any 2 ***different*** coordinates from the list of coordinates.

$$Y \text{ coordinate} = [1, 35, 5, 41, 5]$$
$$X \text{ coordinate} = [123, 2, 74, 213, 2]$$

```
Mindist = 300 [or any high enough initial number]
for a from 1 to 5
    for b from 1 to 5
        if a = b,
            break
        else
            distance = √((x[a] − x[b])² + (y[a] − y[b])²)
            if distance ≤ mindist
                mindist ← distance
    end for.
end for.
print min dist.
```

Section E: [2.5] - Logic & Algorithms II (Checkpoints)

Sub-Section [2.5.1]: Understand the Basics of Logic and Propositional Statements

Question 64

Translate the following to English:
P = I eat healthy.
Q = I exercise regularly.
R = I will lose weight.

$$P \wedge Q \Rightarrow R$$

If I eat healthy and I exercise regularly, then I will lose weight.

Question 65

Translate the following to English:
A = I go jogging.
B = The weather is good.
C = I will feel energised.

$$\neg B \Rightarrow (\neg A \wedge \neg C)$$

If the weather is not good, then I will not go jogging and I will not feel energised.

Question 66

Translate into propositional logic using the correct syntax:

If the team wins the match, then the fans will celebrate and the opposing team will be disappointed.

Let W = The team wins the match.
Let C = The fans celebrate.
Let D = Opposing team is disappointed.
$W \Rightarrow (C \land D)$

Question 67

Translate into propositional logic using the correct syntax:

If the baker uses old flour, then the bread will not rise and the customers will complain.

Let F = The baker uses old flour.
Let B = The bread rises.
Let C = The customers complain.
$F \Rightarrow (\neg B \land C)$

Space for Personal Notes

Sub-Section [2.5.2]: Construct Truth Tables and Recognise Equivalent Logical Expressions

Question 68

Write the truth table for:

$$\sim p \vee q$$

p	q	~p	~p∨q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Question 69

Write the truth table for:

$$(p \wedge q) \vee (p \vee q)$$

p	q	p∧q	p∨q	(p∧q)∨(p∨q)
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

Question 70

Construct a truth table for the statement $(p \oplus q) \Rightarrow r$, where $\oplus$ is the exclusive or.

p	q	r	$p \oplus q$	$(p \oplus q) \Rightarrow r$
T	T	T	F	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

Question 71

Construct a truth table for the statement $\neg(p \wedge q) \oplus r$.

p	q	r	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \oplus r$
T	T	T	T	F	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	T
F	T	T	F	T	F
F	T	F	F	T	T
F	F	T	F	T	F
F	F	F	F	T	T

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Sub-Section [2.5.3]: Represent Logical Expressions using Switching Circuits and Logic Gates

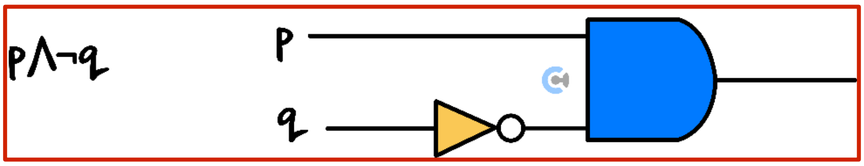
Question 72

Use logic gates to represent the following expression and draw the corresponding truth table:

$$p \wedge \neg q$$

Expression $p \wedge \neg q$:

p	q	$p \wedge \neg q$
1	1	0
1	0	1
0	1	0
0	0	0



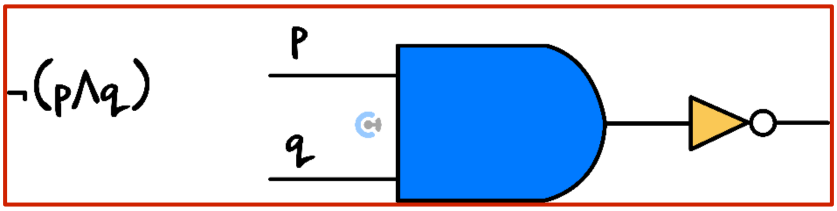
Question 73

Use logic gates to represent the following expression and draw the corresponding truth table:

$$\neg(p \wedge q)$$

Expression $\neg(p \wedge q)$:

p	q	$\neg(p \wedge q)$
1	1	0
1	0	1
0	1	1
0	0	1



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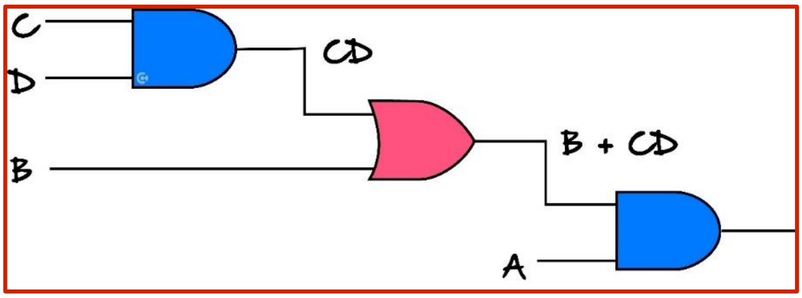
Question 74

Sketch a logic gate for the following expression:

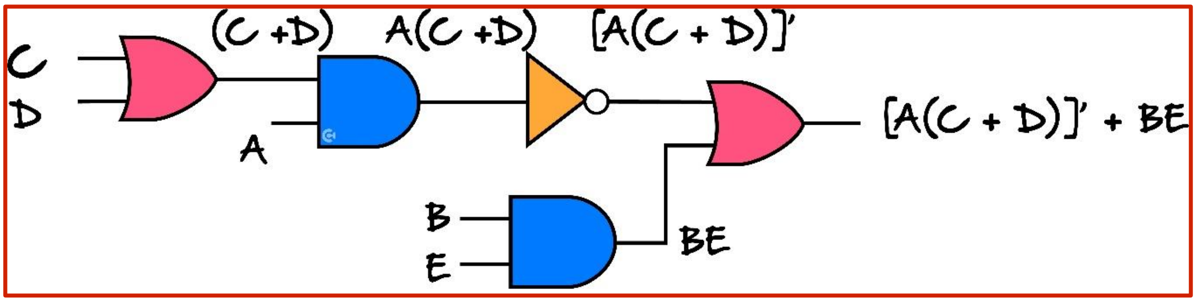
$$A(B + CD)$$



Question 75

Sketch a logic gate for the following expression:

$$[A(C + D)]' + BE$$



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Sub-Section [2.5.4]: Simplify and Evaluate Boolean Algebra Expressions using Algebraic Identities and Karnaugh Maps

Question 76

Simplify each expression by algebraic manipulation.

a. $\bar{a} \cdot 0 =$

0

b. $a + a =$

a

c. $a + \bar{a}b =$

$(a + \bar{a})(a + b) = a + b$

Space for Personal Notes

Question 77

Simplify each expression by algebraic manipulation.

a. $y + y\bar{y} =$

y

b. $xy + x\bar{y} =$

$x(y + \bar{y}) = x$

c. $\bar{x} + y\bar{x} =$

$\bar{x}(1 + y) = \bar{x}$

Space for Personal Notes

Question 78

Simplify each expression by algebraic manipulation.

a. $(w + \bar{x} + y + \bar{z})y =$

y

b. $(x + \bar{y})(x + y) =$

x

c. $w + (w\bar{x}yz) =$

$w(1 + \bar{x}yz) = w$

Question 79

Simplify the following expression by algebraic manipulation:

$$(x + z)(\bar{x} + y)(z + y) =$$

$xy + z\,\bar{x}$

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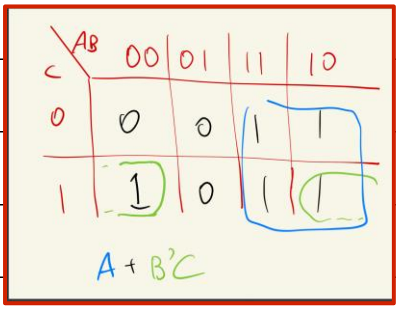
Question 80

Using a Karnaugh map, identify the Boolean expression corresponding to each of the following truth tables:

a.

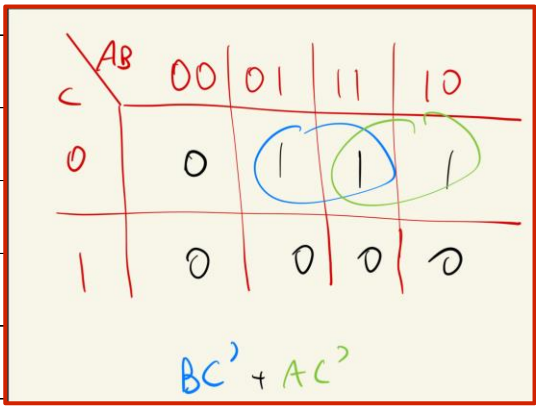
A	B	C	Result
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1



$A + B'C$

b.

A	B	C	Result
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0



$$(A + B)C' = AC' + BC'$$

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Section F: [2.1-2.5] - Exam 1 Overall (Checkpoints) (27 Marks)

Question 81 (3 marks)

Inspired from VCAA Specialist Mathematics Exam 1 2024
https://www.vcaa.vic.edu.au/Documents/exams/mathematics/2024/2024specmaths1-w.pdf#page=3

Prove that if x is an odd integer then $2x^2 - 3x - 7$ is even, using direct proof.

Marks	0	1	2	3	Average
%	8	4	23	65	2.5

As x is an odd integer, we may let $x = 2k+1$ where $k \in Z$.

$$\begin{aligned} 2x^2 - 3x - 7 &= 2(2k+1)^2 - 3(2k+1) - 7 \\ &= 2(4k^2 + 4k + 1) - 6k - 3 - 7 \\ &= 8k^2 + 2k - 8 \\ &= 2(4k^2 + k - 4) \text{ which is even} \end{aligned}$$

Alternatively, it may be observed directly that if x is odd then $2x^2$ is even, $3x$ is odd, and 7 is odd, so that $2x^2 - 3x - 7$ is the sum of one even and two odd integers, hence even.

This question was answered well by students. Substituting $2k+1$ (or $2k-1$) for x in the expression and obtaining $2(4k^2 + k - 4)$ (or $2(4k^2 - 7k - 1)$), hence a multiple of 2 and so even, was a reasonable approach. Occasional arithmetic or algebraic errors were seen.

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Question 82 (4 marks)

Inspired from VCAA Specialist Mathematics Exam 1 2024
https://www.vcaa.vic.edu.au/Documents/exams/mathematics/2024/NHT/2024SM1-nht-w.pdf#page=7

Prove by mathematical induction that $1 \times 7 + 2 \times 15 + 3 \times 23 + \cdots + n(8n - 1) = \frac{1}{6}n(n+1)(16n+5)$ for all $n \in N$.

For the case $n=1$ in the proposition, the left-hand side is $1\times 7=7$, and the right side is $\frac{1}{6}\times 2\times 21=7$.

Therefore, the proposition is true for $n=1$.

Assume that the proposition is true for $n=k$:

$$1\times 7+2\times 15+3\times 23+\ldots+k(8k-1)=\frac{1}{6}k(k+1)(16k+5)$$

Then the left-hand side with $n=k+1$ is

$$1\times 7+2\times 15+3\times 23+\ldots+k(8k-1)+(k+1)(8(k+1)-1)$$
$$=\frac{1}{6}k(k+1)(16k+5)+(k+1)(8(k+1)-1)$$
$$=(k+1)\left(\frac{1}{6}k(16k+5)+(8k+7)\right)$$
$$=\frac{1}{6}(k+1)(16k^2+53k+42)$$
$$=\frac{1}{6}(k+1)(k+2)(16k+21)$$

which is equal to the right-hand side with $n=k+1$.

Therefore, by the principle of mathematical induction, the proposition is true for all $n\in N$.

Question 83 (4 marks)

Prove using induction that for all $n \in \mathbb{N}$, $n < 2^n$.

We begin by verfiying the base case: If $n = 1$ then LHS $= 1 < 2 = 2^n =$ RHS. Now, assume that $n < 2^n$ for some $n \in \mathbb{N}$. We see that $n + 1 < 2^n + 1 < 2^n + 2^n = 2^{n+1}$, where we have also used the fact that $1 < 2^n$ for all $n \in \mathbb{N}$. Therefore, we have shown that $n + 1 < 2^{n+1}$. Using the principle of mathematical induction, we conclude that $n < 2^n$ for all $n \in \mathbb{N}$.

Question 84 (3 marks)

Consider the statement below:

There cannot exist two integers m and n such that $5m + 10n = 3$.

a. Write down a statement to begin a proof by contradiction for the statement above. (1 mark)

There exists $m, n \in \mathbb{Z}$ such that $5m + 10n = 3$.

b. Hence, obtain a contradiction and prove the original statement. (2 marks)

If there are two integers $m, n \in \mathbb{Z}$ so that $5m + 10n = 3$, then the left-hand side is divisible by 5, but the right-hand side is not divisible by 5, which is a contradiction. Therefore, there cannot exist two integers m and n such that $5m + 10n = 3$.

Question 85 (4 marks)

Prove using induction that $6^n + 4$ is divisible by 5 for all $n \in \mathbb{N}$.

Let $f(n) = 6^n + 4$.
Base Case: $f(1) = 6 + 4 = 10$ is divisible by 5.
Inductive step: Suppose that $f(k)$ is divisible by 5 for any $k \in \mathbb{N}$. We then have $f(k) = 5m$ for some $m \in \mathbb{N}$. Now,

$$\begin{aligned} f(k+1) - f(k) &= 6^{k+1} + 4 - (6^k + 4) \\ \implies f(k+1) &= 6^{k+1} - 6^k + 5m \\ &= 6^k(6-1) + 5m \\ &= 5(6^k + m) \\ &= 5p, \quad p \in \mathbb{N} \end{aligned}$$

therefore $f(k+1)$ is divisible by 5 and thus by the priniple of mathematical induction $f(n)$ is divisible by 5 for all $n \in \mathbb{N}$.

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Question 86 (4 marks)

Prove using induction that for all $n \in \mathbb{N}$, it holds that $\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$.

For the base case where $n = 1$, we see that the left-hand side is $1 + \frac{1}{1} = 2 = 1 + 1$, which is equal to the right-hand side. Therefore, the base case has been verified. Now, suppose that $\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$ for some $n \in \mathbb{N}$. We use the induction hypothesis to conclude

$$\begin{aligned}\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right) &= (n+1)\left(1+\frac{1}{n+1}\right)\\ &= (n+1)+1\\ &= n+2\end{aligned}$$

Therefore, by the principle of mathematical induction, it holds that

$$\left(1+\frac{1}{1}\right)\left(2+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$$

for all $n \in \mathbb{N}$.

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Question 87 (5 marks)

Prove the following biconditional statement for $x, y \in \mathbb{Z}$:

$$x + y \text{ is even, if and only if, } x^2 + y^2 \text{ is even.}$$

($\Rightarrow$) Suppose that $x + y$ is even. Then $x + y = 2n$ for some $n \in \mathbb{Z}$. Recall $(x+y)^2 = x^2+2xy+y^2$. Therefore, $x^2+y^2 = (x+y)^2-2xy = 4n^2-2xy = 2(2n^2-xy) = 2k$, where $k = 2n^2 - xy \in \mathbb{Z}$. Therefore, $x^2 + y^2$ is even.

($\Leftarrow$) We shall prove the reverse direction by proving its contrapositive: If $x + y$ is odd, then $x^2 + y^2$ is odd. Thus, assume that $x + y = 2n + 1$ for some $n \in \mathbb{Z}$. Similar to above, $x^2+y^2 = (x+y)^2-2xy = (2n+1)^2-2xy = 4n^2+4n+1-2xy = 2(2n^2+2n-xy)+1 = 2k+1$, where $k = 2n^2 + 2n - xy \in \mathbb{Z}$. Therefore, $x^2 + y^2$ is odd.

Therefore, $x + y$ is even if and only if $x^2 + y^2$ is even.

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Section G: [2.1-2.5] - Exam 2 Overall (Checkpoints) (7 Marks)

Question 88 (1 mark)

Inspired from VCAA Specialist Mathematics Exam 2 2024
https://www.vcaa.vic.edu.au/Documents/exams/mathematics/2024/2024specmaths2-w.pdf#page=2

Consider the statement:

'For any integers m and n, if $m + n \geq 9$ then $m \geq 5$ or $n \geq 5$'.

The contrapositive of this statement is:

Asked for contrapositive – therefore, switch the hypothesis and the conclusion and negate both.

A. If $m < 5$ or $n < 5$, then $m + n < 9$.

B. If $m \geq 5$ or $n \geq 5$, then $m + n \geq 9$.

C. If $m < 5$ and $n < 5$, then $m + n < 9$.

D. If $m \leq 5$ and $n \leq 5$, then $m + n \leq 9$.

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Question 89 (1 mark)

Inspired from VCAA Specialist Mathematics Exam 2 2024
https://www.vcaa.vic.edu.au/Documents/exams/mathematics/2024/NHT/2024SM2-nht-w.pdf#page=2

Consider the following proof:

$$\text{Prove that } \sqrt{15} + \sqrt{7} > \sqrt{19}.$$
$$\text{Assume } \sqrt{15} + \sqrt{7} \leq \sqrt{19}.$$
$$\text{Then } \left(\sqrt{15} + \sqrt{7}\right)^2 \leq 19$$
$$15 + 2\sqrt{105} + 7 \leq 19$$
$$2\sqrt{105} \leq -3$$
$$\text{Hence, } \sqrt{15} + \sqrt{7} > \sqrt{19}.$$

This proof can be best described as a:

A. Direct proof.

B. Proof by contrapositive.

C. Proof by contradiction.

D. Proof by counter-example.

E. Proof by mathematical induction.

Question 90 (1 mark)

The contrapositive to the statement, "If n is even, then n^2 is even." is:

A. If n^2 is odd, then n is odd.

B. If n^2 is even, then n is even.

C. If n is odd, then n^2 is even.

D. If n is odd, then n^2 is odd.

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Question 91 (1 mark)

Consider the following:

$$\text{For all } k > K, 1.5^k < (k-1)!$$

What is the smallest value of $K \in \mathbb{N}$ such that the above holds?

A. 2

B. 3

C. 4

D. 5

Question 92 (1 mark)

The negation of the statement, "All the cars in the carpark are black." is:

A. All the vans in the carpark are black.

B. There exists a car in the carpark that is not black.

C. There exists a bus in the carpark without a mirror.

D. All the cars in the carpark are yellow.

Question 93 (1 mark)

Find the minimum value of $6x^2 + \frac{6}{x^2}$.

A. 6

B. 25

C. 15

D. 12

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Question 94 (1 mark)

Consider the following statement:

If a car in the carpark is black, then it costs a lot of money.

Which of the following is the converse of the above?

A. If a bus in the carpark costs a lot of money, then it is not black.

B. If a car costs a lot of money, then it is in the carpark.

C. If a car in the carpark is not black, then it costs a lot of money.

D. If a car in the carpark costs a lot of money, then it is black in colour.

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