1.4

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VCE Specialist Mathematics ½
Sequence and Series Exam Skills [1.4]
Workbook

Outline:

Section A: Recap

Sub-Section: Sequences

Sequences

t_n 'n'th term

$$t_n = f(n)$$

- **Definition:** A sequence is an ordered list of numbers following a certain **pattern.**
- It is a function of the order n.

Question 1

The sequence is defined by $t_n = 3^n + 1$. Identify the term number for which t_n equals 10.

$3^n + 1 = 10$

$3^n = 9$

$n = 2$

$t_2 = 10$

2^{nd} term.

Sub-Section: Recurrence Relations

What if we define the term t_n with respect to the previous term (t_{n-1})?

Recurrence Relations

- **Definition:**
  - A recurrence relation is when we define **a term** (t_n), in terms of the **previous one** (t_{n-1}).
  - Recurrence relations generate sequences of the form:

$$t_n = f(t_{n-1}) \text{ where } t_1 = a$$

Or

$$t_{n+1} = f(t_n) \text{ where } t_1 = a$$

  - It must always include a first term.

Question 2

Consider the following recurrence relation:

$$t_n = 4t_{n-1} - 1, \; t_1 = 3$$

State the value of t_3.

$t_1 = 3$

$t_2 = 4 \times 3 - 1$
$= 11$

$t_3 = 4 \times 11 - 1$
$= 43$

Sub-Section: Introduction to Series

What does the word "series" mean?

Series

- **Definition:**
  - A series is the sum of the first n terms of a sequence.

$$S_n = \sum_{i=1}^{n} t_i$$

Question 3

Consider the sequence given by $t_n = 2n + 4$.

Evaluate S_4.

$$S_4 = \sum_{i=1}^{4} t_i$$

$$= t_1 + t_2 + t_3 + t_4$$

$$= (2+4) + (4+4) + (6+4) + (8+4)$$

$$= 36$$

Sub-Section: Introduction to Arithmetic Sequence

Arithmetic Sequences

$$a, \quad a+d, \quad a+2d, \quad a+3d$$

2

- **Definition:**
  - An **arithmetic sequence** is one where the **common difference** is added or subtracted to get the next term.

t_1 t_2 t_3

$$t_n = a + (n-1)d$$

  - Where d is the common difference, and a is the first term.

Question 4

Consider the arithmetic sequence defined by $t_n = 4n - 3$.

Identify the common difference, first term and the 8^{th} term.

$a = t_1 = 1$

$d = 4$

$t_8 = 32 - 3 = 29$

NOTE: Read the question carefully. Sometimes, they expand the $n-1$ factor to confuse you.

Sub-Section: Arithmetic Recurrence Relation

What about recurrence relations for arithmetic sequence?

Formula: Recurrence Relation for Arithmetic Sequence

$$t_n = t_{n-1} + d \text{ where } t_1 = a$$

Question 5

Consider the following n^{th} term rule for the arithmetic sequence:

$$t_n = 3 + 2n$$

Find the recurrence relation which corresponds to it.

$a = t_1 = 5$

$d = 2$

$t_{n+1} = t_n + 2$

$t_n = t_{n-1} + 2$

where $t_1 = 5$

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Sub-Section: Arithmetic Mean

The Arithmetic Mean

- **Definition:**

$$\textbf{Arithmetic Mean of } a \textbf{ and } b = \frac{a+b}{2}$$

NOTE: The arithmetic mean is the same as the so-called "normal average".

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Sub-Section: Arithmetic Series

Arithmetic Series (Form 1)

- Use the following formula, if we know the first term, last term and number of terms.

$$S_n = \frac{n}{2}(a + l)$$

$1 + 2 + 3 \ldots + 99 + 100$

- Where n = number of terms, a = first term and l = last term.
- $\frac{n}{2}$ can be thought of as the number of pairs
- $a + l$ can be thought of as the sum of each pair

Question 6

Consider the arithmetic sequence with $t_1 = 2$ and $t_9 = 26$.

Find S_9.

$$S_9 = \frac{9}{2} \times (2 + 26)$$

$$= \frac{9}{2} \times 28$$

$$= 126$$

$$\begin{array}{r} 14 \\ \times\ 9 \\ \hline 126 \end{array}$$

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Now, let's generalise it for all arithmetic sequences!

Arithmetic Series (Form 2)

- Use the following formula, if we know the first term, common difference and number of terms.

$$S_n = \frac{n}{2}(2a + d(n-1))$$

- Where n = number of terms, a = first term and d = common difference.

Question 7

Consider the arithmetic sequence with $t_n = 5n - 7$.

$a = -2$
$d = 5$

Find S_{11}.

$$S_{11} = \frac{11}{2}(2(-2) + 5(10))$$

$$= \frac{11}{2}(46)$$

$$= 253$$

23×11: 23, 230, 253

Sub-Section: Geometric Sequence

Now, let's consider another type of sequence, "Geometric" sequences.

Geometric Sequences

$$a, \quad ar, \quad ar^2, \quad ar^3$$

- **Definition:**
  - A Geometric sequence is one where we keep multiplying or dividing by the common ratio to get the next term.

$$t_n = ar^{n-1}$$

  - Where r is the common ratio, and a is the first term.

NOTE: Geometric sequence is an exponential!

Question 8

Consider the geometric sequence defined by $t_n = 2 \cdot (3)^n$.

Identify the common ratio, first term and the 3rd term.

$a = t_1 = 6$

$r = 3$

$t_3 = 2 \cdot 3^3 = 54$

NOTE: Read the question carefully. Sometimes, they expand the $n-1$ power to confuse you!

Sub-Section: Geometric Recurrence Relation

What about recurrence relations for geometric sequence?

Recurrence Relation for Geometric Sequence

$$t_n = t_{n-1} \times r \text{ where } t_1 = a$$

Question 9

Consider the following n^{th} term rule for the geometric sequence:

$$t_n = 3 \cdot 2^{n-1}$$

Find the recurrence relation which corresponds to it.

$a = 3, \quad r = 2$

$t_{n+1} = t_n \times 2$

$t_n = t_{n-1} \times 2$

where $t_1 = 3$

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Sub-Section: Geometric Mean

How do we find a geometric term between two other geometric terms?

The Geometric Mean

- **Definition:** The **geometric mean** of two numbers a and b is the geometric term in between a and b.

$$a \qquad \sqrt{ab} \qquad b$$

$$\textbf{Geometric Mean of } a \textbf{ and } b = \sqrt{ab}$$

TIP: Remember the similarity!

- Arithmetic Mean: We add the two and divide by 2.
- Geometric Mean: We multiply the two and square root.

Question 10

a. Find the geometric mean of 5 and 45.

$= \sqrt{5 \times 45}$ $= \sqrt{225}$

$= 15$

$5 \times 3 = 15$

b. Explain in words why 15 being a geometric mean makes sense.

$5 \xrightarrow{\times 3} 15 \xrightarrow{\times 3} 45$

Sub-Section: Geometric Series

Geometric Series

- **Definition:** The sum of the first n geometric terms is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

- Where n = number of terms, a = first term and r = common ratio.

Question 11

Consider the geometric sequence with $t_n = 6 \cdot \left(\frac{1}{2}\right)^n$.

Find S_6.

$a = 3$
$r = \frac{1}{2}$

$$S_6 = \frac{3\left(\left(\frac{1}{2}\right)^6 - 1\right)}{\frac{1}{2} - 1} = \frac{3\left(\frac{1}{64} - 1\right)}{-\frac{1}{2}}$$

$$= -\cancel{6}^{\,3}\left(-\frac{63}{\cancel{64}_{\,32}}\right)$$

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$$= \frac{189}{32}$$

Sub-Section: Infinite Geometric Series

The Infinite Geometric Series

- **Definition:** The sum of infinitely many geometric terms is given by: $\frac{1}{2}$

$$S_\infty = \frac{a}{1-r}$$

- **IMPORTANT:** Only works when $-1 < r < 1$.

Question 12

Identify the first term, common ratio and hence, find the sum of the infinite series.

$$\underbrace{1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots}_{\text{series (sum of sequence)}}$$

$\times ?$ $\frac{2}{3}$

I'm just

$a = 1$

$r = 2/3$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-2/3} = 3$

I'm just

fuck ~~cunt~~

NOTE: The common ratio must be between -1 and 1 for an infinite series to be a finite number.

Section B: Sequences & Series Exam Skills

Sub-Section: Find Arithmetic Sequence

Finding Arithmetic Sequence

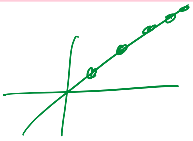
- We can simply take a simultaneous equation approach.
- Or use the formula:

common difference

$$\frac{t_k - t_l}{k - l} = d, \text{ where } k > l$$

Discussion: What does the above formula look like?

Gradient



Question 13 Walkthrough.

Define the arithmetic sequence in terms of n if $t_3 = 10$ and $t_{14} = 43$.

$$\frac{43 - 10}{14 - 3} = d$$

$$\frac{33}{11} = d, \quad d = 3$$

$$t_n = a + d(n-1)$$

$$10 = a + d(2)$$

$$10 = a + 6, \quad a = 4$$

$$t_n = a + d(n-1)$$

$$\boxed{10 = a + 2d} \quad \text{— } ①$$

$$43 = a + 13d \quad \text{— } ②$$

② − ①

$$11d = 33, \quad d = 3$$

⋮

$$t_n = 4 + 3(n-1)$$

Active Recall: Common Difference Formula

$$d = \frac{t_k - t_l}{k - l}, \text{where } k > l$$

Question 14

Define the arithmetic sequence in terms of n if $t_2 = 10$ and $t_{20} = -44$.

$t_n = a + d(n-1)$

$t_2 = 10 \quad (n=2)$

$10 = a + d \quad - (1)$

$t_{20} = -44 \quad (n=20)$

$-44 = a + 19d \quad - (2)$

$(2) - (1)$

$18d = -54$

$d = -3$

Sub $d = -3$ into (1)

$10 = a - 3$

$a = 13$

$= 13 - 3(n-1)$

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https://www.youtube.com/embed/7rnwlEjH2HI

Sub-Section: Find Geometric Sequence

Now, Geometric Sequence!

Finding Geometric Sequence

- We can simply take a simultaneous equation approach.
- Or use the formula:

$$\left(\frac{t_k}{t_l}\right) = r^{k-l}, \text{where } k > l$$

Question 15 Walkthrough.

Define the geometric sequence in terms of n if $t_3 = 12$ and $t_6 = 96$.

$t_n = a r^{n-1}$

$t_3 = 12$

(1) $12 = a r^2$

$t_6 = 96$

(2) $96 = a r^5$

(2) ÷ (1)

$r^3 = 8$

$r = 2$

Sub $r=2$ into (1)

$12 = 4a, \quad a = 3$

$t_n = 3 \times 2^{n-1}$

Active Recall: Common Ratio Formula

$r =$ ________________, *where* $k > l$

Question 16

Define the geometric sequence in terms of n if $t_2 = \frac{2}{3}$ and $t_5 = \frac{2}{81}$.

$$t_n = ar^{n-1}$$

$t_2 = 2/3$

(1) $\frac{2}{3} = ar$

$t_5 = 2/81$

(2) $\frac{2}{81} = ar^4$

(2) $\div$ (1)

$\frac{2}{81} \div \frac{2}{3} = r^3$

$\frac{2}{81} \times \frac{3}{2} = r^3$

$r^3 = \frac{1}{27} \Rightarrow r = 1/3$

Sub $r = 1/3$ in (1)

$\frac{2}{3} = a\frac{1}{3}$

$a = 2$

$t_n = 2 \times \left(\frac{1}{3}\right)^{n-1}$

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Section C: Exam 1 (20 Marks)

Question 1 (4 marks)

$a = 4,\ d = 3$

Consider the arithmetic sequence $t_n = t_{n-1} + 3$ and $t_1 = 4$.

a. Find t_{10}. (1 mark)

$t_1 = 4 \quad t_2 = 7 \quad \times t_3$

$t_n = a + (n-1)d$

$t_{10} = 4 + 9 \times 3 = 31$

b. Find the arithmetic mean of t_4 and t_{10}. (1 mark)

$t_{10} = 31$ (prev qu.)

$t_4 = 4 + 3 \times 3 = 13$

$\text{Mean} = \frac{31+13}{2} = 22$

c. Find the value of x if the arithmetic mean of t_5 and t_x is 49. (2 marks)

$t_5 = 4 + 12 = 16$

$\frac{t_5 + t_x}{2} = 49, \quad \Rightarrow t_x = 82$

$a + (n-1)d$

$4 + (x-1)3 = 82$

$3(x-1) = 78$

$x - 1 = 26$

$x = 27$

Question 2 (4 marks)

It is known that two terms in a geometric sequence are $t_2 = 12$ and $t_5 = 96$.

a. Find the value of r. (2 marks)

$t_n = ar^{n-1}$

$t_2 = 12$ $t_5 = 96$

1 $ar = 12$ 2 $ar^4 = 96$

2 ÷ 1

$r^3 = 8$

$\boxed{r = 2}$

b. Hence, find t_n in terms of n. (1 mark)

Sub $r = 2$ into 1

$2a = 12$

$a = 6$

$t_n = 6 \times (2)^{n-1}$

c. Find the geometric mean of t_1 and t_5. (1 mark)

$t_1 =$ $t_5 =$

$t_3 = 6 \times (2)^2 = 24$

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Question 3 (5 marks)

Consider the following recursive relation:

$$t_n = \frac{2}{3}t_{n-1}, \text{ where } t_1 = 6.$$

$r = \frac{2}{3}$

$a = 6$

Geometric ($\times \frac{2}{3}$ to get the next term)

a. Define t_n in terms of n. (2 marks)

$t_n = 6 \times \left(\frac{2}{3}\right)^{n-1}$

b. Find S_3. (2 marks)

$S_3 = t_1 + t_2 + t_3$

$S_3 = \frac{a(r^n - 1)}{r - 1}$

$= 6 + \frac{12}{3} + \frac{8}{3}$

$= \frac{18}{3} + \frac{20}{3} = \frac{38}{3}$

c. Find S_∞. (1 mark)

$S_\infty = \frac{a}{1-r} = \frac{6}{1-2/3}$

$= \frac{6}{1/3} = 18$

Question 4 (4 marks)

Consider the arithmetic sequence with $t_6 = 3$ and $t_{10} = 15$.

a. Find the rule for the sequence in the form $t_n = a + (n-1)d$. (2 marks)

$t_n = a + (n-1)d$

Using gradient formula

$d = \frac{15-3}{10-6} = \frac{12}{4} = 3$

Using the fact $t_6 = 3$

$3 = a + 5d$

$3 = a + 15$

$a = -12$

$t_n = -12 + 3(n-1)$

b. Find the values of t_{15} and S_{15}. (2 marks)

$t_{15} = -12 + 14(3) = 30$

$S_{15} = \frac{n}{2}(2a + \cancel{d}(n-1))$

$= \frac{n}{2}(a + \ell)$

$= \frac{15}{2}(-12+30)$

$= \frac{15}{2} \times 18 = 135$

$n = 15$
$a = -12$
$\ell = 30$

Question 5 (3 marks)

The sum of the first 4 terms of a geometric is 10 times the sum of the first 2 terms and $t_1 = \frac{3}{2}$.

Find a rule for the series t_n.

$S_n = \frac{a(r^n - 1)}{r-1}$

$S_4 = 10 S_2$

$\frac{S_4}{S_2} = 10$

$\frac{\frac{a(r^4-1)}{r-1}}{\frac{a(r^2-1)}{r-1}} = 10$

$\frac{r^4-1}{r^2-1} = 10$

$\frac{(r^2+1)(r^2-1)}{r^2-1} = 10$

$r^2 + 1 = 10$

$\boxed{r = \pm 3}$

(rej −ve)

$S_4 = 10S_2$

$t_n = \frac{3}{2} \times (3)^{n-1}$

Section D: Tech Active Exam Skills

Calculator Commands: Generating sequences

- **TI-Nspire**
  - seq() and seqn() functions:

```
seq(Expr, Var, Low, High [,Step])
```

```
seqn(Expr(u, n) [, ListOfInitTerms [, nMax [,
```

  Must use seqn for recursive sequences.

  - E.g. Find the first 10 terms of the sequence $t_n = 2n + 5$.
  - Find the first 5 terms of the sequence $t_n = t_{n-1} + 3$ where $t_1 = 2$.
  - **NOTE:** Must use the variables u and n for this function.

- **Mathematica**
  - Table[] and RecurrenceTable[] commands.
  - E.g. Find the first 10 terms of the sequence $t_n = 2n + 5$.

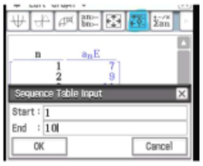
```
In[67]:= Table[2 n + 5, {n, 1, 10}]
Out[67]= {7, 9, 11, 13, 15, 17, 19, 21, 23, 25}
```

  - E.g. Find the first 5 terms of the sequence $t_n = t_{n-1} + 3$ where $t_1 = 2$.

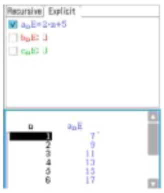
```
In[70]:= RecurrenceTable[{t[n] == t[n - 1] + 3, t[1] == 2}, t, {n, 1, 5}]
Out[70]= {2, 5, 8, 11, 14}
```

- **Casio Classpad**
  - Go to Sequences from the Menu.

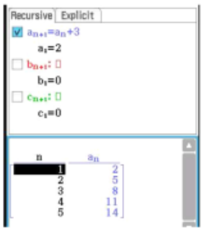
  - Pick between Recursive or Explicit sequence definition.
  - Change sequence range using:



  - E.g. Find the first 10 terms of the sequence $t_n = 2n + 5$.



- E.g. Find the first 5 terms of the sequence $t_n = t_{n-1} + 3$ where $t_1 = 2$.



Calculator Commands: Calculating sums

TI-Nspire

- Use the big Sigma found.

- E.g. Calculate the sum of the first 5 terms of the sequence $t_n = 2n + 3$.

$\sum_{n=1}^{5}(2\cdot n+3)$ 45

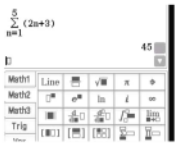
Mathematica

- Use the Sum[] command.
- E.g. Calculate the sum of the first 5 terms of the sequence $t_n = 2n + 3$.

```
In[71]:= Sum[2 n + 3, {n, 1, 5}]
Out[71]= 45
```

Casio Classpad

- Find the big Sigma under Math2.
- E.g. Calculate the sum of the first 5 terms of the sequence $t_n = 2n + 3$.



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Section E: Exam 2 (19 Marks)

Question 6 (1 mark)

Write an equation for the n^{th} term of the given arithmetic sequence.

$$2,7,12,17,\ldots$$

$t_2 = 7$

$n=1$

A. $a_n = 5n - 2$

B. $a_n = 6n + 1$

C. $a_n = 5n - 3$

$t_1 = 2$

D. $a_n = 7n - 5$

Question 7 (1 mark)

A tennis ball is dropped from a height of 10 metres. After the ball hits the floor, it rebounds to 80% of its previous height.

How high will the ball rebound after its third bounce? Round to the nearest tenth of a metre.

A. 5.1 metres

$a = 8$

B. 6.4 metres

$r = 0.8$

C. 4.8 metres

D. 3.2 metres

$t_n = 8 \times 0.8^{n-1}$

$t_3 =$

Question 8 (1 mark)

Write an equation for the n^{th} term of the given geometric sequence.

$\times 3$ $\times 3$

$$3,9,27,\ldots,a_n$$

$t_1 = 3$

A. $3\left(\frac{1}{3}\right)^{n+1}$

B. $3(3)^{n-1}$

C. $3(2)^{n-1}$

D. $9(3)^{n-1}$

Question 9 (1 mark)

A sequence is generated by the rule $t_{n+1} = 2t_n + 5$, for $n \in \mathbb{N}$. If the third term of the sequence is 15, then the first term is:

$t_n = \frac{t_{n+1} - 5}{2}$

$t_2 = \frac{15 - 5}{2} = 5$

$t_1 = \frac{5 - 5}{2} = 0$

A. 10

B. 5

C. 0

D. -5

Question 10 (1 mark)

$a = 10$
$r = \frac{1}{2}$

Consider the sequence $10, 5, 2.5, \ldots$. Let S_n be the sum of the first n terms of the sequence.

$S_\infty - S_4$ is equal to:

$\frac{a}{1-r} - \frac{a(r^4 - 1)}{r - 1}$

A. $\frac{5}{4}$

B. 5

C. $\frac{5}{2}$

D. $\frac{28}{16}$

Question 11 (1 mark)

For the arithmetic sequence $-12, -9, -6, \ldots$, where $t_1 = -12$, there is an n such that $t_{n-k} + t_{n+k} = 0$ for all $k \in \mathbb{N}$. What is the value of n?

1, 2, 3, 4
$-12, -9, -6, -3, 0, 3, 6, 9, 12$

A. 2

B. 5

C. 6

D. 4

Question 12 (1 mark)

Find the sum of all numbers divisible by 7 between 200 and 600 inclusive.

A. 23503

B. 22665

C. 22743

D. 28620

$\frac{203}{7}$ 29

$\frac{600-200}{7}$ 57.1429

$a = 203$

$n = 57$

$d = 7$

$S_n = \frac{n}{2}\left(2a + d(n-1)\right)$

Question 13 (6 marks)

In the year 2002, a shop sold 120 computers. Each following year, the shop sells 20 more computers than the year before, forming an arithmetic sequence, 120, 140, 160, ...

a. Find how many computers the shop sold in 2010. (1 mark)

b. Calculate the total number of computers sold by the shop from 2002 to 2015 inclusive. (2 marks)

In the year 2002, the selling price of each computer was $1260. The selling price falls by $30 each year forming an arithmetic sequence, $1260, 1230, 1200, \ldots$.

c. In a particular year, the selling price of each computer in dollars was equal to three times the number of computers sold in the shop that year. Find the year in which this occurred. (3 marks)

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Question 14 (6 marks)

Jacob runs a farm where he keeps chickens. He plans to increase the number of chickens he has each year. The number of chickens on the farm will form a geometric sequence with a common ratio of 1.5 (where any non-integer result is rounded up to the nearest integer). Jacob initially has 12 chickens on the farm (year 0).

a.

i. Write an expression that can be used to predict the number of chickens on the farm after n years. (1 mark)

ii. How many chickens will be on the farm after 2 years? (1 mark)

iii. After how many whole years will the number of chickens exceed 200? (1 mark)

b. Let C_n be the number of chickens on the farm after n years. Write a recursive equation in terms of C_{n+1} and C_n, that can be used to predict the number of chickens on the farm after n years. (1 mark)

Jacob also has some ducks on the farm.

c. The following are more realistic models for the number of chickens (C_n) and ducks (D_n) on Jacob's farm after n years:

$$C_{n+1} = 1.4C_n - 5, C_0 = 24$$

and

$$D_{n+1} = 1.8D_n - 7, D_0 = 10$$

Find the number of years it will be before the number of ducks on the farm first exceeds the number of chickens on the farm. (2 marks)

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