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VCE Specialist Mathematics ½
Sequences & Series Exam Skills [1.4]
Homework Solutions

Homework Outline:

Compulsory Questions	Pg 2 – Pg 18
Supplementary Questions	Pg 19 – Pg 38

Section A: Compulsory Questions

Sub-Section [1.4.1]: Find Sequences From Two Terms

Question 1

Define the arithmetic sequence in terms of n if $t_6 = 12$ and $t_{12} = 30$.

The difference is $\frac{30-12}{12-6} = 3$, thus our sequences is $t_n = a + 3(n-1)$

We can solve for a since if $n = 6$, then $t_n = 12$, thus $a = -3$

Hence our sequence is $t_n = -6 + 3n$

Question 2

Define two possible geometric sequences in terms of n if $t_3 = -\frac{2}{3}$ and $t_5 = -\frac{2}{27}$.

Our ratio r satisfies, $r^2 = \left(-\frac{2}{27}\right) \div \left(-\frac{2}{3}\right) = \frac{1}{9}$. Hence $r = \frac{1}{3}$ or $r = -\frac{1}{3}$.

If $r = \frac{1}{3}$ our sequence is $t_n = a\left(\frac{1}{3}\right)^{n-1}$ We can solve for a since $t_3 = -\frac{2}{3}$, thus $a = -6$

and our sequence could be $t_n = -18\left(\frac{1}{3}\right)^n$.

If $r = -\frac{1}{3}$ our sequence is $t_n = a\left(-\frac{1}{3}\right)^{n-1}$ We can solve for a since $t_3 = -\frac{2}{3}$, thus $a = -6$

and our sequence could be $t_n = 18\left(-\frac{1}{3}\right)^n$.

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Question 3

Let g_n be a geometric sequence and a_n be an arithmetic sequence.

It is known that $g_1 = a_1 = 2$ and that $g_2 = a_2$ and $g_3 = a_4$.

Describe g_n and a_n in terms of n.

We know that $a_n = 2 + d(n-1)$ and $g_n = 2r^{n-1}$.
From $g_2 = a_2$ and $g_3 = a_4$ we get the following pair of simultaneous equations.

$$2 + d = 2r \qquad (1)$$
$$2 + 3d = 2r^2 \qquad (2)$$

Subtracting $3 \times (1)$ from (2) yields, $-4 = 2r^2 - 6r$.
We solve this equation, $r^2 - 3r + 2 = (r-2)(r-1) = 0$ to get $r = 2$ or $r = 1$.
If $r = 1$, then $2 + d = 2r$ and $d = 0$. Thus $a_n = g_n = 2$ for all values of n.
If $r = 2$, then $2 + d = 4 \implies d = 2$. Thus $a_n = 2n$ and $g_n = \boxed{2^n}$

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Sub-Section [1.4.2]: Apply Recurrence Relation To Different Types of Sequences

- We define the Fibonacci Sequence f_t via the following recursive definition.

$$f_1 = f_2 = 1$$

$$f_{n+1} = f_n + f_{n-1} \quad \text{for } n \geq 2$$

Question 4

a. Find f_5.

$f_3 = 1 + 1 = 2$, $f_4 = 1 + 2 = 3$ and $f_5 = 2 + 3 = 5$.

b. Find $f_5 + f_6 + f_7 + f_8$.

$f_6 = 3 + 5 = 8$, $f_7 = 5 + 8 = 13$ and $f_8 = 13 + 8 = 21$. Thus,

$$f_5 + f_6 + f_7 + f_8 = 5 + 8 + 13 + 21 = 47$$

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Question 5

a. Show that f_n is not an arithmetic sequence.

Observe that $f_3 - f_2 = f_1 = 1$ and $f_2 - f_1 = 0$. Since the difference between two pairs of consecutive terms is not the same, f_n cannot be an arithmetic sequence.

b. Show that f_n is not a geometric sequence.

Observe that $f_3 \div f_2 = 2$ and $f_2 \div f_1 = 1$ Since the ratio between two pairs of consecutive terms is not the same, f_n cannot be a geometric sequence.

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Question 6

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ be roots of the equation $x^2 = x + 1$.

a. Show that $\phi^{n+1} = \phi^n + \phi^{n-1}$ (Hint: Use the fact that ϕ satisfies the equation $x^2 = x + 1$).

Since ϕ is a root of the equation $x^2 = x + 1$ we know that $\phi^2 = \phi + 1$.
We can multiply this expression by ϕ^{n-1} to get, $\phi^{n+1} = \phi^n + \phi^{n-1}$.

b. Show that $a\phi^{n+1} + b\psi^{n+1} = a(\phi^n + \phi^{n-1}) + b(\psi^n + \psi^{n-1})$ for all real a, b.

ψ is also a root of the equation $x^2 = x + 1$, hence by a similar logic as in part a, we know that $\psi^{n+1} = \psi^n + \psi^{n-1}$.
Now we can multiply the ϕ equation by a to get $a\phi^{n+1} = a\phi^n + a\phi^{n-1}$, and multiply the ψ equation by b to get $b\psi^{n+1} = b\psi^n + b\psi^{n-1}$.
We add the two equations together to get $a\phi^{n+1} + b\psi^{n+1} = a(\phi^n + \phi^{n-1}) + b(\psi^n + \psi^{n-1})$.

c. Hence, show that $f_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$.

We let $f_n = a\phi^n + b\psi^n$. From part **b** we know that $f_{n+1} = f_n + f_{n-1}$, thus we simply need to set a and b to satisfy $f_1 = f_2 = 1$.
Since,

$$f_1 = a\frac{1+\sqrt{5}}{2} + b\frac{1-\sqrt{5}}{2} = 1$$

$$\text{and} \qquad f_2 = a\left(\frac{1+\sqrt{5}}{2} + 1\right) + b\left(\frac{1-\sqrt{5}}{2} + 1\right) = 1$$

We can subtract the first equation from the second to get $0 = a + b \implies a = -b$. Now we substitute this back into the first equation to get,

$$a\frac{1+\sqrt{5}}{2} - a\frac{1-\sqrt{5}}{2} = \sqrt{5}a = 1 \implies a = \frac{1}{\sqrt{5}} \implies b = -\frac{1}{\sqrt{5}}$$

Hence $f_n = \dfrac{\phi^n - \psi^n}{\sqrt{5}}$.

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Sub-Section: Exam 1 Questions

Question 7 (5 marks)

Consider an arithmetic sequence a_n with the following properties:

$$a_1 = 5, a_2 = 8, a_p = 23$$

a. In closed form, $a_n = b \times n + d$. Find the values of b and d. (2 marks)

$b = a_2 - a_1 = 3$, and $5 = a_1 = 3 + d$, hence $d = 2$.

b. Hence, find the value of p. (1 mark)

We know that $a_p = 3p + 2 = 23 \implies 3p = 21 \implies p = 7$.

c. Let $s_n = \sum_{i=1}^{n} a_i$ be the series corresponding to the sequence a_n.

Find the value of s_{10}. (2 marks)

We know that $s_n = \frac{n}{2}(a_{10} + a_1) = 5(32 + 5) = 185$

Question 8 (5 marks)

Consider the following sequence, $g_n = 2 \times \left(-\frac{1}{3}\right)^n$.

a. Evaluate the geometric mean of $g_1, \ldots, g_9$. (2 marks)

$$\sqrt[9]{\prod_{n=1}^{10} 2 \times \left(-\frac{1}{3}\right)^n} = 2 \times \sqrt[9]{\left(-\frac{1}{3}\right)^{\sum_{n=1}^{9} n}} = 2 \times \left(-\frac{1}{3}\right)^{\frac{45}{9}} = -\frac{2}{243}$$

b. Evaluate $g_3 + g_4 + \dots$. (2 marks)

The common ratio is $-\frac{1}{3}$ and the term g_3 is equal to $-\frac{2}{27}$. Thus our sum is,

$$-\frac{2}{27} \times \frac{1}{1-(-\frac{1}{3})} = -\frac{1}{18}$$

c. The arithmetic mean of $g_1, \dots, g_n$ tends towards some number as $n \to \infty$. Find that number. (1 mark)

Since $g_1 + \dots + g_n$ tends towards some number (not infinity). And our arithmetic mean is this number divided by n, the arithmetic mean must tend towards 0 as $n \to \infty$.

Question 9 (3 marks)

Let a_n and g_n be an arithmetic and geometric sequence respectively.

Show that if $a_1 = g_1$, $a_2 = g_2$ and $a_3 = g_3$, then $a_n = g_n = a_1$ for all natural numbers n.

We will let $a_n = d(n-1) + a_1$ and $g_n = g_1 r^{n-1}$.
Since $a_1 = g_1$ and $a_2 = g_2$, we know that $a_2 = d + a_1 = g_2 = a_1 r$.
Similarly, since $a_3 = g_3$, we know that $2d + a_1 = a_1 r^2$.
Subtracting the second equation from 2 times the first equation yields, $a_1 = 2a_1 r - a_1 r^2 \implies r^2 - 2r + 1 = (r-1)^2 = 0$. Hence $r = 1$.
Substituting this into $d + a_1 = a_1 r$ yields $d + a_1 = a_1 \implies d = 0$.
Since $r = 1$ we see that $g_n = g_1 = a_1$. Since $d = 0$, we see that $a_n = a_1$.

Question 10 (4 marks)

Consider a geometric sequence $g_n = a \times r^n$.

a. Show that $g_1 + g_2 + ... + g_n = \frac{a(r - r^{n+1})}{1-r}$. (2 marks)

$$\begin{aligned} g_1 + g_2 + \cdots + g_n &= a(r^1 + r^2 + \cdots + r^n) \\ &= a \times \frac{(r + r^2 + \cdots + r^n)(1-r)}{1-r} \\ &= a\frac{r - r^{n+1}}{1-r} \end{aligned}$$

b. Use the above formula to evaluate, $1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{10}}$. (1 mark)

$a = 2$, $r = \frac{1}{2}$ and $n = 11$. Thus,

$$1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{10}} = 2 \times \frac{\frac{1}{2} - \frac{1}{2^{11}}}{1 - \frac{1}{2}} = \frac{2^{12} - 4}{2^{11}}$$

c. Explain why $-1 < r < 1$ for us to be able to evaluate the series $g_1 + g_2 + \ldots$. (1 mark)

If $-1 < r < 1$ then we can replace n with ∞ in the above formula to get $\frac{ar - ar^{\infty}}{1-r} = \frac{ar}{1-r}$, since $r^n \to 0$ as $n \to \infty$.
If $r = -1$, then r^n will oscilate between -1 and $+1$ and we don't know what's going on.
If $r = 1$ then in our denominator we divide by $1 - r = 0$, thus our expression does not make sense.
If $r > 1$ or $r < -1$, then r^n tends towards $\pm\infty$ as $n \to \infty$, thus our expression also doesn't make sense.

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Question 11 (3 marks)

Consider the sequence $a_n = 2n + 1$, and the derived sequence $b_1 = 1$ and $b_n = b_{n-1} + a_n$ for $n \geq 2$.

a. Express b_n in terms of s_n. (2 marks)

We note that $b_1 = a_1 - 2$, and that $b_2 = b_1 + a_2 = a_1 + a_2 - 2$. Inductively we see that $b_n = a_1 + a_2 + \cdots + a_n - 2 = s_n - 2$.

b. Hence, or otherwise find b_{10}. (1 mark)

As $s_{10} = \frac{10}{2}(a_1 + a_{10}) = 5(3 + 21) = 120$ we see that $b_{10} = 118$.

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Sub-Section: Exam 2 Questions

Question 12 (1 mark)

The sequence with consecutive entries 1, -2, 4 could be:

A. An arithmetic sequence.

B. A geometric sequence.

C. Neither an arithmetic nor a geometric sequence.

D. Either an arithmetic or a geometric sequence.

Question 13 (1 mark)

Consider the geometric sequence, g_n with the following entries, $g_3 = 4$ and $g_5 = 16$.

A possible closed form for g_n is:

A. $g_n = 6n - 14$

B. $g_n = 2 \times 2^n$

C. $g_n = 6n - 2$

D. $g_n = -\frac{1}{2} \times (-2)^n$

Question 14 (1 mark)

The infinite sum, $10 - 1 + \frac{1}{10} - \frac{1}{100} + \ldots$ is equal to:

A. $\frac{100}{11}$

B. $\frac{100}{9}$

C. $-\frac{1000}{9}$

D. $-\frac{10}{9}$

Question 15 (1 mark)

The value of $1 + 2 + ... + 100$ is:

A. 5000

B. 10100

C. 10000

D. 5050

Question 16 (1 mark)

Consider the sequence $a_n = 2^n + 3n + 1$, the corresponding series to this sequence has a closed form of:

A. $s_n = 5^{n+1} - 5$

B. $s_n = \frac{5n^2+7n}{2}$

C. $s_n = 2^{n+1} + \frac{3n^2+5n-4}{2}$

D. $s_n = 2^n + \frac{3n^2+5n}{2}$

Question 17 (9 marks)

An annuity is an annual payment of $\$d$ dollars at the start of every year for n years.

Due to inflation \$1 dollar today is worth \$1.03 dollars exactly one year from now, thus the present value of an annuity, p (i.e., how much the annuity p is worth today) is equal to:

$$p = d + d \times 1.03^{-1} + d \times 1.03^{-2} + \ldots + d \times 1.03^{n-1}$$

For this question assume the current date is January first. Thus a payment at the start of a year usually pays off immediately.

a. In today's dollar, how much is \$1000 dollars worth one year from now? (1 mark)

$1000 \times 1.03^{-1} \approx \$970.87$

b. In today's dollar, how much is \$1000 dollars worth 10 years from now? Give your answer correct to 2 decimal places. (1 mark)

$$1000 \times 1.03^{-10} \approx 774.09$$

c. For this part of the question you may use the identity $1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$.

i. Calculate the present value of an annuity that pays \$2000 dollars each year at the **start** of every year for the next 10 years. Give your answer correct to 2 decimal places. (2 marks)

The PV is $2000(1 + 1.03^{-1} \cdots + 1.03^{-9}) = 2000 \times \dfrac{1 - 1.03^{-10}}{1 - 1.03^{-1}} \approx \$17572.22$

ii. Calculate the present value of an annuity that pays \$2000 dollars each year at the **end** of every year for the next 10 years. Give your answer correct to 2 decimal places. (1 mark)

Each payment will be worth 1.03^{-1} times the payment as in part **i**. Thus we multiply our answer by 1.03^{-1}.
Hence our present value is \$17060.41.

iii. Calculate the present value of an annuity that pays \$2000 every year for 10 consecutive years, starting exactly 3 years from now. Give your answer correct to 2 decimal places. (1 mark)

Each payment will be worth 1.03^{-3} times the payment in part **i.** Hence our answer is $17572.22 \times 1.03^{-3} = \$16081.07$.

d. Express in closed form the present value of an annuity that pays $\$d$ dollars every 2 years, at the start of the year for $2n$ years. (2 marks)

The present value is $d + 1.03^2 d + \cdots + 1.03^{2n-1} d = d \times \dfrac{1 - 1.03^{2n}}{1 - 1.03^2}$

e. A perpetuity is an annuity that pays $\$d$ dollars at the start of a year every year, forever. Calculate the present value of a perpetuity that pays $100 dollars. (1 mark)

The present value of the perpetuity is $100 + 100 \times 1.03^{-1} + \ldots$.
The value of it is $\$100 \times \dfrac{1}{1 - 1.03^{-1}} = \$\dfrac{103}{0.03} = \$\dfrac{10300}{3}$

Question 18 (10 marks)

Consider the following arithmetic sequence, a_n, with $a_1 = 2$ and $a_n = ba_{n-1} + c$.

a. For what values of b is a_n an arithmetic sequence? (1 mark)

$b = 1$

b. For what value(s) of b and c does a_n have the following values? (3 marks)

$$a_3 = \frac{1}{2}, a_5 = \frac{1}{8}$$

$c = 0$ and $b = \pm\frac{1}{2}$

c. Let $b = 2$ and $c = -1$.

i. Find a_3. (1 mark)

$a_2 = 2a_1 - 1 = 4 - 1 = 3$
$a_3 = 2a_2 - 1 = 6 - 1 = 5.$

ii. Find $a_1 + a_2 + a_3$. (1 mark)

$a_1 + a_2 + a_3 = 2 + 3 + 5 = 10$

iii. Is there some $m \in R$ such that $a_n < m$ for all natural numbers n? Why or why not? (1 mark)

If $a_n \geq 2$, then $a_{n+1} = 2a_n - 1 = a_n + a_n - 1 \geq a_n + 1$.
Since $a_1 = 2$ we know that $a_2 \geq a_1 + 1 \geq 2$. Thus inductively $a_n \geq 2$ hence $a_{n+1} \geq a_n + 1$.
Since our sequence always increases by at least 1, there cannot be such an m that bounds our sequence.

d. Let $b = \frac{1}{2}$ and $c = 0$.

i. Find the geometric mean of $a_3, \ldots a_{13}$. (2 marks)

Our geometric mean is $\sqrt[11]{\prod_{n=3}^{13} 2 \times \left(\frac{1}{2}\right)^n} = 2\sqrt{\left(\frac{1}{2}\right)^{\sum_{n=3}^{13} n}} = 2 \times \left(\frac{1}{2}\right)^{\frac{11}{2}(13+3)\frac{1}{11}} = 2 \times \left(\frac{1}{2}\right)^8 = \frac{1}{128}$

ii. Let m_n denote the geometric mean of $a_1, \ldots, a_n$. Find the smallest possible number q, such that $m_n \leq q$ for all natural numbers n. (1 mark)

Observe that our geometric mean is decreasing. Hence we only require $m_1 = 2 \leq q$. Hence $q = 2$.

Section B: Supplementary Questions

Sub-Section [1.4.1]: Find Sequences From Two Terms

Question 19

Define the arithmetic sequence in terms of n if $t_3 = -10$ and $t_{13} = 10$.

The common difference d is equal to $\frac{10-(-10)}{13-3} = 2$.

Thus $t_n = a + 2(n-1)$ Since $t_3 = a + 6 = -10$ we see that $a = -14$ Hence $t_n = -16 + 2n$.

Question 20

Define a possible geometric sequences in terms of n if $t_4 = \frac{1}{4}$ and $t_7 = \frac{27}{4}$.

The common ratio r satisfies $r^3 = t_7/t_4 = 27$. Thus $r = 3$.

Since $t_n = a \times 3^{(n-1)}$ and $t_4 = 27a = \frac{1}{4}$ we see that $a = \frac{1}{108}$ and $t_n = \frac{3^n}{324}$.

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Question 21

Consider the arithmetic, a_n sequence with the following properties, $a_3 = 8$, $a_6 = -\frac{5}{2}$.

g_n is a geometric sequence with the property that $g_3 = a_3$ and $g_5 = a_5$.

Find g_n in terms of n.

We know that the common difference for a_n is $d = \frac{-\frac{5}{2} - 8}{6 - 3} = -\frac{7}{2}$. Thus $a_5 = a_3 + 2d = 8 - 7 = 1$.

Now since $g_3 = 8$ and $g_5 = 1$ we know that the common ratio for g_n, r satisfies $r^2 = \frac{1}{8}$, hence $r = \pm\frac{1}{2\sqrt{2}}$.

If $r = \frac{1}{2\sqrt{2}}$ we can see that $g_n = 128\sqrt{2}\left(\frac{1}{2\sqrt{2}}\right)^n$.

If $r = -\frac{1}{2\sqrt{2}}$ we can see that $g_n = -128\sqrt{2}\left(-\frac{1}{2\sqrt{2}}\right)^n$.

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Question 22

Consider the following sequence, $a_n = b^n + c + dn$.

It is known that $a_1 = 0$, $a_2 = 1$ and $a_3 = 4$.

Find the values of b, c and d.

We construct a system of equations from the given information.

$$a_1 = b + c + d = 0 \quad (1)$$
$$a_2 = b^2 + c + 2d = 1 \quad (2)$$
$$a_3 = b^3 + c + 3d = 4 \quad (3)$$
$$(4)$$

By subtracting (1) from (2) we get the equation $b^2 - b + d = 1$.
By subtracting (1) from (3) we get the equation $b^3 - b + 2d = 4$.
By subtracting 2 $\times$ the former equation from the latter equation we get $b^3 - 2b^2 + b = 2$.
We can factorise the above cubic to get $b^2(b-2) + b - 2 = (b-2)(b^2+1) = 0$ to see that $b = 2$.
Substituting that value of b into one of the new equations we created yields $4 - 2 + d = 1 \implies d = -1$.
Substituting both of those values into (1) yields $2 + c - 1 = 0 \implies c = -1$.

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Sub-Section [1.4.2]: Apply Recurrence Relation To Different Types of Sequences

Question 23

Consider the sequence a_n, with the property that $a_3 = -5$ and $a_n = 2a_{n-1} + 1$.

a. Find a_1.

We know that $-5 = a_3 = 2a_2 + 1$. Thus $2a_2 = -6$ and $a_2 = -3$.
Thus $-3 = a_2 = 2a_1 + 1$ hence $2a_1 = -4$ and $a_1 = -2$.

b. Now assume that $a_1 = b$ and $a_n = 2a_{n-1} + 1$. Find a value of b such that $a_n = b$ for all values of n.

It is sufficient to have $a_2 = a_1 = b$.
Thus we solve $b = 2b + 1$ for b to get $b = -1$.

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Question 24

Consider the sequence defined by the following recursive relationship:

$$f_{n+1} = \frac{f_n + f_{n-1}}{4}$$

The sequence can be expressed in the form $f_n = a^n$. Find all possible values of a.

If $n = 1$, we know that $4a^2 = a + 1$ thus $a = \frac{-1 \pm \sqrt{1+16}}{8} = \frac{-1 \pm \sqrt{17}}{8}$

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Question 25

Consider the Fibonacci Sequence, f_n defined as such:

$$f_1 = f_2 = 1$$

$$f_{n+1} = f_n + f_{n-1} \quad \text{for } n \geq 2$$

Now consider the sequence $a_n = a2^n$.

Show that for a suitable value of a, $a_n > f_n$ for all values of n.

If $a = 1$ we see that $a_1 = 2 > f_1 = 1$ and $a_2 = 4 > f_1 = 1$.
Now assume that $a_{n-1} > f_{n-1}$. Since the Fibonacci sequence is increasing, we also know that $a_{n-1} > f_{n-2}$.
Thus $a_n = 2a_{n-1} = a_{n-1} + a_{n-1} > f_{n-1} + f_{n-2} = f_n$.
Since our original statement is true for $n = 1$ and $n = 2$, the above logic shows that it will be true for $n = 3$ and hence $n = 4$ and hence any value of n.

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Question 26

Find a sequence, a_n that satisfies the recursive relationship, $a_n = 4a_{n-1} + 2a_{n-2} - 12a_{n-3} - 9a_{n-4}$, as well as the conditions:

$$a_2 = 2 \text{ and } a_3 = 4$$

Hint: $((x-1)^2 - 4)^2 = x^4 - 4x^3 - 2x^2 + 12x + 9$

If θ and ϕ are roots of $x^4 - 4x^3 - 2x^2 + 12x + 9$ we know that a sequence $a_n = a\theta^n + b\phi^n$ will satisfy our recursive relationship for any real a and b.
We solve the equation $((x-1)^2 - 4)^2 = 0$.

$$\begin{aligned} ((x-1)^2 - 4)^2 = 0 &\implies (x-1)^2 = 4 \\ &\implies x = 1 \pm 2 \\ &\implies x = -1, 3 \end{aligned}$$

Thus we consider a sequence $a_n = a(-1)^n + b(3)^n$ and solve for a and b.
Since $2 = a + 9b$ and $4 = -a + 27b$ we see that $6 = 36b \implies b = \frac{1}{6}$. Substituting this back into an equation we see that $a = \frac{1}{2}$. Hence

$$a_n = \frac{(-1)^n}{2} + \frac{3^n}{6}$$

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Sub-Section: Exam 1 Questions

Question 27 (3 marks)

Consider the arithmetic sequence, a_n with the following properties:

$$a_5 = 7 \text{ and } a_8 = 19$$

a. Find $a_2 - a_1$. (1 mark)

$$a_2 - a_1 = \frac{a_8 - a_5}{3} = \frac{19 - 7}{3} = 4$$

b. Find a_1. (1 mark)

$$a_1 = a_5 - 4 \times 4 = 7 - 16 = -9.$$

c. Hence, find a_n for any natural number n. (1 mark)

$$a_n = -13 + 4n.$$

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Question 28 (4 marks)

Consider the following geometric progression, $b_n = 2 \times \left(-\frac{2}{3}\right)^{n-3}$.

a. Find the geometric mean of $b_1, b_2, \ldots, b_5$. (2 marks)

$$2 \times \left(-\frac{2}{3}\right)^{-3} \times \sqrt[5]{\left(-\frac{2}{3}\right)^{\sum_{n=1}^{5} n}}$$
$$= -\frac{27}{4} \times \sqrt[5]{\left(-\frac{2}{3}\right)^{15}}$$
$$= -\frac{27}{4} \times \left(-\frac{2}{3}\right)^{3} = 2$$

b. Evaluate $5b_1 - 5b_2 + 5b_3 + \ldots$. (2 marks)

$$5b_1 - 5b_2 + 5b_3 + \ldots = 5 \times 2 \times \left(\frac{9}{4}\right)\left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \ldots\right)$$
$$= \frac{45}{2} \times \frac{1}{1 - \frac{2}{3}} = \frac{135}{2}$$

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Question 29 (4 marks)

Consider a positive sequence a_n with $a_n > 0$ for all natural numbers n.

a. If $a_1 + a_2 + \ldots + a_n < 5$ for all values of n, show that there exists an integer k, such that for all $n > k$, $a_n < 1$. (3 marks)

Assume such an integer k does not exist. Then there is some n_1 such that $a_{n_1} \geq 1$. And there will be an $n_2 > n_1$ such that $a_{n_2} \geq 1$, and an $n_3 > n_2$ such that $a_{n_3} \geq 1$, and an $n_4 > n_3$ such that $a_{n_4} \geq 1$, and finally an $n_5 > n_4$ such that $a_{n_5} \geq 1$.
From here we see that since $a_n > 0$ for all n, $a_1 + a_2 + \ldots a_{n_5} \geq a_{n_1} + a_{n_2} + a_{n_3} + a_{n_4} + a_{n_5} \geq 5$ a contradiction.
Hence the statement in the question must be true.

b. Explain why a_{1000} is not necessarily less than 0.1. (1 mark)

We can set $a_n = 0$ for all $n \neq 1000$ and $a_{1000} = 1 > 0.1$. This satisties the statement $a_1 + a_2 + \cdots + a_n < 5$ for all values of n as well as the statement $a_{1000} > 0.1$.

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Question 30 (4 marks)

Consider a sequence, $\phi_n = ab^n + cd^n$, defined by the following recursive relationship:

$$\phi_{n+1} = 5\phi_n - 6\phi_{n-1}$$

If $\phi_2 = 7$ and $\phi_3 = 17$, find possible values of a, b, c and d.

We note that the sequences $(b^n)_n$ and $(d^n)_n$ must also satisfy the recursive relationship.
Hence b and d both satisfy the polynomial equation, $x^2 - 5x + 6 = (x-3)(x-2) = 0$.
We can then set $b = 3$ and $d = 2$.
Now we simply solve for a and c.
Since $\phi_2 = 7$ we get $7 = 9a + 4c$, and since $\phi_3 = 17$ we get $17 = 27a + 8c$.
From here we see that $3 = 9a$ thus $a = \frac{1}{3}$, and hence $c = 1$.

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Question 31 (5 marks)

Consider the following two sequences:

$$a_n = 3n - 1 \text{ and } b_n = 3 \times 2^{-n}$$

a. Express the sequence $c_n = b a_n$ in terms of n. (1 mark)

$$c_n = 3 \times 2^{-(3n-1)} = 6 \times 2^{-3n}$$

b. The arithmetic mean of $a_1, \ldots a_p$ is 17.

i. Find the value of p. (1 mark)

By the arithmetic mean formula we know that $a_1 + a_p = 2 + 3p - 1 = 34$.
Hence $3p = 33 \implies p = 11$.

ii. Hence, or otherwise find the geometric mean of $c_1, \ldots c_p$. (2 marks)

The geometric mean of $c_1, \ldots, c_{11}$ is

$$\sqrt[11]{3 \times 2^{-a_1} \times \cdots \times 3 \times 2^{-a_{11}}} = 3 \times \boxed{2^{-17}}$$

c. Evaluate $c_1 + c_2 + \ldots$. (1 mark)

We know that $c_1 = \frac{6}{8}$ and the common ratio is $\frac{1}{8}$, hence our sum is equal to,

$$\frac{6}{8} \times \frac{1}{1 - \frac{1}{8}} = \frac{6}{7}$$

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Sub-Section: Exam 2 Questions

Question 32 (1 mark)

Consider the following sequence a_n defined recursively.

$$a_1 = 2$$

$$a_2 = 3$$

$$a_n = a_{n-1} + a_{n-2}$$

Evaluate a_{10}.

A. 55

B. 89

C. 144

D. 233

Question 33 (1 mark)

Consider the geometric sequence, a_n.

It is known that $a_1 + a_2 + a_3 + ... = 4$ and that $a_1 = 2$.

The geometric mean of $a_1, a_2 ... a_8$ is:

A. $\frac{1}{4\sqrt{2}}$

B. $\frac{1}{32}$

C. $\frac{1}{1048576}$

D. $\frac{1}{2}$

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Question 34 (1 mark)

The sequence with consecutive entries $1, -3, 5, -7$ could be:

A. An arithmetic sequence.

B. A geometric sequence.

C. Either an arithmetic or a geometric sequence.

D. Neither an arithmetic nor a geometric sequence.

Question 35 (1 mark)

How many entries are sufficient to uniquely determine all the entries in an arithmetic progression?

A. 1

B. 2

C. 3

D. 4

Question 36 (1 mark)

Let a_n be an arithmetic sequence and let $b_n = 2^n$ be a geometric sequence.

Define the sequence $c_n = b_{a_n}$.

The arithmetic mean of $a_1, a_2, \ldots a_p$ is 3.

The geometric mean of $c_1, c_2, \ldots, c_p$ is:

A. 9

B. 5

C. 8

D. Impossible to tell with the current information.

Question 37 (9 marks)

An island has 10 fertile immortal monkeys. Every year, each pair of two fertile monkeys produces another monkey.

Let m_n denote the population of monkeys on the Island at the start of the year n.

a. Show that $m_n = 10 \times \left(\frac{3}{2}\right)^{n-1}$. (1 mark)
Hint: For this question simply approximate all answers using series and round at the end of calculations

Every year, 2 monkeys turn into 3 monkeys. Hence the ratio of increase is $\frac{3}{2}$.
Hence $m_n = a \times \left(\frac{3}{2}\right)^n$
At the start of the first year there are 10 monkeys, i.e. $m_1 = 10$.
We solve for a to get $a = 10 \times \frac{2}{3}$.
Hence $m_n = 10 \times \left(\frac{3}{2}\right)^{n-1}$.

b. At the end of every year, 20 additional sterile immortal monkeys (they can't reproduce) are introduced.

i. Find the number of monkeys on the Island by the start of the 5th year. (1 mark)

We know that 4 groups of sterile monkeys will be introduced. Hence the number of monkeys will be

$$m_5 + 80 \approx 131$$

ii. After how many years will there be more fertile monkeys than sterile monkeys? (2 marks)

The number of fertile monkeys at the start of year n is m_n. The number of sterile monkeys at the start of year n is $-20 + 20n$.
The number of fertile monkeys should eventually eclipse the number of sterile monkeys. Thus we solve $m_n = -20 + 20n$ for n. This yields $n = 7.21$ which we round up to $n = 8$.

c. At the end of each year, monkeys who have been on the Island for at least a year pay their taxes to the Jade Emperor (the initial monkeys pay tax at the end of the first year). At the end of 10 years how many times has the Jade Emperor received a tax form? (3 marks)

At the end of the n'th year there will be m_n fertile monkeys submitting their taxes. Thus the number of tax forms submitted by the fertile monkeys will be,

$$10 \times \left(1 + \frac{3}{2} + \cdots + \left(\frac{3}{2}\right)^9\right) = 10 \times \frac{1 - \left(\frac{3}{2}\right)^9}{1 - \frac{3}{2}} \approx 749$$

At the end of the n'th year there will be $-20 + 20n$ infertile monkey's submitting their taxes.
Thus $5 \times (0 + 180) = 900$ tax forms will be submitted by infertile monkeys.
Overall the Jade emperor will get 1649 tax forms.

d. After p years the infertile monkeys start attacking the fertile monkeys, killing 1000 monkeys a year. State the possible values of p, such that the population of fertile monkeys does not decrease. (2 marks)

The population m_p must be such that $\frac{3}{2}(m_p - 1000) \geq m_p$.
This means that $m_p \geq 3000$.
This will occur if $p \geq 16$.

Question 38 (9 marks)

Consider the harmonic sequence, $h_n = \frac{1}{n}$ and its associated series $H_n = \sum_{i=1}^{n} h_i$.

a. Find H_5. (1 mark)

$$H_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$$

b.

i. Show that $h_{2^n+1} + h_{2^n+2} + \ldots + h_{2^{n+1}} > \frac{1}{2}$. (2 marks)

Observe that there are 2^n integers between 2^{n+1} and $2^n + 1$ (inclusive). Thus

$$\begin{aligned} h_{2^n+1} + h_{2^n+2} + \cdots + h_{2^{n+1}} &= \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &> \underbrace{\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}}_{2^n \text{ times}} \\ &= 2^n \times \frac{1}{2^{n+1}} \\ &= \frac{1}{2} \end{aligned}$$

ii. Hence, or otherwise find the smallest value of n such that $H_n > 3$. (1 mark)

We see that $h_1 = 1$ and $h_2 = \frac{1}{2}$. Then $h_3 + h_4 \geq \frac{1}{2}$, and $h_5 + \cdots + h_8 \geq \frac{1}{2}$ and lastly $h_9 + h_{16} \geq \frac{1}{2}$.

Thus $H_{16} \geq \frac{1}{2}$.

We will then go back from 16 to find the largest value of n such that $H_n < 3$.

This value turns out to be 10, with $H_{10} = 2.93$.

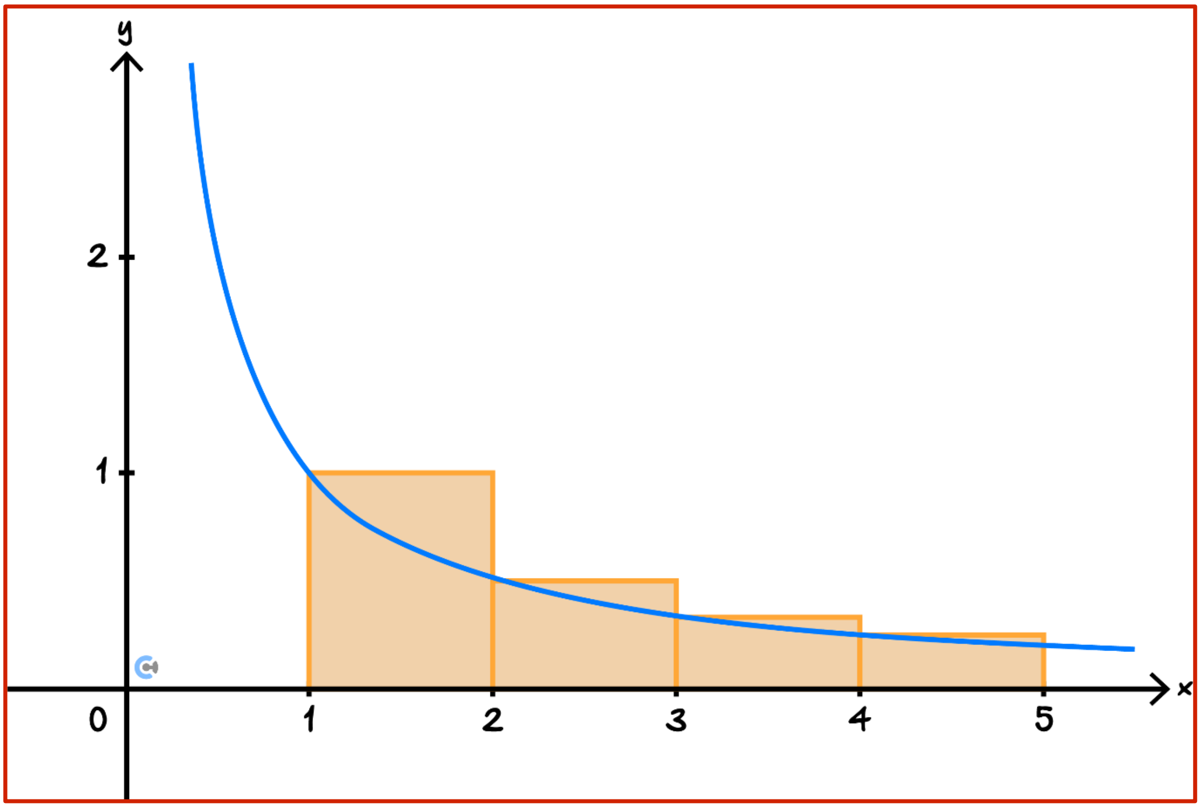
Thus our smallest value of n is 11.

iii. Argue why for all real m there exists some n such that $H_n > m$. (1 mark)

From part **i** we know that $H_{2^n} > \frac{1}{2}n$. Thus for any m we know that $H_{2^{2\lceil m \rceil}} > \lceil m \rceil > m$.

c. The area bounded by the graph $y = \frac{1}{x}$, the x-axis, and the lines $x = 1, x = a$ for $a > 1$ is equal to $\log_e(a)$.

The graph of $y = \frac{1}{x}$ is shown below.



Draw a region with an area H_5 and use that region to argue why for all $m \in R$ there exists an n such that $H_n > m$. (4 marks)

From the graph above we see that H_n is greater than the area bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = n + 1$.
This area is equal to $\log_e(n + 1)$.
Hence we see that $H_{\lceil e^m \rceil - 1} > m$.

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