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VCE Specialist Mathematics ½
Sequences & Series Exam Skills [1.4]
Homework

Homework Outline:

Compulsory Questions	Pg 2 – Pg 18
Supplementary Questions	Pg 19 – Pg 38

Section A: Compulsory Questions

Sub-Section [1.4.1]: Find Sequences From Two Terms

Question 1

Define the arithmetic sequence in terms of n if $t_6 = 12$ and $t_{12} = 30$.

Question 2

Define two possible geometric sequences in terms of n if $t_3 = -\frac{2}{3}$ and $t_5 = -\frac{2}{27}$.

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Question 3

Let g_n be a geometric sequence and a_n be an arithmetic sequence.

It is known that $g_1 = a_1 = 2$ and that $g_2 = a_2$ and $g_3 = a_4$.

Describe g_n and a_n in terms of n.

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Sub-Section [1.4.2]: Apply Recurrence Relation To Different Types of Sequences

- We define the Fibonacci Sequence f_t via the following recursive definition.

$$f_1 = f_2 = 1$$

$$f_{n+1} = f_n + f_{n-1} \quad \text{for } n \geq 2$$

Question 4

a. Find f_5.

b. Find $f_5 + f_6 + f_7 + f_8$.

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Question 5

a. Show that f_n is not an arithmetic sequence.

b. Show that f_n is not a geometric sequence.

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Question 6

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ be roots of the equation $x^2 = x + 1$.

a. Show that $\phi^{n+1} = \phi^n + \phi^{n-1}$ (Hint: Use the fact that ϕ satisfies the equation $x^2 = x + 1$).

b. Show that $a\phi^{n+1} + b\psi^{n+1} = a(\phi^n + \phi^{n-1}) + b(\psi^n + \psi^{n-1})$ for all real a, b.

c. Hence, show that $f_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$.

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Sub-Section: Exam 1 Questions

Question 7 (5 marks)

Consider an arithmetic sequence a_n with the following properties:

$$a_1 = 5, a_2 = 8, a_p = 23$$

a. In closed form, $a_n = b \times n + d$. Find the values of b and d. (2 marks)

b. Hence, find the value of p. (1 mark)

c. Let $s_n = \sum_{i=1}^{n} a_i$ be the series corresponding to the sequence a_n.

Find the value of s_{10}. (2 marks)

Question 8 (5 marks)

Consider the following sequence, $g_n = 2 \times \left(-\frac{1}{3}\right)^n$.

a. Evaluate the geometric mean of $g_1, \ldots, g_9$. (2 marks)

b. Evaluate $g_3 + g_4 + \ldots$. (2 marks)

c. The arithmetic mean of $g_1, \ldots, g_n$ tends towards some number as $n \to \infty$. Find that number. (1 mark)

Question 9 (3 marks)

Let a_n and g_n be an arithmetic and geometric sequence respectively.

Show that if $a_1 = g_1$, $a_2 = g_2$ and $a_3 = g_3$, then $a_n = g_n = a_1$ for all natural numbers n.

Question 10 (4 marks)

Consider a geometric sequence $g_n = a \times r^n$.

a. Show that $g_1 + g_2 + ... + g_n = \frac{a(r - r^{n+1})}{1-r}$. (2 marks)

b. Use the above formula to evaluate, $1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{10}}$. (1 mark)

c. Explain why $-1 < r < 1$ for us to be able to evaluate the series $g_1 + g_2 + ...$ (1 mark)

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Question 11 (3 marks)

Consider the sequence $a_n = 2n + 1$, and the derived sequence $b_1 = 1$ and $b_n = b_{n-1} + a_n$ for $n \geq 2$.

a. Express b_n in terms of s_n. (2 marks)

b. Hence, or otherwise find b_{10}. (1 mark)

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Sub-Section: Exam 2 Questions

Question 12 (1 mark)

The sequence with consecutive entries $1, -2, 4$ could be:

A. An arithmetic sequence.

B. A geometric sequence.

C. Neither an arithmetic nor a geometric sequence.

D. Either an arithmetic or a geometric sequence.

Question 13 (1 mark)

Consider the geometric sequence, g_n with the following entries, $g_3 = 4$ and $g_5 = 16$.

A possible closed form for g_n is:

A. $g_n = 6n - 14$

B. $g_n = 2 \times 2^n$

C. $g_n = 6n - 2$

D. $g_n = -\frac{1}{2} \times (-2)^n$

Question 14 (1 mark)

The infinite sum, $10 - 1 + \frac{1}{10} - \frac{1}{100} + \ldots$ is equal to:

A. $\frac{100}{11}$

B. $\frac{100}{9}$

C. $-\frac{1000}{9}$

D. $-\frac{10}{9}$

Question 15 (1 mark)

The value of $1 + 2 + ... + 100$ is:

A. 5000

B. 10100

C. 10000

D. 5050

Question 16 (1 mark)

Consider the sequence $a_n = 2^n + 3n + 1$, the corresponding series to this sequence has a closed form of:

A. $s_n = 5^{n+1} - 5$

B. $s_n = \frac{5n^2+7n}{2}$

C. $s_n = 2^{n+1} + \frac{3n^2+5n-4}{2}$

D. $s_n = 2^n + \frac{3n^2+5n}{2}$

Question 17 (9 marks)

An annuity is an annual payment of $\$d$ dollars at the start of every year for n years.

Due to inflation \$1 dollar today is worth \$1.03 dollars exactly one year from now, thus the present value of an annuity, p (i.e., how much the annuity p is worth today) is equal to:

$$p = d + d \times 1.03^{-1} + d \times 1.03^{-2} + \ldots + d \times 1.03^{n-1}$$

For this question assume the current date is January first. Thus a payment at the start of a year usually pays off immediately.

a. In today's dollar, how much is \$1000 dollars worth one year from now? (1 mark)

b. In today's dollar, how much is $1000 dollars worth 10 years from now? Give your answer correct to 2 decimal places. (1 mark)

c. For this part of the question you may use the identity $1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$.

i. Calculate the present value of an annuity that pays $2000 dollars each year at the **start** of every year for the next 10 years. Give your answer correct to 2 decimal places. (2 marks)

ii. Calculate the present value of an annuity that pays $2000 dollars each year at the **end** of every year for the next 10 years. Give your answer correct to 2 decimal places. (1 mark)

iii. Calculate the present value of an annuity that pays $2000 every year for 10 consecutive years, starting exactly 3 years from now. Give your answer correct to 2 decimal places. (1 mark)

d. Express in closed form the present value of an annuity that pays $\$d$ dollars every 2 years, at the start of the year for $2n$ years. (2 marks)

e. A perpetuity is an annuity that pays $\$d$ dollars at the start of a year every year, forever. Calculate the present value of a perpetuity that pays $\$100$ dollars. (1 mark)

Question 18 (10 marks)

Consider the following arithmetic sequence, a_n, with $a_1 = 2$ and $a_n = ba_{n-1} + c$.

a. For what values of b is a_n an arithmetic sequence? (1 mark)

b. For what value(s) of b and c does a_n have the following values? (3 marks)

$$a_3 = \frac{1}{2}, a_5 = \frac{1}{8}$$

c. Let $b = 2$ and $c = -1$.

i. Find a_3. (1 mark)

ii. Find $a_1 + a_2 + a_3$. (1 mark)

iii. Is there some $m \in R$ such that $a_n < m$ for all natural numbers n? Why or why not? (1 mark)

d. Let $b = \frac{1}{2}$ and $c = 0$.

i. Find the geometric mean of $a_3, \ldots a_{13}$. (2 marks)

ii. Let m_n denote the geometric mean of $a_1, \ldots, a_n$. Find the smallest possible number q, such that $m_n \leq q$ for all natural numbers n. (1 mark)

Section B: Supplementary Questions

Sub-Section [1.4.1]: Find Sequences From Two Terms

Question 19

Define the arithmetic sequence in terms of n if $t_3 = -10$ and $t_{13} = 10$.

Question 20

Define a possible geometric sequences in terms of n if $t_4 = \frac{1}{4}$ and $t_7 = \frac{27}{4}$.

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Question 21

Consider the arithmetic, a_n sequence with the following properties, $a_3 = 8$, $a_6 = -\frac{5}{2}$.

g_n is a geometric sequence with the property that $g_3 = a_3$ and $g_5 = a_5$.

Find g_n in terms of n.

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Question 22

Consider the following sequence, $a_n = b^n + c + dn$.

It is known that $a_1 = 0$, $a_2 = 1$ and $a_3 = 4$.

Find the values of b, c and d.

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Sub-Section [1.4.2]: Apply Recurrence Relation To Different Types of Sequences

Question 23

Consider the sequence a_n, with the property that $a_3 = -5$ and $a_n = 2a_{n-1} + 1$.

a. Find a_1.

b. Now assume that $a_1 = b$ and $a_n = 2a_{n-1} + 1$. Find a value of b such that $a_n = b$ for all values of n.

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Question 24

Consider the sequence defined by the following recursive relationship:

$$f_{n+1} = \frac{f_n + f_{n-1}}{4}$$

The sequence can be expressed in the form $f_n = a^n$. Find all possible values of a.

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Question 25

Consider the Fibonacci Sequence, f_n defined as such:

$$f_1 = f_2 = 1$$

$$f_{n+1} = f_n + f_{n-1} \quad \text{for } n \geq 2$$

Now consider the sequence $a_n = a2^n$.

Show that for a suitable value of a, $a_n > f_n$ for all values of n.

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Question 26

Find a sequence, a_n that satisfies the recursive relationship, $a_n = 4a_{n-1} + 2a_{n-2} - 12a_{n-3} - 9a_{n-4}$, as well as the conditions:

$$a_2 = 2 \text{ and } a_3 = 4$$

Hint: $((x-1)^2 - 4)^2 = x^4 - 4x^3 - 2x^2 + 12x + 9$

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Sub-Section: Exam 1 Questions

Question 27 (3 marks)

Consider the arithmetic sequence, a_n with the following properties:

$$a_5 = 7 \text{ and } a_8 = 19$$

a. Find $a_2 - a_1$. (1 mark)

b. Find a_1. (1 mark)

c. Hence, find a_n for any natural number n. (1 mark)

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Question 28 (4 marks)

Consider the following geometric progression, $b_n = 2 \times \left(-\frac{2}{3}\right)^{n-3}$.

a. Find the geometric mean of $b_1, b_2, \ldots, b_5$. (2 marks)

b. Evaluate $5b_1 - 5b_2 + 5b_3 + \ldots$. (2 marks)

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Question 29 (4 marks)

Consider a positive sequence a_n with $a_n > 0$ for all natural numbers n.

a. If $a_1 + a_2 + ... + a_n < 5$ for all values of n, show that there exists an integer k, such that for all $n > k$, $a_n < 1$. (3 marks)

b. Explain why a_{1000} is not necessarily less than 0.1. (1 mark)

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Question 30 (4 marks)

Consider a sequence, $\phi_n = ab^n + cd^n$, defined by the following recursive relationship:

$$\phi_{n+1} = 5\phi_n - 6\phi_{n-1}$$

If $\phi_2 = 7$ and $\phi_3 = 17$, find possible values of a, b, c and d.

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Question 31 (5 marks)

Consider the following two sequences:

$$a_n = 3n - 1 \text{ and } b_n = 3 \times 2^{-n}$$

a. Express the sequence $c_n = ba_n$ in terms of n. (1 mark)

b. The arithmetic mean of $a_1, \ldots a_p$ is 17.

i. Find the value of p. (1 mark)

ii. Hence, or otherwise find the geometric mean of $c_1, \ldots c_p$. (2 marks)

c. Evaluate $c_1 + c_2 + \ldots$. (1 mark)

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Sub-Section: Exam 2 Questions

Question 32 (1 mark)

Consider the following sequence a_n defined recursively.

$$a_1 = 2$$

$$a_2 = 3$$

$$a_n = a_{n-1} + a_{n-2}$$

Evaluate a_{10}.

A. 55

B. 89

C. 144

D. 233

Question 33 (1 mark)

Consider the geometric sequence, a_n.

It is known that $a_1 + a_2 + a_3 + \dots = 4$ and that $a_1 = 2$.

The geometric mean of $a_1, a_2 \dots a_8$ is:

A. $\frac{1}{4\sqrt{2}}$

B. $\frac{1}{32}$

C. $\frac{1}{1048576}$

D. $\frac{1}{2}$

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Question 34 (1 mark)

The sequence with consecutive entries $1, -3, 5, -7$ could be:

A. An arithmetic sequence.

B. A geometric sequence.

C. Either an arithmetic or a geometric sequence.

D. Neither an arithmetic nor a geometric sequence.

Question 35 (1 mark)

How many entries are sufficient to uniquely determine all the entries in an arithmetic progression?

A. 1

B. 2

C. 3

D. 4

Question 36 (1 mark)

Let a_n be an arithmetic sequence and let $b_n = 2^n$ be a geometric sequence.

Define the sequence $c_n = b_{a_n}$.

The arithmetic mean of $a_1, a_2, \ldots a_p$ is 3.

The geometric mean of $c_1, c_2, \ldots, c_p$ is:

A. 9

B. 5

C. 8

D. Impossible to tell with the current information.

Question 37 (9 marks)

An island has 10 fertile immortal monkeys. Every year, each pair of two fertile monkeys produces another monkey.

Let m_n denote the population of monkeys on the Island at the start of the year n.

a. Show that $m_n = 10 \times \left(\frac{3}{2}\right)^{n-1}$. (1 mark)
Hint: For this question simply approximate all answers using series and round at the end of calculations

b. At the end of every year, 20 additional sterile immortal monkeys (they can't reproduce) are introduced.

i. Find the number of monkeys on the Island by the start of the 5^{th} year. (1 mark)

ii. After how many years will there be more fertile monkeys than sterile monkeys? (2 marks)

c. At the end of each year, monkeys who have been on the Island for at least a year pay their taxes to the Jade Emperor (the initial monkeys pay tax at the end of the first year). At the end of 10 years how many times has the Jade Emperor received a tax form? (3 marks)

d. After p years the infertile monkeys start attacking the fertile monkeys, killing 1000 monkeys a year. State the possible values of p, such that the population of fertile monkeys does not decrease. (2 marks)

Question 38 (9 marks)

Consider the harmonic sequence, $h_n = \frac{1}{n}$ and its associated series $H_n = \sum_{i=1}^{n} h_i$.

a. Find H_5. (1 mark)

b.

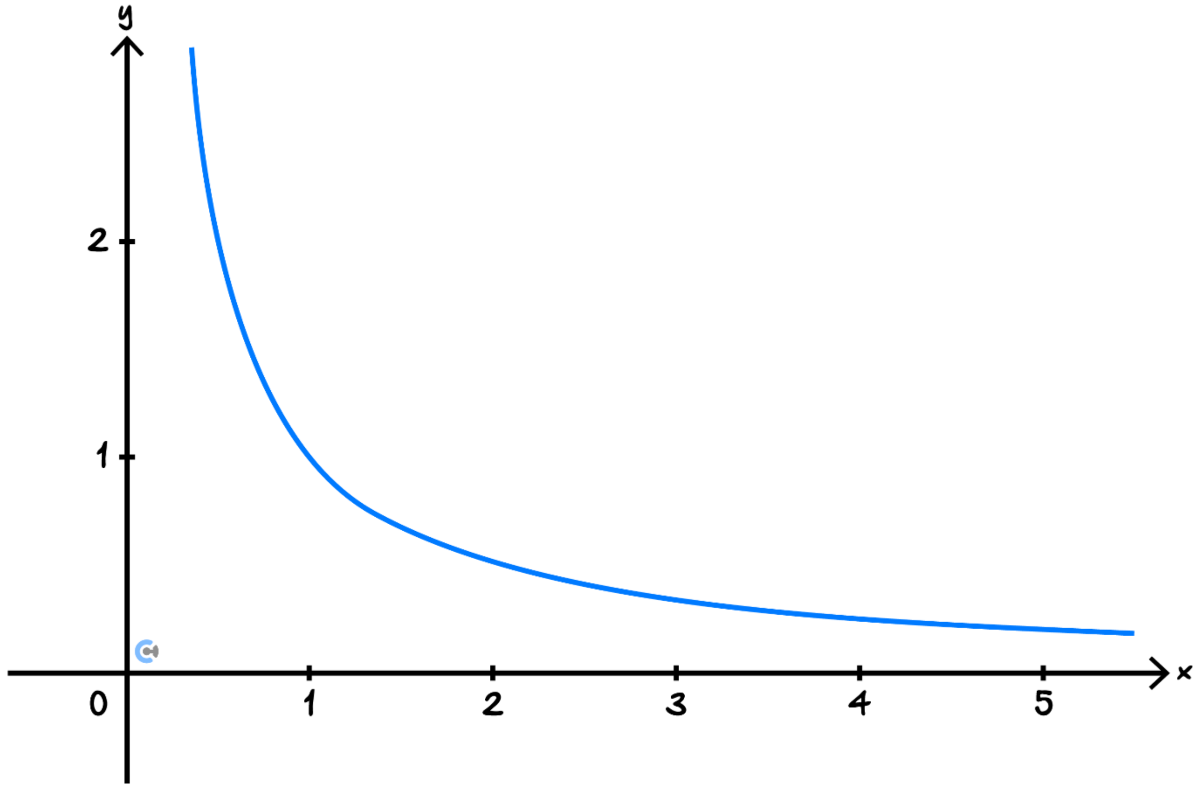
i. Show that $h_{2^n+1} + h_{2^n+2} + \ldots + h_{2^{n+1}} > \frac{1}{2}$. (2 marks)

ii. Hence, or otherwise find the smallest value of n such that $H_n > 3$. (1 mark)

iii. Argue why for all real m there exists some n such that $H_n > m$. (1 mark)

c. The area bounded by the graph $y = \frac{1}{x}$, the x-axis, and the lines $x = 1, x = a$ for $a > 1$ is equal to $\log_e(a)$.

The graph of $y = \frac{1}{x}$ is shown below.



Draw a region with an area H_5 and use that region to argue why for all $m \in R$ there exists an n such that $H_n > m$. (4 marks)

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