1.3

CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Specialist Mathematics ½
Sequence and Series [1.2]
Workbook

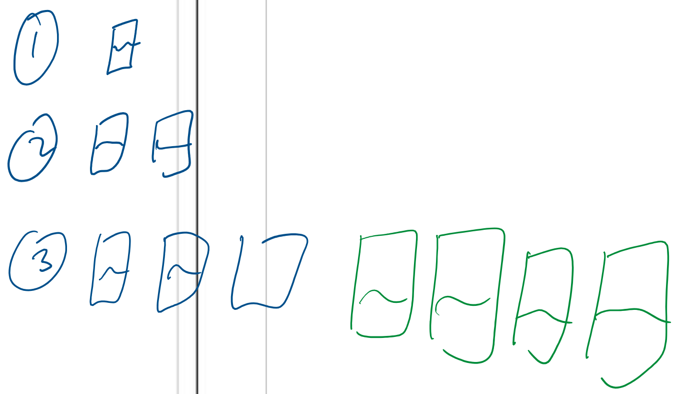
Outline:

Section A: Introduction to Sequences and Series

Sub-Section: Sequences

Sequences

- Consider Samuel, our beloved Contour Tutor.



- Lets say he eats a chocolate bar on the first day.
- The next day, he eats two chocolate bars.
- How many chocolate bars do you think he eats on the third day?
- Hands up if you think it's 3 chocolate bars!
  - You guys have taken an arithmetic approach!
- Hands up if you think it's 4 chocolate bars!
  - You guys have taken a geometric approach!
  - **We will consider the two types of sequences today!**

Sequences

$$t_n = f(n)$$

- **Definition:** A sequence is an ordered list of numbers following a certain **pattern.**
- It is a function of the order n.

Question 1 Walkthrough.

Construct the first 3 terms for the sequence given by $t_n = 2n + 1$.

$t_1 = 3$
$t_2 = 5$
$t_3 = 7$

t_n

$t(n)$

NOTE: t_n stands for the n^{th} term. Eg: For t_3, our value of n is equal to 3.

ALSO NOTE: This was a sequence of odd numbers!

Question 2

The sequence is defined by $t_n = 2^n + 1$. Identify the term number for which t_n equals 9.

$t_9 = ?$ $t_? = 9$

$2^n + 1 = 9$

$n = 3$

3^{rd} term in the sequence is 9.

Space for Personal Notes

Sub-Section: Recurrence Relations

What if we define the term t_n with respect to the previous term (t_{n-1})?

Recurrence Relations

- **Definition:**
  - A recurrence relation is when we define **a term** (t_n), in terms of the previous one (t_{n-1}).
  - Recurrence relations generate sequences of the form:

$$t_n = f(t_{n-1}) \text{ where } t_1 = a$$

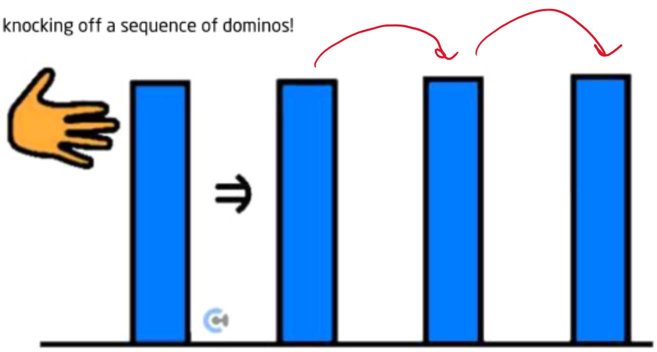
Or

$$t_{n+1} = f(t_n) \text{ where } t_1 = a$$

  - It must always include a first term.

Analogy: Recurrence Relations

- It's like knocking off a sequence of dominos!



- We focus on the relationship between the two terms (dominos).

Question 3

Consider the following recurrence relation.

$$t_n = 3t_{n-1} + 2, \quad t_1 = 2$$

State the value of t_3.

$t_1 = 2$

$t_2 = 8$

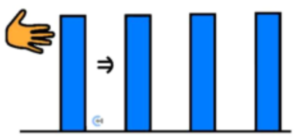
$t_3 = 3 \times 8 + 2$

$t_3 = 26$

t_1 t_2 t_3

Why do we always need a first term?

Analogy: Reason for why recurrence relations always need a first term.



- Can you knock off any of these dominos without knocking over the first one?
- Similarly, how can we solve for any term in recurrence relation without the first term?
It's impossible!

Sub-Section: Introduction to Series

What does the word "series" mean?

Series

- **Definition:**
  - A series is the sum of the first n terms of a sequence.

$$S_n = \sum_{i=1}^{n} t_i$$

Question 4 Walkthrough.

Consider the sequence given by $\boxed{t_n = 3n - 4}$.

Evaluate S_2.

$$S_2 = \sum_{i=1}^{2} t_i$$

$$= t_1 + t_2$$

$t_1 = 3 - 4$
$= -1$

$t_2 = 6 - 4$
$= 2$

$S_2 = (-1) + 2$
$= 1$

Question 5

Consider the sequence given by $t_n = 2n + 2$.

Evaluate S_4.

$$S_4 = \sum_{i=1}^{4} t_i$$

$$= t_1 + t_2 + t_3 + t_4$$

$$= 4 + 6 + 8 + 10$$

$$= 28$$

<u>**Key Takeaways**</u>

- Sequence follows a certain pattern.
- Recurrence relation is a relationship between the next term and the current term.
- Series is the sum of the first n terms.

Section B: Arithmetic Sequence and Series

Sub-Section: Introduction to Arithmetic Sequence

Let's go back to the previous context

Arirthmetic Sequence

- Consider Samuel, our beloved Contour Tutor.

- He ate one chocolate bar on the first day.
- The next day, he ate two chocolate bars.
- Arithmetic sequence suggests Sam eats 3 chocolate bars on the third day !

Discussion: In an arithmetic sequence, how do we go from one term to the next?

Add the same number

Arithmetic Sequences

1 2 3 4

$$a, \quad a+d, \quad a+2d, \quad a+3d$$

- **Definition**

  - An **arithmetic sequence** is one where the **common difference** is added or subtracted to get the next term.

$$t_n = a + (n-1)d$$

  - Where d is the common difference, and a is the first term.

Discussion: Why do we add $(n-1)d$ instead of nd? How many differences do we add from t_1 to t_n?

$t_5 = a + 4d$

$5d$

Question 6 Walkthrough.

Consider the arithmetic sequence defined by $t_n = 12 + 6(n-1)$.

Identify the common difference, first term and the 5[th] term.

t_5

$t_1 = 12 + 6(1-1)$

$= 12$

$d = 6$

$t_2 = 12 + 6(2-1)$

$= 18$

Question 7

Consider the arithmetic sequence defined by $t_n = -1 + 6n$.

Identify the common difference, first term and the 8[th] term.

$d = 6$

$t_1 = 5$

$t_8 = 47$

NOTE: Read the question carefully. Sometimes, they expand the $n - 1$ factor to confuse you.

Space for Personal Notes

Sub-Section: Arithmetic Recurrence Relation

What about recurrence relations for arithmetic sequence?

Discussion: What must be the relationship between the current term (t_n) and the previous term (t_{n-1}) for an arithmetic sequence?

Add the common difference

Formula: Recurrence Relation for Arithmetic Sequence

$$t_n = t_{n-1} + d \text{ where } t_1 = a$$

Question 8

Consider the following n^{th} term rule for the arithmetic sequence:

$$t_n = 10 - 3(n-1)$$

Find the recurrence relation which corresponds to it.

$a = t_1 = 10 - 3 \times 0$

$a = 10$

$d = -3$

$t_n = t_{n-1} - 3$, where $t_1 = 10$

Sub-Section: Arithmetic Mean

What do we mean by the Arithmetic Mean?

Exploration: Finding arithmetic mean.

- Consider three terms of an arithmetic sequence with the common difference of d.

$$t_1 = a, t_2 = a + d \text{ and } t_3 = a + 2d.$$

- Say that t_2 is an arithmetic mean (average) of t_1 and t_3.
- Try finding the sum of t_1 and t_3.

$$t_1 + t_3 = 2a + 2d$$

- What should we do to $t_1 + t_3$ to find t_2?

$$t_2 = \frac{t_1 + t_3}{2}$$

The Arithmetic Mean

- **Definition:**

$$\text{Arithmetic Mean of } a \text{ and } b = \frac{a+b}{2}$$

NOTE: Arithmetic mean is same as the so called "normal average".

Space for Personal Notes

Sub-Section: Arithmetic Series

Discussion: What would be the most efficient way of adding all the whole numbers $1 - 100$?

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

$t_n = n$

$$\sum_{i=1}^{100} t_i$$

Question 9

Find the sum of all the odd numbers from 1 to 99.

$$1 + 3 + 5 + \cdots + 95 + 97 + 99.$$

$100 \times 25 = \boxed{2500}$

Now let's generalise it for all arithmetic sequences!

Arithmetic Series (Form 1)

- Use the following formula, if we know the first term, last term and number of terms.

$$S_n = \frac{n}{2}(a + l)$$

  - Where n =number of terms, a = first term and l = last term.
  - $\frac{n}{2}$ can be thought as the number of pairs
  - $a + l$ can be thought as the sum of each. pair

Question 10 Walkthrough.

Consider the arithmetic sequence with $t_1 = -5$ and $t_4 = 10$

Find S_4.

$$S_4 = \frac{n}{2}(a + l)$$

$$= \frac{4}{2}(-5 + 10)$$

$$= 2 \times 5 = 10$$

Question 11

Consider the arithmetic sequence with $t_1 = 3$ and $t_9 = 19$.

Find S_9.

$$S_9 = \frac{9}{2}(3+19)$$

$$= \frac{9}{2}(22)$$

$$= 99$$

Now let's generalise it for all arithmetic sequences!

Exploration: Arithmetic series for when we don't know the last term (l).

- Recall the series formula-

$$S_n = \frac{n}{2}(a + l)$$

- Which term would l be? $l = t_n$
- How can we define t_n for an arithmetic sequence?

$$t_n = a + (n-1)d$$

- On the space below, substitute $l = a + d(n-1)$ to the series formula!

$$S_n = \frac{n}{2}\left(a + a + (n-1)d\right)$$

$$= \frac{n}{2}\left(2a + d(n-1)\right)$$

Arithmetic Series (Form 2)

- Use the following formula, if we know the first term, common difference and number of terms.

$$\boxed{S_n = \frac{n}{2}(2a + d(n-1))}$$

  - Where n = number of terms, a = first term and d = common difference.

Question 12 Walkthrough.

Consider the arithmetic sequence with $t_n = 2 + 3(n-1)$. t_1

Find S_{10}.

$a = 2$

$d = 3$

$n = 10$

$$S_{10} = \sum_{i=1}^{10} t_i = \frac{10}{2}(2\times2 + 3(9))$$

$$= 5(4 + 27)$$

$$= 5 \times 31 = 155$$

Space for Personal Notes

Question 13

Consider the arithmetic sequence with $t_n = -2 + \frac{1}{3}(n-1)$.

Find S_{13}.

$n = 13$

$a = -2$

$d = \frac{1}{3}$

$S_{13} = \frac{13}{2}\left(2 \times -2 + \frac{1}{3}(12)\right)$

$= \frac{13}{2}(-4 + 4)$

$S_{13} = 0$

Key Takeaways

- Arithmetic sequence has a common difference between the next term and the current one.
- Arithmetic sequence is given by $t_n = a + (n-1)d$.
- Arithmetic mean of a and b is $\frac{a+b}{2}$.
- Arithmetic sum is given by $\frac{n}{2}(a + l)$ or $\frac{n}{2}(2a + (n-1)d)$.

Section C: Geometric Sequence and Series

Sub-Section: Geometric Sequence

Now let's consider another type of sequence, "Geometric" sequences.

Arirthmetic Sequence

- Remember again
- He ate one chocolate bar on the first day.
- The next day, he ate two chocolate bars.
- **Geometric sequence will suggest that Sam eats 4 chocolate bars on the 3^{rd} day!**

Discussion: In geometric sequence, how do we go from one term to the next?

Multiply by a number

Geometric Sequences

$$a, \quad ar, \quad ar^2, \quad ar^3$$

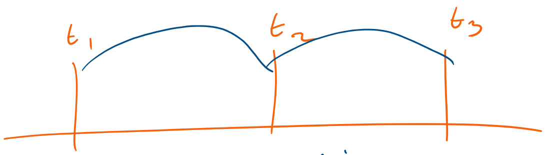
- **Definition:**
  - A Geometric sequence is one where we keep multiplying or dividing by **the common ratio** to get the next term.

$$t_n = ar^{n-1}$$

$t_n = a + (n-1)d$

  - Where r is the common ratio, and a is the first term.

Discussion: Why do we have a power of $n - 1$ instead of n? How many ratios do we multiply from t_1 to t_n?



2 times

Question 14 Walkthrough.

Consider the geometric sequence defined by $t_n = 2 \cdot \left(\frac{1}{3}\right)^{n-1}$.

Identify the common ratio, first term and the 4th term.

$r = \frac{1}{3}$

$a = t_1 = 2 \times \left(\frac{1}{3}\right)^0$

$= 2$

$t_4 = 2 \times \left(\frac{1}{3}\right)^3 = 2 \times \frac{1}{27} = \frac{2}{27}$

NOTE: Geometric sequence is an exponential!

Question 15

Consider the geometric sequence defined by $t_n = 6 \cdot (2)^n$.

Identify the common ratio, first term and the 2nd term.

$r = 2$

$a = t_1 = 6 \times 2 = 12$

$t_2 = 6 \times 2^2 = 24$

NOTE: Read the question carefully. Sometimes, they expand the $n - 1$ power to confuse you!

Space for Personal Notes

Sub-Section: Geometric Recurrence Relation

What about recurrence relations for geometric sequence?

Discussion: What must be the relationship between the current term (t_n) and the previous term (t_{n-1}) for a geometric sequence?

Always multiply by r

Recurrence Relation for Geometric Sequence

$$t_n = t_{n-1} \times r \text{ where } t_1 = a$$

Question 16

Consider the following n^{th} term rule for the geometric sequence

$$t_n = 2 \cdot 4^{n-1}$$

Find the recurrence relation which corresponds to it.

$r = 4$

$a = t_1 = 2$

$t_n = t_{n-1} \times 4$, where $t_1 = 2$

Sub-Section: Geometric Mean

How do we find a geometric term between two other geometric terms?

Exploration: Finding geometric mean.

- Consider three terms of a geometric sequence with the common ratio of r.

$$t_1 = a, t_2 = ar \text{ and } t_3 = ar^2$$

- Geometric mean simply means a middle geometric term

  Here we can say, t_2 is a geometric mean (average) of t_1 and t_3

- Find the product of t_1 and t_3.

$$t_1 \cdot t_3 = a^2 r^2$$

- What should we do to $t_1 \cdot t_3 = a^2r^2$ to find $t_2 = ar$?

$$t_2 = \sqrt{t_1 \times t_3}$$

The Geometric Mean

- **Definition:** The **geometric mean** of two numbers a and b is the geometric term in between a and b.

$$a \qquad \sqrt{ab} \qquad b$$

$$\textbf{Geometric Mean of } a \textbf{ and } b = \sqrt{ab}$$

Space for Personal Notes

TIP: Remember the similarity!

- Arithmetic Mean: We add the two and divide by 2.
- Geometric Mean: We multiply the two and square root.

Question 17

a. Find the geometric mean of 5 and 20.

$\sqrt{5 \times 20} = 10$

b. Explain in words why 10 being a geometric mean makes sense.

$5 \xrightarrow{\times 2} 10 \xrightarrow{\times 2} 20$

Space for Personal Notes

Sub-Section: Geometric Series

Geometric Series

- **Definition:** Sum of first n geometric terms is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

- Where n = number of terms, a = first term and r = common ratio.

Question 18 Walkthrough.

Consider the geometric sequence $t_n = 2 \cdot (3)^{n-1}$.

Find S_4.

$n = 4$

$r = 3$

$a = t_1 = 2$

$$S_4 = \frac{2 \times (3^4 - 1)}{3 - 1} = \frac{2 \times (80)}{2} = 80$$

Question 19

Consider the ~~arithmetic~~ geometric sequence with $t_n = 4 \cdot \left(\frac{1}{2}\right)^{n-1}$.

Find S_5.

$n = 5$

$a = 4$

$r = \frac{1}{2}$

$$S_5 = \frac{4\left(\left(\frac{1}{2}\right)^5 - 1\right)}{\frac{1}{2} - 1}$$

$$= \frac{4\left(\frac{1}{32} - 1\right)}{-\frac{1}{2}}$$

$$= -8\left(-\frac{31}{32}\right)$$

$$= \frac{31}{4}$$

Space for Personal Notes

Sub-Section: Infinite Geometric Series

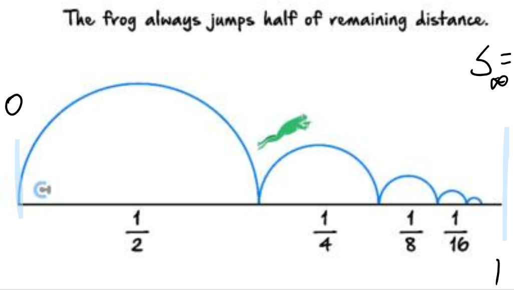
What does infinite geometric series even mean?

Context

- Imagine a frog jumping from one end of the pond to the other.

  Here's the catch!

The frog always jumps half of remaining distance.



$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1/2}{1/2} = 1$$

Question 20

Construct a geometric sequence for which its terms represent the % of the distance between the two ends of the pond that the frog covers in his n^{th} jump.

$d = t_{n+1} - t_n$

$a = \frac{1}{2}$

$r = \frac{1/4}{1/2} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$

$r = \frac{t_{n+1}}{t_n}$ $\qquad t_n = \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1}$

NOTE: Notice how, even if the frog jumps infinitely, its distance covered is still a finite number.

- Even if we add infinitely many geometric terms, the series(sum) can still be finite.
- This is called "Zeno's Paradox".

NOTE: If the common ratio is higher than 1 like the above discussion, the infinite series will not be a finite number.

The Infinite Geometric Series

- **Definition:** The sum of infinitely many geometric terms is given by

$$S_\infty = \frac{a}{1-r}$$

IMPORTANT: Only works when $\boxed{-1 < r < 1.}$

Question 21 Walkthrough.

Identify the first term, common ratio and hence find the infinite series.

$$1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

$\frac{1}{10}$

$r = \frac{1/10}{1}$

$= 1/10$

$a = 1$

$r = 1/10$

$S_\infty = \frac{1}{1 - 1/10}$

$= \frac{1}{9/10} = \frac{10}{9}$

Question 22

Identify the first term, common ratio and hence find the infinite series.

$$1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\cdots$$

$a = 1$

$r = \frac{-\frac{2}{3}}{1} = \frac{4/9}{-\frac{2}{3}} = -2/3$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{2}{3}\right)}$

$= \frac{1}{5/3} = 3/5$

NOTE: The common ratio must be between -1 and 1 for an infinite series to be a finite number.

Key Takeaways

- Geometric sequence has a common ratio between the next term and the current one.
- Geometric sequence is given by $t_n = ar^{n-1}$.
- Geometric mean of a and b is $\sqrt{ab}$.
- Geometric sum is given by $\frac{a(r^n-1)}{r-1}$.
- Infinite geometric sum is given by $\frac{a}{1-r}$, where $-1 < r < 1$.

CONTOUREDUCATION
Website: contoureducation.com.au | **Phone:** 1800 888 300 | **Email:** hello@contoureducation.com.au

VCE Specialist Mathematics ½

1-on-1 Maths Consults

What Are 1-on-1 Maths Consults?

- Individual 30-minute consultations via Zoom with a Contour tutor, where you can ask questions, clarify doubts, get tips, advice, and support in a one-on-one format.
- Complimentary (yes, entirely free) with your Contour enrolment (as long as you're enrolled in a Maths subject at Contour). You can book up to a week in advance.

SAVE THE LINK, AND MAKE THE MOST OF THIS (FREE) SERVICE!

Booking Link
bit.ly/maths-consults