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VCE Specialist Mathematics ½
Sequence and Series [1.3]
Workbook

Outline:

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Section A: Introduction to Sequences and Series

Sub-Section: Sequences

Sequences

- Consider Samuel, our beloved Contour Tutor.

- Lets say he eats a chocolate bar on the first day.
- The next day, he eats two chocolate bars.
- How many chocolate bars do you think he eats on the third day?
- Hands up if you think it's 3 chocolate bars!
  - You guys have taken an **arithmetic** approach!
- Hands up if you think it's 4 chocolate bars!
  - You guys have taken a **geometric** approach!
  - **We will consider the two types of sequences today!**

$+1 \quad +1$
$1, 2, 3$
$\times 2 \quad \times 2$
$1, 2, 4$

Definition

Sequences

t_1, t_2, t_3

term

$$t_n = f(n)$$

- **Definition:** A sequence is an ordered list of numbers following a certain pattern.
- It is a function of the order n.

Question 1 Walkthrough.

Construct the first 3 terms for the sequence given by $t_n = 2n + 1$.

$t_1 = 2(1)+1 = 3$

$t_2 = 2(2)+1 = 5$

$t_3 = 2(3)+1 = 7$

$\{3, 5, 7\}$

NOTE: t_n stands for the n^{th} term. Eg: For t_3, our value of n is equal to 3.

ALSO NOTE: This was a sequence of odd numbers!

Question 2

The sequence is defined by $t_n = 2^n + 1$. Identify the term number for which t_n equals 9.

$t_n = 2^n + 1$

$9 = 2^n + 1$

$2^n = 8$

$n = 3$

3rd term

Space for Personal Notes

Sub-Section: Recurrence Relations

What if we define the term t_n with respect to the previous term (t_{n-1})?

Recurrence Relations

- **Definition:**
  - A recurrence relation is when we define **a term** (t_n), in terms of the **previous one** (t_{n-1}).
  - Recurrence relations generate sequences of the form:

$$t_n = f(t_{n-1}) \text{ where } t_1 = a$$

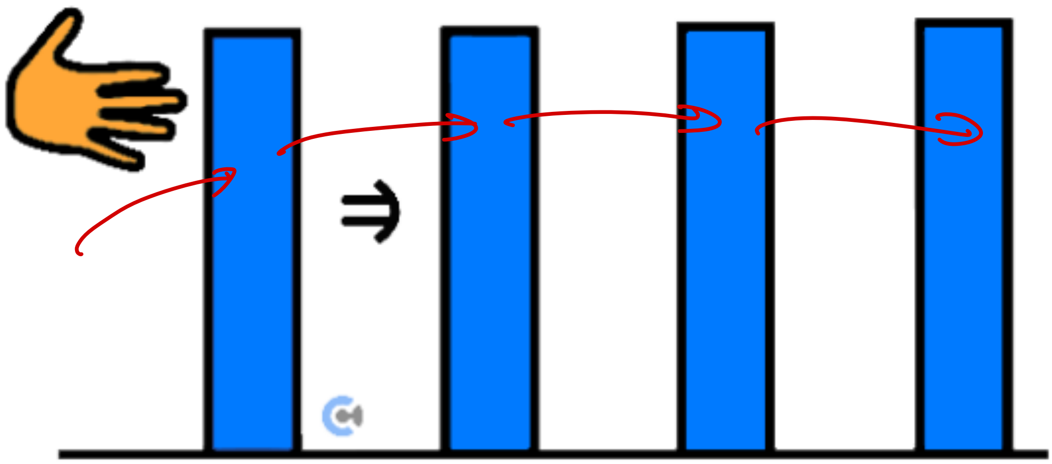
Or

$$t_{n+1} = f(t_n) \text{ where } t_1 = a$$

  - It must always include a t_1.

Analogy: Recurrence Relations

- It's like knocking off a sequence of dominos!



- We focus on the Relationship between the two terms (dominos).

Question 3

Consider the following recurrence relation.

$$t_n = 3\boxed{t_{n-1}} + 2,\ t_1 = 2$$

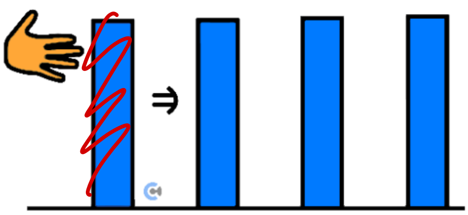
State the value of t_3.

$t_2 = 3(2) + 2 = 8$

$t_3 = 3(8) + 2 = 26$

Why do we always need a first term?

<u>Analogy</u>: **Reason for why recurrence relations always need a first term.**



- Can you knock off any of these dominos without knocking over the first one?
- Similarly, how can we solve for any term in recurrence relation without the first term? It's impossible!

→ Rule → First term

Sub-Section: Introduction to Series

What does the word "series" mean?

$\{1,2,3,4\}$ $\sum = 10$

$\{1,2,3,4,5,6,7\}$ $\sum_{i=1}^{3}$

Series

- **Definition:**

A series is the sum of the first n terms of a sequence. sum.

n-th

$$S_n = \sum_{i=1}^{n} t_i$$

first

$$\sum_{i=1}^{4} i^2$$

$= t_1 + t_2 + t_3 + t_4$

$= 1^2 + 2^2 + 3^2 + 4^2$

Question 4 Walkthrough.

Consider the sequence given by $t_n = 3n - 4$.

Evaluate S_2.

$= \sum_{n=1}^{2} 3n - 4$

$= t_1 + t_2$

$= 3(1) - 4 + 3(2) - 4$

$= 3 - 4 + 6 - 4$

$= 1$

Question 5

Consider the sequence given by $t_n = 2n + 2$.

Evaluate S_4.

$$S_4 = \sum_{n=1}^{4} 2n+2$$

$$= t_1 + t_2 + t_3 + t_4$$

$$= 4 + 6 + 8 + 10$$

$$= 28$$

Key Takeaways

- ☑ Sequence follows a certain pattern.
- ☑ Recurrence relation is a relationship between the next term and the current term.
- ☑ Series is the sum of the first n terms.

Section B: Arithmetic Sequence and Series

Sub-Section: Introduction to Arithmetic Sequence

Let's go back to the previous context

Arirthmetic Sequence

- Consider Samuel, our beloved Contour Tutor.

- He ate one chocolate bar on the first day.
- The next day, he ate two chocolate bars.
- Arithmetic sequence suggests Sam eats 3 chocolate bars on the third day !

Discussion: In an arithmetic sequence, how do we go from one term to the next?

Add the same number
(minus)

Arithmetic Sequences

Definition

$$\overset{+d}{}\quad\overset{+d}{}\quad\overset{+d}{}$$

$$\underset{t_1}{a,}\quad \underset{t_2}{a+d,}\quad \underset{t_3}{a+2d,}\quad \underset{t_4}{a+3d}$$

- An **arithmetic sequence** is one where the **common difference** is added or subtracted to get the next term.

$$t_n = \overset{t_1}{a} + (n-1)d$$

common difference

- Where d is the common difference, and a is the first term.

Discussion: Why do we add $(n-1)d$ instead of nd? How many differences do we add from t_1 to t_n?

-1

First term → zero $+d$

Second term → one $+d$

-1

Question 6 Walkthrough.

Consider the arithmetic sequence defined by $\boxed{t_n = 12 + 6(n-1)}$.

Identify the common difference, first term and the 5^{th} term.

t_1 common diff

$n=5$

$t_5 = 12 + 6\times(5-1)$

$= 12 + 6\times 4$

$= 12 + 24$

$= 36$

Question 7

Consider the arithmetic sequence defined by $t_n = -1 + 6n$.

$a + (n-1)d$.

Identify the common difference, first term and the 8^{th} term.

$= 6$

$n=1$

$t_1 = -1 + 6(1)$

$t_1 = 5$

$n=8$

$t_8 = -1 + 6(8)$

$= -1 + 48$

$= 47$

NOTE: Read the question carefully. Sometimes, they expand the $n - 1$ factor to confuse you.

t_1 $n=1$

Space for Personal Notes

Sub-Section: Arithmetic Recurrence Relation

+/ −

What about recurrence relations for arithmetic sequence?

Discussion: What must be the relationship between the current term (t_n) and the previous term (t_{n-1}) for an arithmetic sequence?

$t_n = t_{n-1} + d$

Formula: Recurrence Relation for Arithmetic Sequence

$$t_n = t_{n-1} + d \text{ where } t_1 = a$$

+2 +2 +2

1, 3, 5, 7

Question 8

Consider the following n^{th} term rule for the arithmetic sequence:

$$t_n = \underbrace{10}_{t_1}\overbrace{\boxed{-3}}^{d}(n-1)$$

t_1 t_2 t_3

Find the recurrence relation which corresponds to it.

$t_n = t_{n-1} - 3$, $t_1 = 10$

1, 3, 5, 7

$t_n = 1 + 2(n-1)$

Sub-Section: Arithmetic Mean

What do we mean by the Arithmetic Mean?

Exploration: Finding arithmetic mean.

Average

- Consider three terms of an arithmetic sequence with the common difference of d.

$$t_1 = a, t_2 = a + d \text{ and } t_3 = a + 2d.$$

- Say that t_2 is an arithmetic mean (average) of t_1 and t_3 .
- Try finding the sum of t_1 and t_3.

$$t_1 + t_3 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- What should we do to $t_1 + t_3$ to find t_2?

$$t_2 = \frac{t_1 + t_3}{2}$$

The Arithmetic Mean

- **Definition:**

$$\textbf{Arithmetic Mean of } a \textbf{ and } b = \frac{a+b}{2}$$

NOTE: Arithmetic mean is same as the so called "normal average".

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Sub-Section: Arithmetic Series

→ add

$\{1, 2, 3, 4, 5\}$

Discussion: What would be the most efficient way of adding all the whole numbers $1 - 100$?

+1

50, 51

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

101

101

101

50×101

Question 9

Find the sum of all the odd numbers from 1 to 99.

49, 51

$$1 + 3 + 5 + \cdots + 95 + 97 + 99.$$

100

100

1, 2, 3, 4, 5, 6, 7, 8

... 48 | 49 50

24

1

25 pairs

$25 \times 100 = 2500$

Now let's generalise it for all arithmetic sequences!

Arithmetic Series (Form 1)

- Use the following formula, if we know the first term, last term and number of terms.

t_1 t_n

$$\boxed{S_n = \frac{n}{2}(a + l)}$$

- Where n =number of terms, a = first term and l = last term.
- $\frac{n}{2}$ can be thought as the number of pairs.
- $a + l$ can be thought as the sum of each pair.

Question 10 Walkthrough.

Consider the arithmetic sequence with $t_1 = -5$ and $t_4 = 10$.

Find S_4.

$S_n = \frac{n}{2}(a+l)$

t_1 t_n

$S_4 = \frac{4}{2}(-5+10)$

$= 2(5)$

$= 10$ #

Question 11

Consider the arithmetic sequence with $t_1 = 3$ and $t_9 = 19$.

Find S_9.

$$S_n = \frac{n}{2}(a+l)$$
$$= \frac{9}{2}(3+19)$$
$$= \frac{9}{2} \times 22$$
$$= 9 \times 11$$
$$= 99$$

Now let's generalise it for all arithmetic sequences!

Exploration: **Arithmetic series for when we don't know the last term (l).**

- Recall the series formula-

$$\boxed{S_n = \frac{n}{2}(a+l)}$$

- Which term would l be? t_n
- How can we define t_n for an arithmetic sequence?

$$t_n = a + d(n-1)$$

- On the space below, substitute $l = a + d(n-1)$ to the series formula!

$$S_n = \frac{n}{2}(a + a + d(n-1))$$
$$= \frac{n}{2}(2a + d(n-1))$$

Arithmetic Series (Form 2)

- Use the following formula, if we know the first term, common difference and number of terms.

$$\boxed{S_n = \frac{n}{2}(2a + d(n-1))}$$

- Where $n =$ number of terms, $a =$ first term and $d =$ common difference.

Question 12 Walkthrough.

$a, d.$

Consider the arithmetic sequence with $\boxed{t_n = 2 + 3(n-1)}$.

Find S_{10}.

a d

$$S_{10} = \frac{10}{2}(2a + d(10-1))$$
$$= 5(2(2) + 3(9))$$
$$= 5(4 + 27)$$
$$= 5 \times 31 \quad = 155$$

$t_1 = 2$

$t_{10} = 2 + 3(9)$
$= 29$

$S_n = \frac{10}{2}(2 + 29)$
$= 5(31)$
$= 155$

Space for Personal Notes

Question 13

Consider the arithmetic sequence with $t_n = -2 + \boxed{\frac{1}{3}}(n-1)$.

a d

Find S_{13}.

$S_{13} = \frac{13}{2}(2a + d(n-1))$

$= \frac{13}{2}(2(-2) + \frac{1}{3}(12))$

$= \frac{13}{2}(-4 + 4)$

$= \frac{13}{2} \times 0$

$= 0$

Key Takeaways

- Arithmetic sequence has a common difference between the next term and the current one.
- Arithmetic sequence is given by $t_n = a + (n-1)d$.
- Arithmetic mean of a and b is $\frac{a+b}{2}$.
- Arithmetic sum is given by $\frac{n}{2}(a + l)$ or $\frac{n}{2}(2a + (n-1)d)$.

Section C: Geometric Sequence and Series

Sub-Section: Geometric Sequence

Now let's consider another type of sequence, "Geometric" sequences.

Arirthmetic Sequence

- Remember again
- He ate one chocolate bar on the first day. $\times 2$
- The next day, he ate two chocolate bars. $\times 2$
- **Geometric sequence will suggest that Sam eats 4 chocolate bars on the 3^{rd} day!**

Discussion: In geometric sequence, how do we go from one term to the next?

multiply by the same number

Geometric Sequences

$$\overset{\times r}{\curvearrowright}\quad\overset{\times r}{\curvearrowright}\quad\overset{\times r}{\curvearrowright}$$

$$a, \quad ar, \quad ar^2, \quad ar^3$$

$$t_1 \quad t_2$$

- **Definition:**
  - A Geometric sequence is one where we keep multiplying or dividing by **the common ratio** to get the next term.

$$t_n = a \times r^{\boxed{n-1}}$$

common ratio

  - Where r is the common ratio, and a is the first term.

Discussion: Why do we have a power of $n-1$ instead of n? How many ratios do we multiply from t_1 to t_n?

Question 14 Walkthrough.

Consider the geometric sequence defined by $t_n = 2 \cdot \boxed{\left(\frac{1}{3}\right)}^{n-1}$.

r

Identify the common ratio, first term and the 4th term.

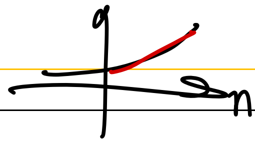
$n=1$

$$t_1 = 2 \times \left(\frac{1}{3}\right)^0 = 2 \times 1 = 2$$

$$t_4 = 2 \times \left(\frac{1}{3}\right)^{4-1} = 2 \times \left(\frac{1}{3}\right)^3$$

$$= \frac{2}{27}$$

NOTE: Geometric sequence is an exponential! ⇒ only uses whole numbers



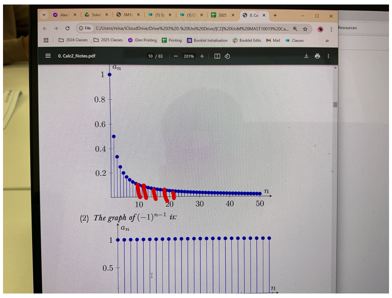
exponential

Question 15

Consider the geometric sequence defined by $t_n = 6 \cdot (2)^n$.

Identify the common ratio, first term and the 2^{nd} term.

2 12 24



NOTE: Read the question carefully. Sometimes, they expand the $n-1$ power to confuse you!

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Sub-Section: Geometric Recurrence Relation

↳ term before

What about recurrence relations for geometric sequence?

Discussion: What must be the relationship between the current term (t_n) and the previous term (t_{n-1}) for a geometric sequence?

$t_n = r \times t_{n-1}$

new = ratio × old

Recurrence Relation for Geometric Sequence

$$t_n = t_{n-1} \times r \text{ where } t_1 = a$$

Question 16

Consider the following n^{th} term rule for the geometric sequence

$$t_n = 2 \cdot \boxed{4}^{n-1}$$

2 3 4

Find the recurrence relation which corresponds to it.

$r = 4$

$t_n = t_{n-1} \times 4 \quad t_1 = 2$

Sub-Section: Geometric Mean

How do we find a geometric term between two other geometric terms?

Exploration: Finding geometric mean.

t_1, t_2, t_3

- Consider three terms of a geometric sequence with the common ratio of r.

$$t_1 = a, t_2 = ar \text{ and } t_3 = ar^2$$

- Geometric mean simply means a middle term of 2 terms.

  Here we can say, t_2 is a geometric mean (average) of t_1, t_3.

- Find the product of t_1 and t_3.

$a \quad ar^2$

$$t_1 \cdot t_3 = a^2r^2 = (ar)^2$$

t_2

- What should we do to $t_1 \cdot t_3 = a^2r^2$ to find $t_2 = ar$?

$$t_2 = \sqrt{t_1 \times t_3}$$

$t_1 \times t_3 = t_2^2$

The Geometric Mean

① Multiply ② square root

- **Definition:** The **geometric mean** of two numbers a and b is the geometric term in between a and b.

$$a \qquad \sqrt{ab} \qquad b$$

$$\text{Geometric Mean of } a \text{ and } b = \sqrt{ab}$$

Space for Personal Notes

TIP: Remember the similarity!

- Arithmetic Mean: We add the two and divide by 2.
- Geometric Mean: We multiply the two and square root.

Question 17

a. Find the geometric mean of 5 and 20.

$\sqrt{5\times 20} = \sqrt{100} = 10$

b. Explain in words why 10 being a geometric mean makes sense.

$5 \xrightarrow{\times 2} 10 \xrightarrow{\times 2} 20$

There is a common ratio of 2 and hence 10 is the middle of the geometric sequence containing 5, 20

Space for Personal Notes

Sub-Section: Geometric Series

Geometric Series

- **Definition:** Sum of first n geometric terms is given by: common ratio

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

t_1

- Where n = number of terms, a = first term and r = common ratio.

Question 18 Walkthrough.

Consider the geometric sequence $t_n = 2 \cdot (3)^{n-1}$.

Find S_4.

$$a = t_1 = 2 \cdot (3)^{1-1} = 2 \times 3^0 = 2 \times 1 = 2$$

$$S_4 = \frac{a(r^n - 1)}{r - 1} = \frac{2(3^4 - 1)}{3 - 1} = \frac{2 \times (81 - 1)}{2} = 80$$

Question 19

Consider the geometric sequence with $t_n = 4 \cdot \left(\frac{1}{2}\right)^{n-1}$.

$r = \frac{1}{2}$

$a = t_1 = 4 \times \left(\frac{1}{2}\right)^0 = 4$

Find S_5.

$$S_5 = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{4\left(\left(\frac{1}{2}\right)^5 - 1\right)}{\frac{1}{2} - 1}$$

$$= \frac{4 \times \left(\frac{1}{32} - 1\right)}{-1/2} = 8\left(\frac{31}{32}\right)$$

$$= \frac{31}{4}$$

Space for Personal Notes

multiply (terms)

Sub-Section: Infinite Geometric Series

add

S_∞

What does infinite geometric series even mean?

Context

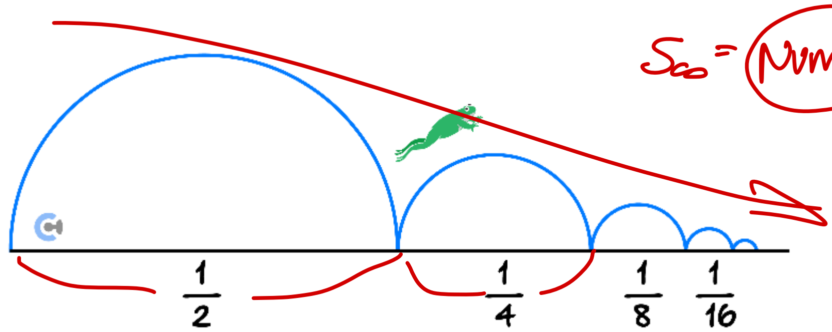
- Imagine a frog jumping from one end of the pond to the other.

  Here's the catch!

The frog always jumps half of remaining distance.

The frog always jumps half of remaining distance.

$S_\infty =$ Number



Question 20 $t_n = a \times r^{n-1}$

Construct a geometric sequence for which its terms represent the % of the distance between the two ends of the pond that the frog covers in his n^{th} jump.

$a = t_1 = \frac{1}{2}$

$r = \frac{1}{2}$

$t_n = \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1}$

$\to 0 < r < 1$

NOTE: Notice how, even if the frog jumps infinitely, its distance covered is still a finite number.

- Even if we add infinitely many geometric terms, the series(sum) can still be finite.
- This is called "Zeno's Paradox".

NOTE: If the common ratio is higher than 1 like the above discussion, the infinite series will not be a finite number.

Definition

The Infinite Geometric Series

- **Definition:** The sum of infinitely many geometric terms is given by

$$S_\infty = \frac{a}{1-r}$$

IMPORTANT: Only works when $-1 < r < 1$.

Question 21 Walkthrough.

Identify the first term, common ratio and hence find the infinite series.

$$1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

$a = 1$

$r = \frac{1}{10}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{1}{9/10} = \frac{10}{9}$

Question 22

Identify the first term, common ratio and hence find the infinite series.

$$1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\cdots$$

$a=1$

$r=-\frac{2}{3}$

$S_\infty=\frac{a}{1-r}=\frac{1}{1+\frac{2}{3}}=\frac{1}{5/3}=3/5$

NOTE: The common ratio must be between -1 and 1 for an infinite series to be a finite number.

<u>**Key Takeaways**</u>

- Geometric sequence has a common ratio between the next term and the current one.
- Geometric sequence is given by $t_n = ar^{n-1}$.
- Geometric mean of a and b is $\sqrt{ab}$.
- Geometric sum is given by $\frac{a(r^n-1)}{r-1}$.
- Infinite geometric sum is given by $\frac{a}{1-r}$, where $-1 < r < 1$.

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