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VCE Specialist Mathematics ½
Modulus & Partial Fractions Exam Skills [1.2]
Workbook

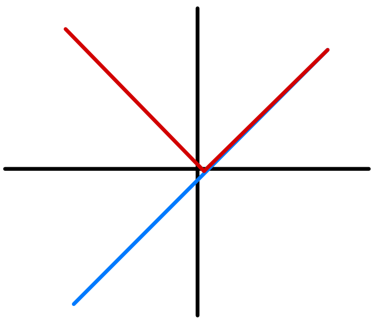
Outline:

Section A: Recap of [1.1] – Modulus and Partial Fractions

Modulus Functions

- **Definition:**

$$f(x) = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$



- Is a **hybrid function.**
- **Purpose:** Always return a non-negative number.
- **Range:** $[0, \infty)$.

Alternative Definition of Modulus Functions

$$\sqrt{x^2} = |x|$$

NOTE: Important not to forget the modulus in the exams!

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Sub-Section: Solving Modulus Equations

Solving Equations Involving Modulus Functions

$$|f(x)| = b$$

$$f(x) = \underline{\pm b}$$

Interpretation:

- The size of $f(x)$ equals to b.
- $f(x)$ can be either $\pm b$.

TIP: Check your solutions by substituting them back into the equation!

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Sub-Section: Solving Modulus Inequalities

Solving Modulus Inequalities

$$|x-a| \le b$$

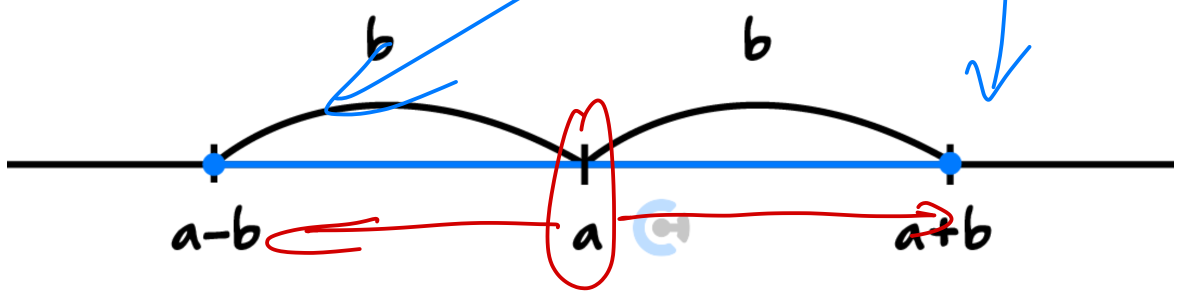
$x - a = \pm b$

$x = a \pm b$

- Interpretation:

x has a distance from 'a' that is less than or equal to 'b'

- Visualise:



$a - b \le x \le a + b$

TIP: Always sketch a number line!

Question 1 Walkthrough.

Solve the following inequality.

$|x-a| \le b$

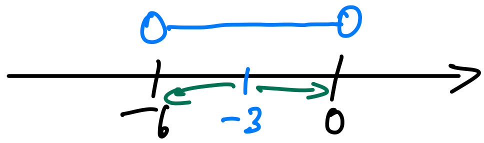
$|x-(-3)| < 3$

$$|x+3| < 3$$

$x + 3 = \pm 3$

$x = -3 \pm 3$

$x = -6, 0$



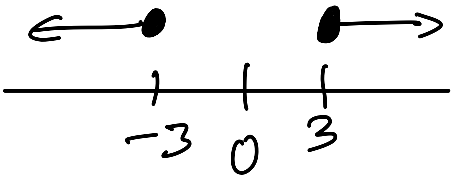
$x \in (-6, 0)$

Question 2

Solve each of the following inequalities for x:

a. $|x| \geq 3$ $\longrightarrow |x-a| \geq b$

$x = \pm 3$



$x \in (-\infty, -3] \cup [3, \infty)$

b. $|2x - 4| - 3 \geq 5$

$|2x-4| \geq 8$

$|2| \times |x-2| \geq 8$

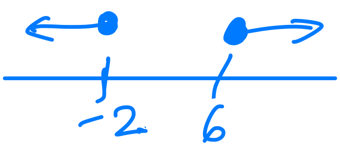
$2|x-2| \geq 8$

$|x-2| \geq 4$

$x - 2 = \pm 4$

$x = 2 \pm 4$

$x = -2, 6$



$x \in (-\infty, -2] \cup [6, \infty)$

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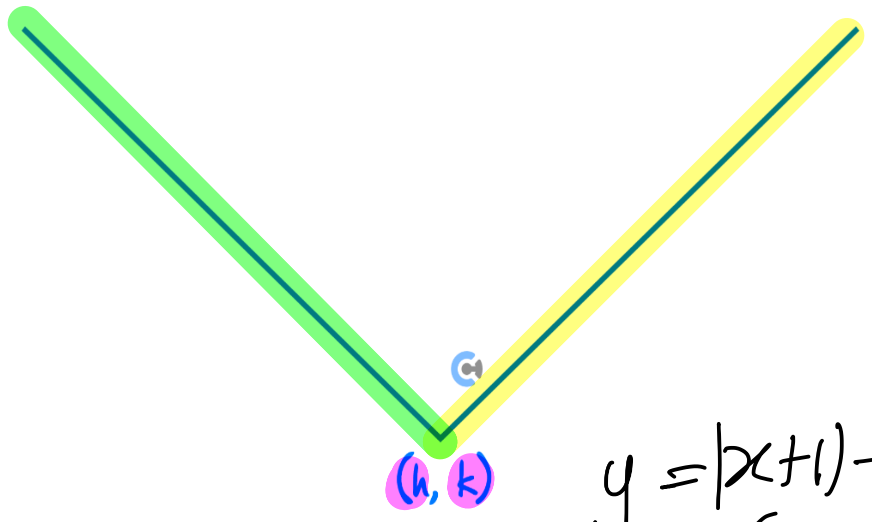
Sub-Section: Sketching Modulus Functions

Let's now consider the graph of modulus functions!

Graph of The Modulus Function



$y = |x+1| - 4$

$(-1, -4)$

- **General form:**

$$y = a|x - h| + k$$

$x + 1 = 0$

$x = -1$

Vertex is at (h, k).

- **Hybrid form:**

$$y = \begin{cases} a(x-h)+k, & x \geq h \\ -a(x-h)+k, & x < h \end{cases}$$

TIP: Think of modulus functions as a "**straightened quadratic**".

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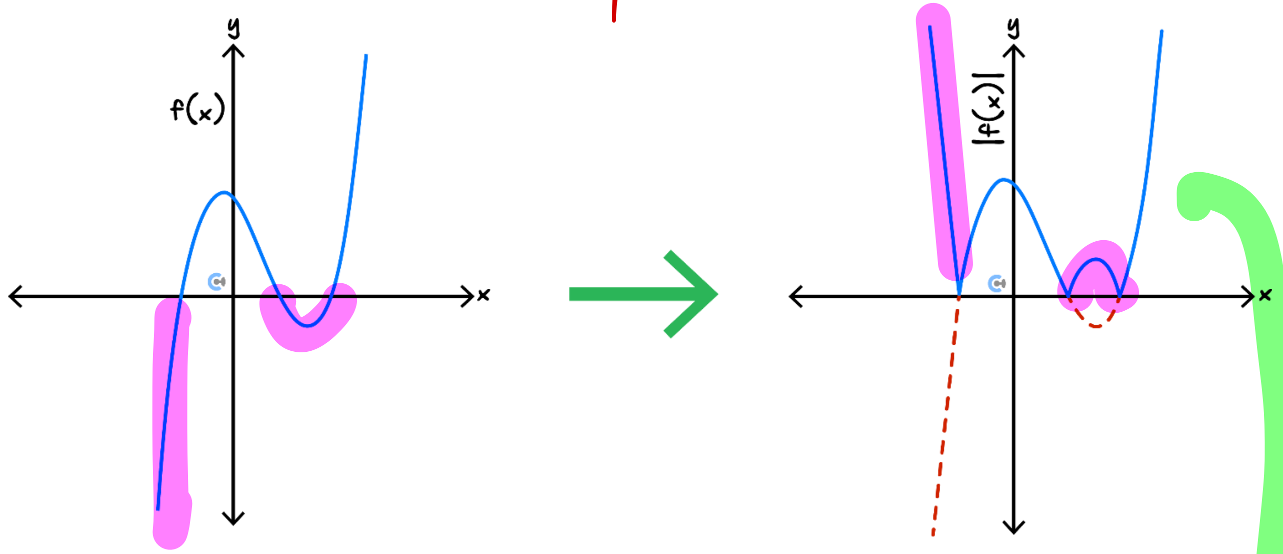
Sub-Section: Graphing Composite of Modulus Functions

Graphs of Composite Modulus Functions

- Modulus is the outer function.

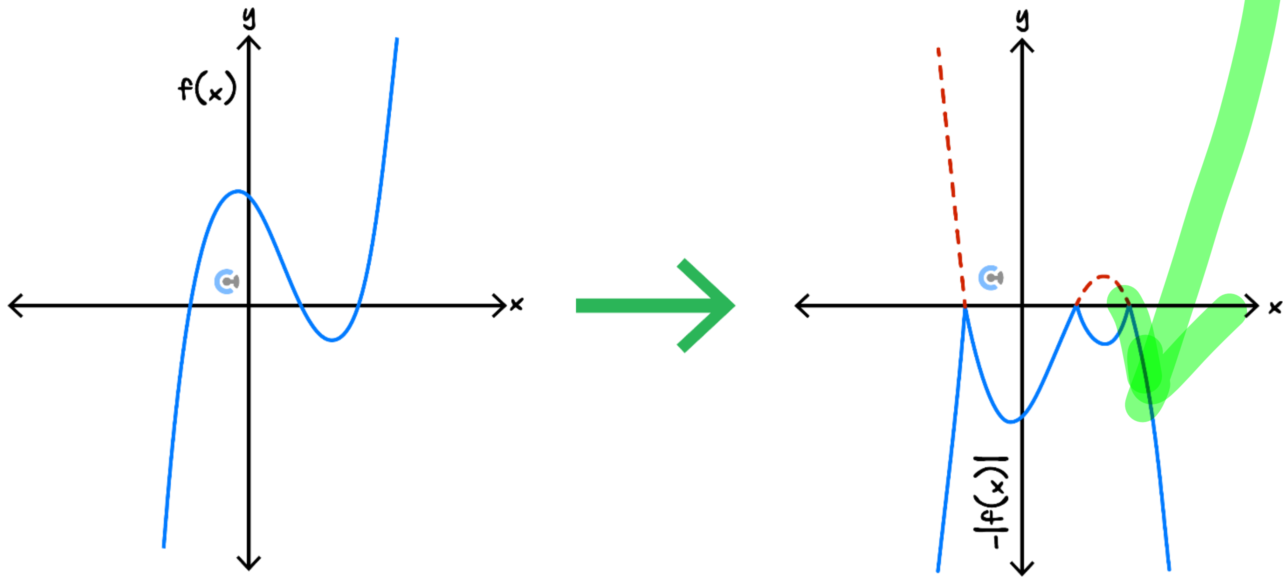
$$y = |f(x)|$$

- All negative y-values are flipped to be positive.



$$y = -|f(x)|$$

- The graph of $y = |f(x)|$ has undergone a reflection in the [x-axis] / [y-axis].

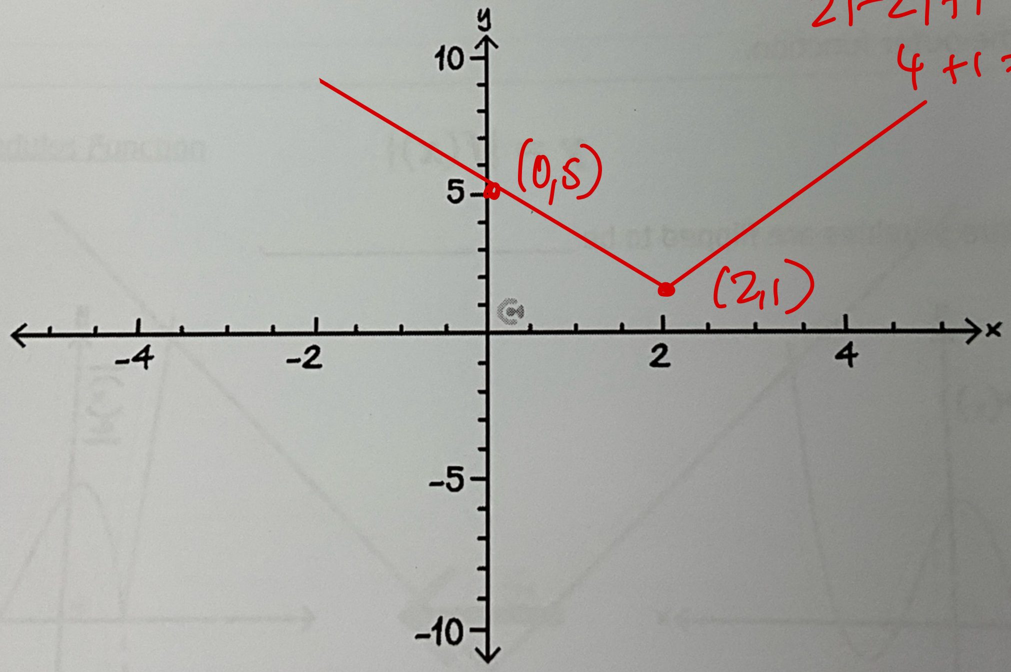


Question 3

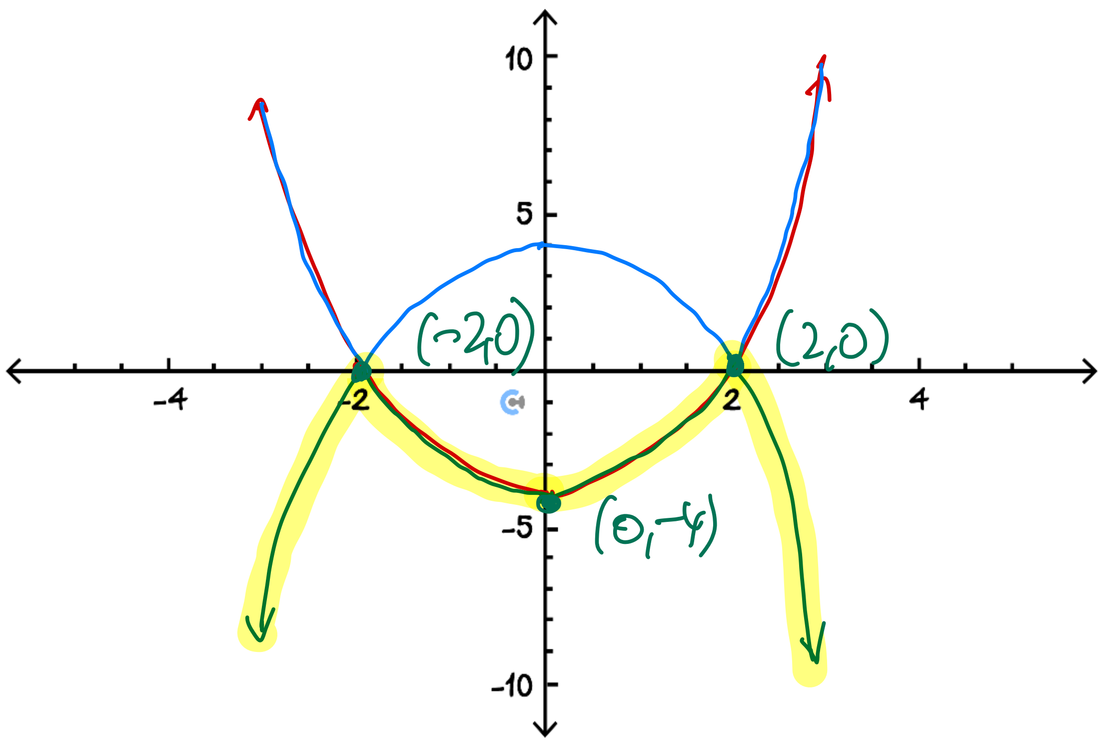
Sketch the following graph over the specified domain. Label all key intercepts.

$$y = 2|x - 2| + 1$$

$x-2=0$
$x=2$
$2|-2|+1$
$4+1=5$



b. $y = -|(x+2)(x-2)|$



Discussion: What would happen if $f(x)$ turned into $f(|x|)$?

- f will always take a [positive] / [negative] value, even if the x-value is negative.

x (anything) → $|x|$ $[0, \infty)$ → $f(|x|)$

- At:
  - $x = -2$: $f(-2) = f(2)$
  - $x = 2$: $f(2)$

Discussion: Since $f(|-2|) = f(|2|)$, where is $f(|x|)$ symmetrical about?

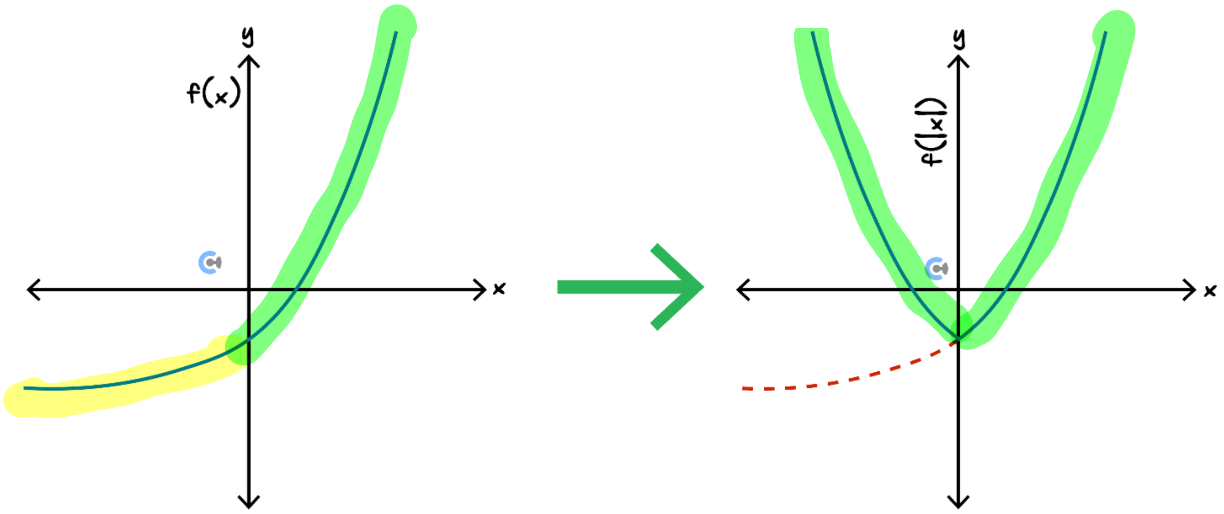
y-axis, $x = 0$

Graphs of Composite Modulus Functions

- Modulus is the inside function.

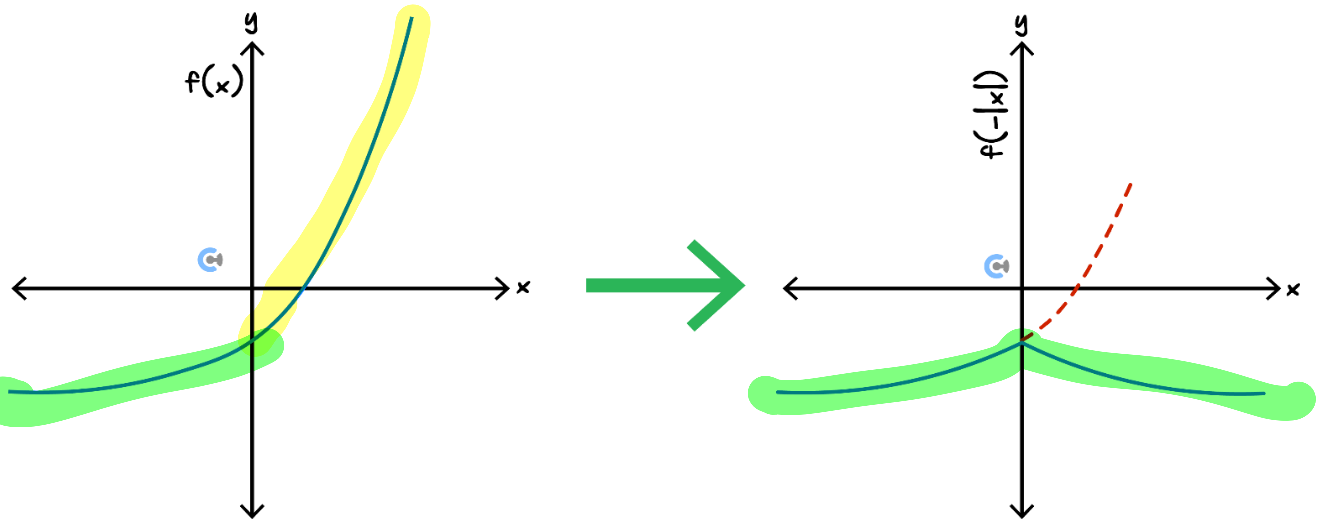
$$y = f(|x|)$$

- Take the positive side and flip it to the other side.



Take the negative side and flip it to the other side.

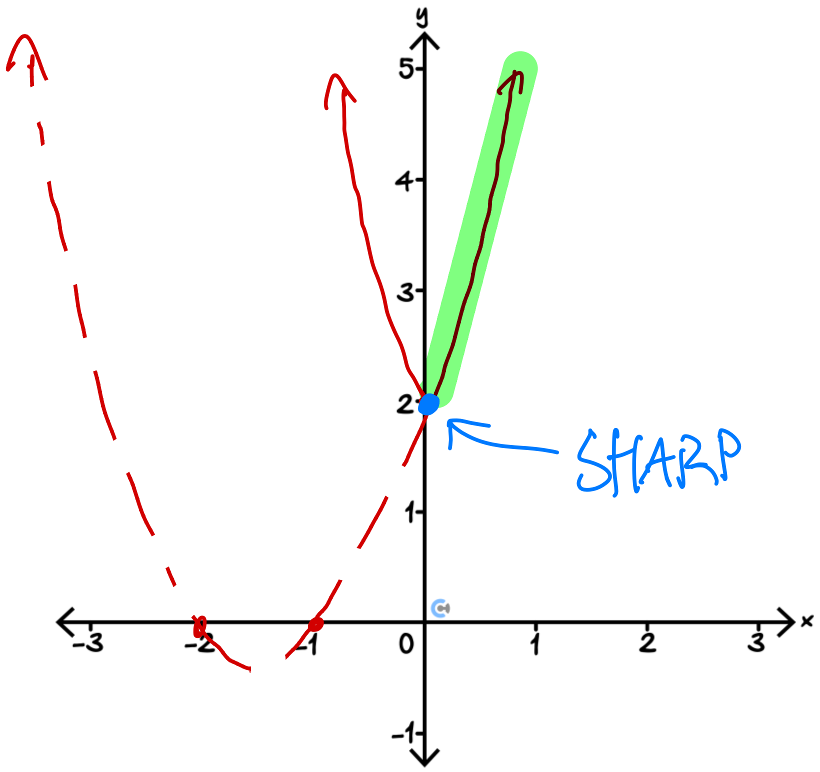
$$y = f(-|x|)$$



Question 4 Walkthrough.

Sketch the graph below.

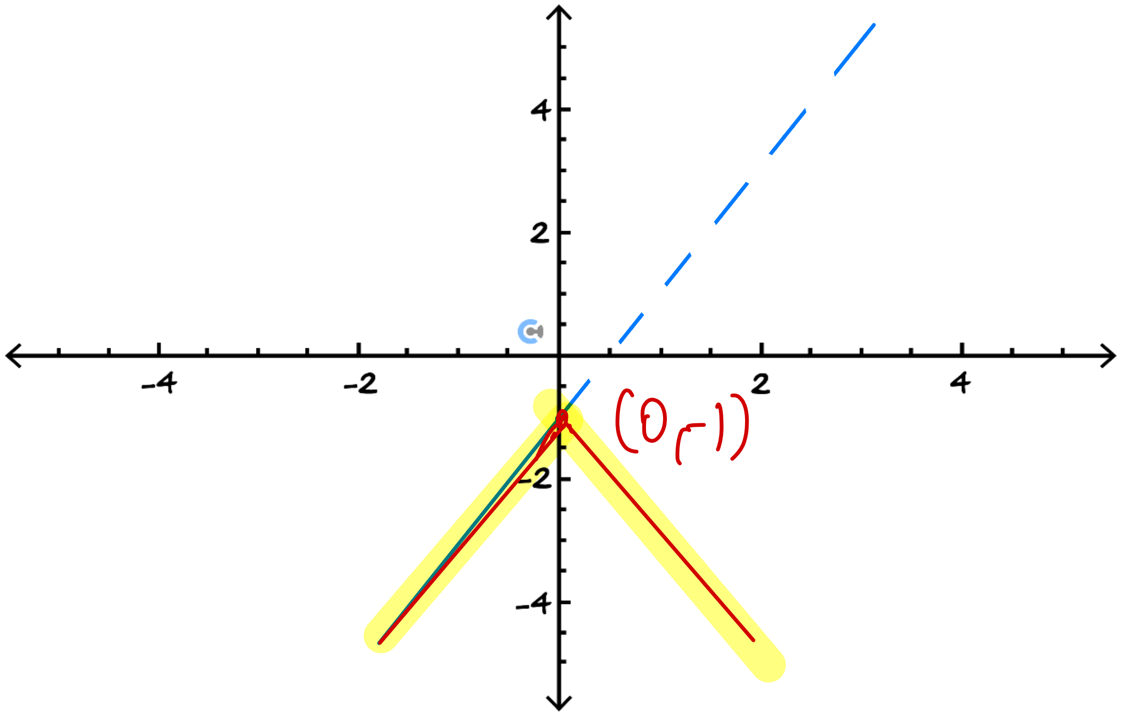
$$y = f(|x|), \text{ where } f(x) = (x+1)(x+2)$$



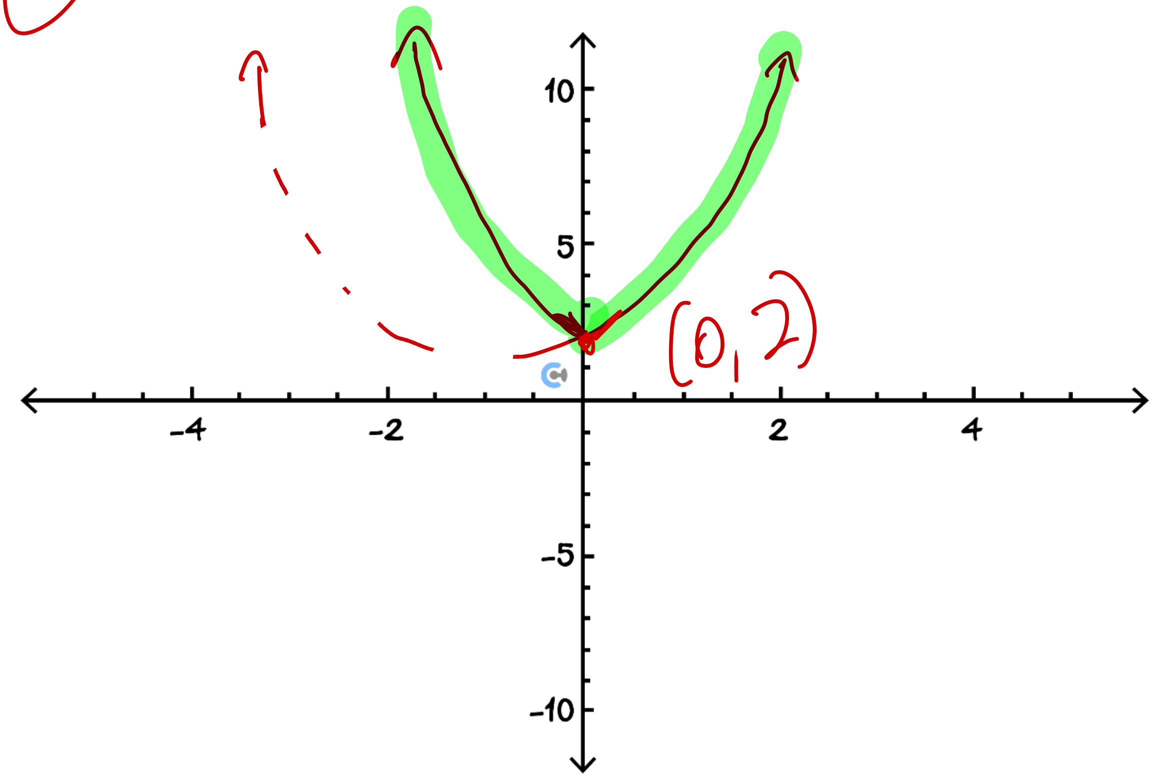
Question 5

Sketch the graph below.

a. $y = f(-|x|)$, where $f(x) = 2x - 1$.



b. $y = f(|x|)$, where $f(x) = (x+1)^2 + 1$.



Section B: Partial Fractions

Sub-Section: Introduction to Partial Fractions

Definition

Partial Fractions

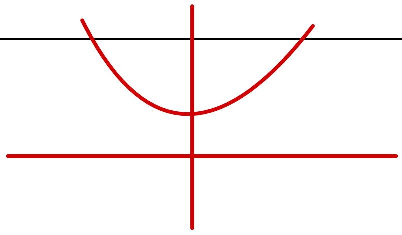
$$\frac{A}{4\,\square\,\bigcirc} = \frac{B}{4} + \frac{C}{\square} + \frac{D}{\bigcirc}$$

- The rules for partial fractions:

For every factor of this form in the denominator of the function...	There will be a partial fraction(s) of this form:
Linear factors: $\frac{1}{(ax+b)(cx+d)}$	$\frac{A}{ax+b} + \frac{B}{cx+d}$
Repeated linear factor: $\frac{1}{(cx+d)^n}$	$\frac{A}{cx+d} + \frac{B}{(cx+d)^2} + \dots + \frac{Z}{(cx+d)^n}$
Irreducible quadratic: $\frac{1}{(ax^2+bx+c)}$	$\frac{Dx+E}{ax^2+bx+c}$

x^2+1

Must do long division before using any of the rules above.



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Sub-Section: Case 1

Let's consider when we have two linear factors in the denominator!

Question 6 Walkthrough.

Perform partial fraction decomposition for $f(x) = \frac{2x+9}{(x-3)(x+2)}$. $= \frac{A}{x-3} + \frac{B}{x+2}$

$$\frac{2x+9}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$2x+9 = A(x+2)+B(x-3)$

Let $x=-2$

$-4+9 = A(-2+2)+B(-2-3)$

$5 = 0-5B$, $B=-1$

Let $x=3$

$6+9 = 5A+0$

$5A = 15$

$A=3$

$\therefore \boxed{f(x) = \frac{3}{x-3} - \frac{1}{x+2}}$

NOTE: ALWAYS factorise the denominator by its factors first!

Question 7

Perform partial fraction decomposition for the following function.

$$\frac{2x+8}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}$$

$2x+8 = A(x-5)+B(x-1)$

Let $x=1$

$10 = -4A$

$A = -\frac{5}{2}$

$\hookrightarrow RHS = Ax-5A+Bx-B$

$= (A+B)x - 5A - B$

$-\frac{5}{2}+B = 2$

$B = \frac{9}{2}$

$f(x) = \frac{-5}{2(x-1)} + \frac{9}{2(x-5)}$

Sub-Section: Case 2

How about repeated linear factors?

Question 8 Walkthrough.

Perform partial fraction decomposition for $f(x) = \frac{x^2-5x+8}{x(x-3)^2}$. $= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

$$\frac{x^2-5x+8}{x(x-3)^2} = \frac{A(x-3)^2+Bx(x-3)+Cx}{x(x-3)^2}$$

$$x^2-5x+8 = A(x-3)^2+Bx(x-3)+Cx$$

Let $x=0$

$8 = 9A+0+0$

$A = \frac{8}{9}$

Let $x=3$

$9-15+8 = 0+0+3C$

$2 = 3C$

$C = 2/3$

$B = \frac{1}{9}$

$\therefore \frac{8}{9x} + \frac{1}{9(x-3)} + \frac{2}{3(x-3)^2}$

NOTE: When a linear factor is repeated, we repeat the splitting by that power.

Question 9

Perform partial fraction decomposition for the following function.

$$\frac{8}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$8 = A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

Let $x=1$

$8 = 4A+0+0$

$A = 2$

Let $x=-1$

$8 = 0+0-2C$

$C = -4$

Let $x=0$

$8 = 2+B\times-1\times1+(-4)\times-1$

$8 = 2-B+4$

$B = -2$

$$f(x) = \frac{2}{x-1} - \frac{2}{x+1} - \frac{4}{(x+1)^2}$$

Sub-Section: Case 3

Finally, non-factorisable quadratic factors!

Question 10 Walkthrough.

Perform partial fraction decomposition for $f(x) = \frac{x^2+3}{(x-1)(x^2+1)}$. $= \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

linear irreducible quad.

$$\frac{x^2+3}{(x-1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$x^2+3 = A(x^2+1)+(Bx+C)(x-1)$

RHS $= Ax^2+A+Bx^2-Bx+Cx-C$

$= (A+B)x^2+(C-B)x+A-C$

Let $x=1$

$4 = 2A+0$

$A=2$

Let $x=0$

$3 = A + C\times-1$

$3 = 2-C$

$C=-1$

$A+B=1$

$2+B=1$

$B=-1$

$\therefore \boxed{\frac{2}{x-1} + \frac{-x-1}{x^2+1}}$

NOTE: For quadratic factors that cannot be factorised, we split it as it is.

Question 11

Perform partial fraction decomposition for the following function.

$$\frac{2x-4}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

$2x-4 = A(x^2+4)+(Bx+C)(x+2)$

Let $x=-2$

$-8 = 8A+0$

$A=-1$

Let $x=0$

$-4 = 4A+2C$

$-4 = -4+2C$

$C=0$

Let $x=1$

$2-4 = 5A+B\times3$

$-2 = -5+3B$

$3B=3$

$B=1$

$\therefore \frac{-1}{x+2} + \frac{x}{x^2+4}$

Section C: Modulus and Partial Fractions Exam Skills

Sub-Section: Solving Advanced Modulus Equations and Inequalities

How do we find an intersection between two modulus functions?

Misconception

"To solve $|x-a|+b=-|x-h|+k$, we just remove the modulus and put $\pm$!"

TRUTH: We must check which equation will give us a valid solution. (Case Check)

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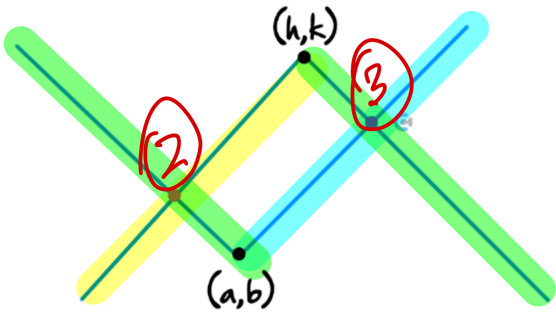
Exploration: Case Checking for Finding Intersections Between Two Modulus Functions

- Consider the following equation which finds an intersection between two modulus functions.



$$|x-a|+b=-|x-h|+k$$

- As previously mentioned, if we remove the modulus and change it to $\pm$, we get four equations.

1 $$(x-a)+b=(x-h)+k$$

2 $$-(x-a)+b=(x-h)+k$$

3 $$(x-a)+b=-(x-h)+k$$

4 $$-(x-a)+b=-(x-h)+k$$

- How many different solutions would we get from the four equations above?

potentially 4

- Looking at the graph above, how many solutions should we get?

2

- Look at the graph above, and think about which of the above equations will give us a valid solution.

  Highlight the two equations above!

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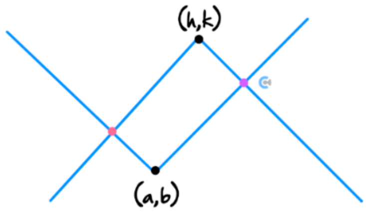
Definition

Solving Advanced Modulus Equations



$$|x-a|+b=-|x-h|+k$$

- Two corresponding equations are:

$$-(x-a)+b=(x-h)+k$$

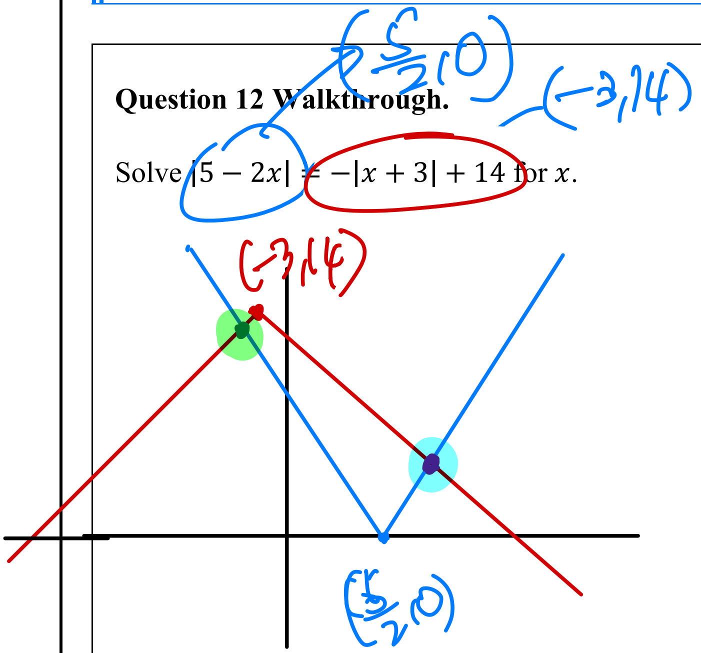
$$(x-a)+b=-(x-h)+k$$

- Steps:

1. Sketch the two modulus functions
2. Find the correct equation by looking at positive/negative linear lines.

Question 12 Walkthrough.

Solve $|5-2x|=-|x+3|+14$ for x.



$5-2x=0$
$x=5/2$

$5-2x=x+3+14$
$3x=-12$
$x=-4$

$2x-5=-x-3+14$
$3x=16$
$x=16/3$

Active Recall: Steps for Solving Advanced Modulus Equations

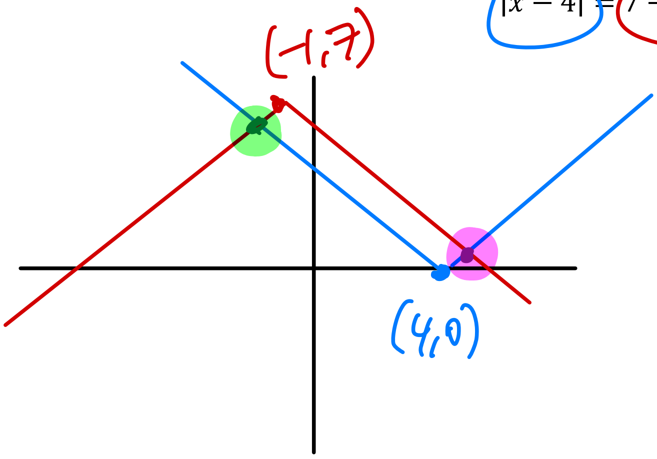
1. Sketch the two modulus functions
2. Find the correct equation by looking at positive/negative linear lines.

Question 13

Find the value(s) of x for the following equations:

$(4,0)$ $(-1,7)$

$$|x-4| = 7 - |x+1|$$



$-x+4 = 7+x+1$

$2x = -4$

$x = -2$

$x-4 = 7-x-1$

$2x = 10$

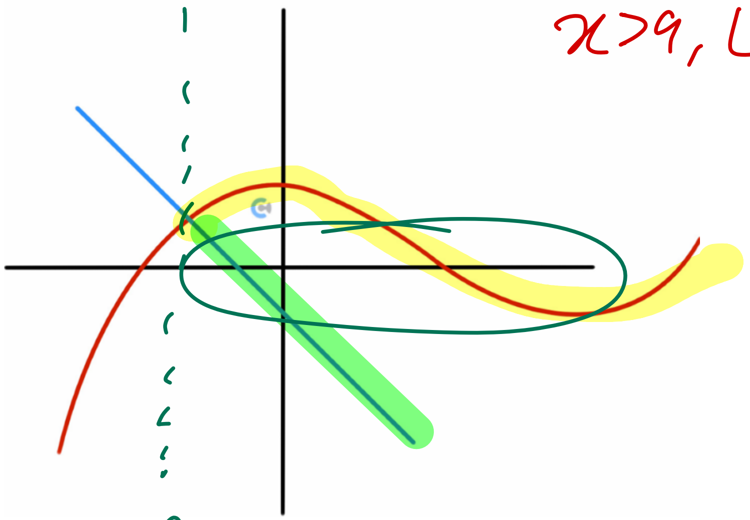
$x = 5$

What about modulus inequalities now?

Modulus Inequalities

LHS > RHS

$x > a$, LHS > RHS



$x = a$

- Steps:

1. Graph either side of the inequality.
2. Find the x-values when one side is higher than the other.

Question 14 Walkthrough.

Solve the inequality $x - 1 \leq |x^2 - 5x + 4|$ for x.

$x - 1 = -x^2 + 5x - 4$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

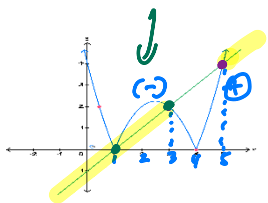
$x = 1, 3$

$x - 1 = x^2 - 5x + 4$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 5, 1$



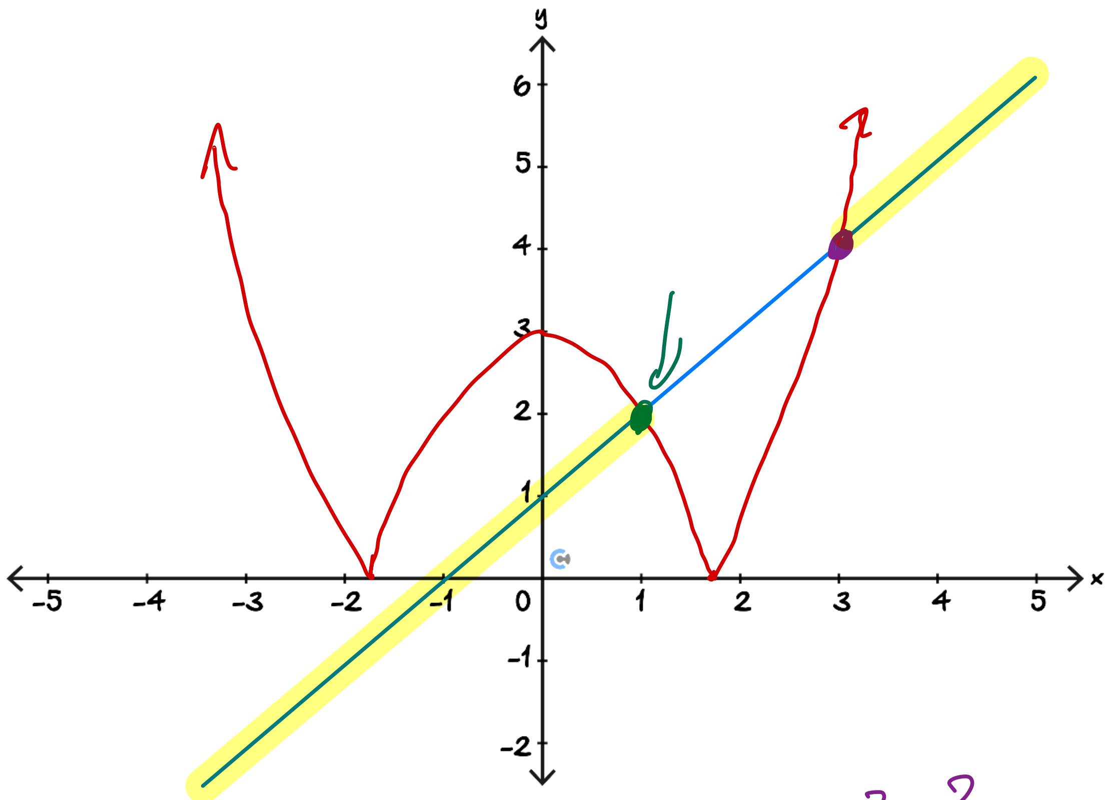
$x \in (-\infty, 3] \cup [5, \infty)$

Active Recall: Steps for Solving Modulus Inequality

1. Graph either side of the inequality.
2. Find the x-values when one side is higher than the other.

Question 15

Solve the inequality $x + 1 \leq |x^2 - 3|$ for x.



$x+1 = -x^2+3$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2, 1$

$x=1$

$x+1 = x^2-3$

$x^2-x-4=0$

$x = \frac{1 \pm \sqrt{1+16}}{2}$

$= \frac{1 \pm \sqrt{17}}{2}$

$x>0$

$\therefore x = \frac{1+\sqrt{17}}{2}$

$x \in (-\infty, 1] \cup \left[\frac{1+\sqrt{17}}{2}, \infty\right)$

Key Takeaways

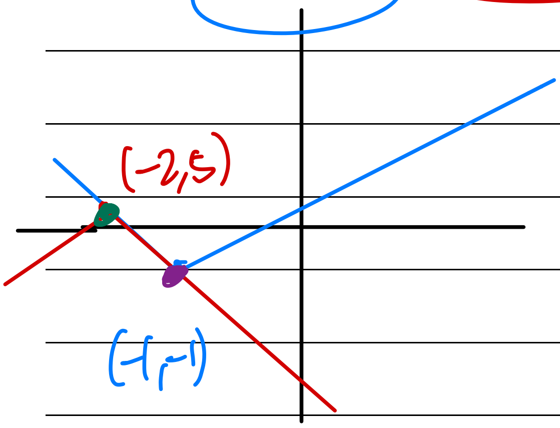
- ☑ Solving an intersection between two modulus functions requires graphing to eliminate invalid solutions.
- ☑ Finding inequalities is best done using graphs.

Section D: Exam 1 Questions (20 Marks) 18, 20

Question 16 (3 marks)

$\boxed{C\emptyset 2}$

Solve the equation $3|2x + 2| - 1 = -2|3x + 6| + 5$ for $x \in \mathbb{R}$.



$-3(2x+2)-1 = 2(3x+6)+5$

$-6x-6-1 = 6x+12+5$

$12x = -24$

$x = -2$

$x = -1$

$x \in [-2,-1]$

Question 17 (3 marks)

Solve the inequality $x - 1 \leq |x^2 - x - 6|$ for $x \in \mathbb{R}$.

Question 18 (7 marks)

a. Perform partial fraction decomposition for $f(x) = \frac{2x^2+2}{(x+1)^2(x-1)}$. (3 marks)

b. Express $\frac{2x^3+3x^2+2x+3}{(x-2)(x+3)}$ in the form $Ax + b + \frac{c}{x-2} + \frac{d}{x+3}$. (4 marks)

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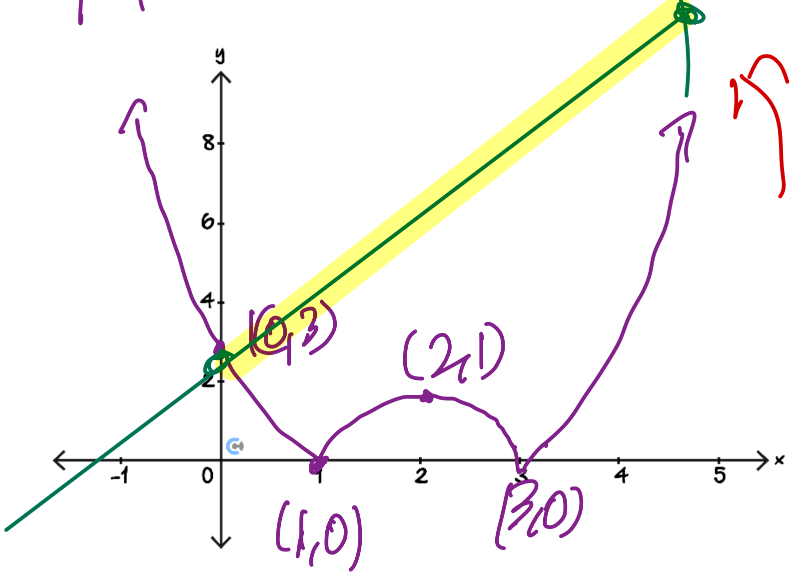
Question 19 (7 marks)

Let $f(x) = -x^2 + 4x - 3$.

$-(x^2-4x+3)$

$-(x-3)(x-1)$

a. Sketch the graph of $y = |f(x)|$ on the axis below. Label all axes intercepts. (3 marks)



b. Find the value of k such that $|f(x)| = k$ has exactly 3 solutions. (2 marks)

$k=1$

c. Solve the inequality $x + 3 > |-x^2 + 4x - 3|$ for $x \in \mathbb{R}$. (2 marks)

$x+3 = x^2-4x+3$

$x^2-5x=0$

$x(x-5)=0$

$x=0, x=5$

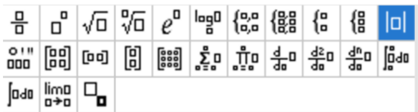
$x \in (0,5)$

Section E: Tech Active Exam Skills

Calculator Commands: Solving Modulus Equations and Inequalities

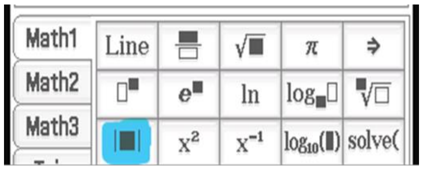
- **Mathematica**
  - Abs[].

- **TI-Nspire**



  - Under button situated next to the book button.

- **Casio Classpad**
  - Under Math1.



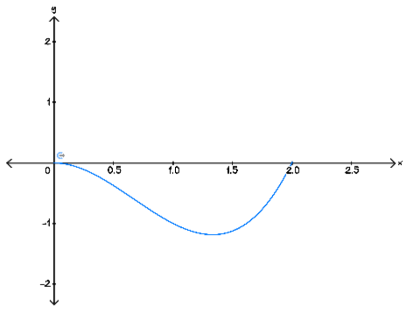
Question 20 Tech-Active.

Solve the inequality $|x-2| < |x^2-4|$.

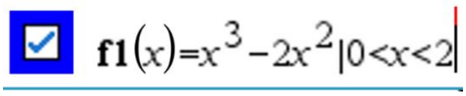
Calculator Commands: Graphing

Mathematica

- Plot[function,{x,$x$min,$x$max}, PlotRange→{$y$min, $y$max}].
- PlotRange is optional but makes the scale appropriate for the question.

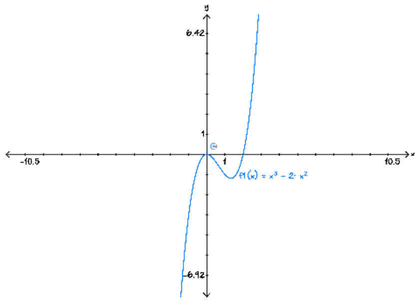


- Menu→ 6 (Analyse) to find min/max x and y intercepts
- Restrict domain to $0 < x < 2$ use the bar can get it from Ctrl+ = .

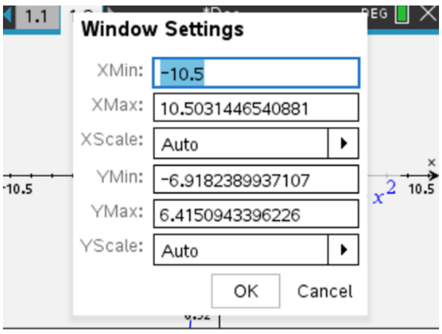


TI-Nspire

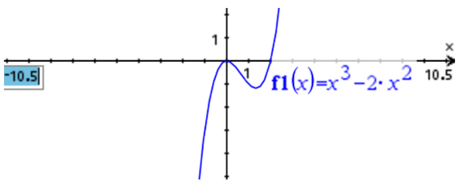
- Open a graph page and plot your function.



- Zoom settings: Menu→ 4 (window/zoom)→ 1 enter your x and y ranges.

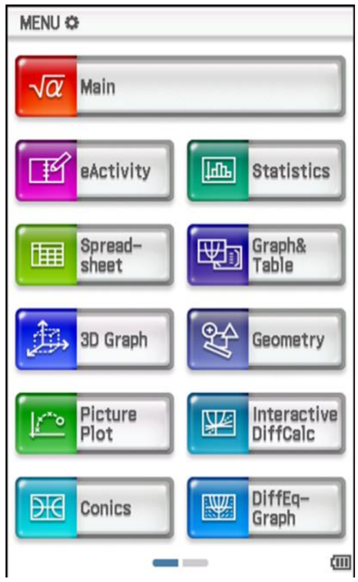


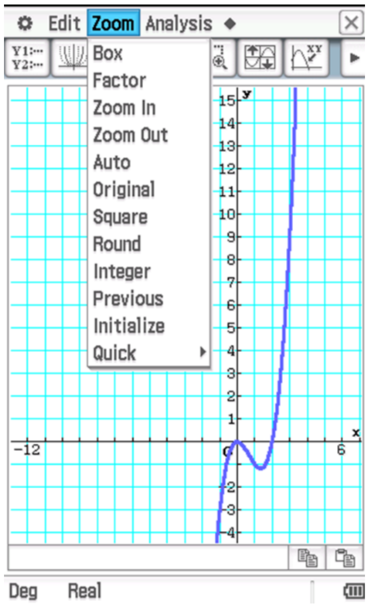
- Can also click the axis numbers on the graph and alter them directly.

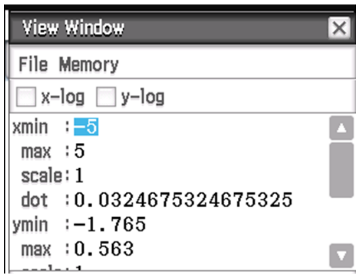


Casio Classpad

- Click Graph & Table, and enter the function.

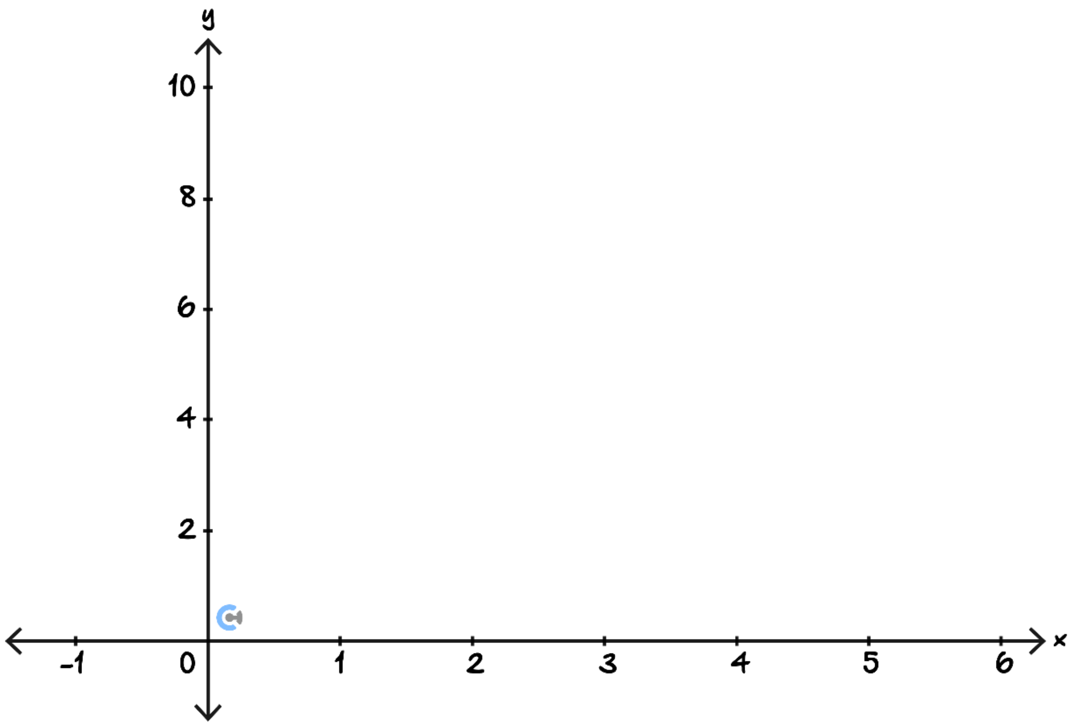




		Analysis→G-Solve to find intercepts. Use this button to set the view window.  Use | to restrict domain → find it in Math 3 $y1=x^3-2\cdot x^2 \mid 0<x<2$

Question 21 Tech-Active.

Let $f(x) = x^2 - 5x + 4$. Sketch the graph of $y = |f(x)|$ on the axis below.



Calculator Commands: Partial Fractions

Mathematica	TI-Nspire	Casio Classpad
Apart[].	Expand.	expand(func, x) or Interactive → Transformation → expand → Partial Fraction.

Question 22 Tech-Active.

Find the partial fraction decomposition of $f(x) = \frac{3}{x^2-5x+4}$.

Section F: Exam 2 Questions (18 Marks)

Question 23 (1 mark)

Which one of the following, where A, B, C, and D are non-zero real numbers, is a partial fraction form for the expression?

$$\frac{x-3}{(x^2+1)(x-2)^2}$$

A. $\frac{A}{x^2+1} - \frac{B}{(x-2)^2}$

B. $\frac{Ax+B}{x^2+1} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$

C. $\frac{Ax+B}{x^2+1} + \frac{C}{(x-2)^2} + \frac{Dx}{x-2}$

D. $\frac{A}{x^2+1} + \frac{C}{(x-2)^2} + \frac{D}{x-4}$

Question 24 (1 mark)

For the interval $\frac{1}{2} \leq x \leq\ 3$, the graph of $y = |2x-1| + |x-3|$ is the same as the graph of:

A. $y = -x - 2$

B. $y = 3x - 4$

C. $y = x + 2$

D. $y = 3x + 2$

Space for Personal Notes

Question 25 (1 mark)

The equation $|2x - 3| = -|x + 2| + 6$, where $x \in \mathbb{R}$, has solution(s):

A. $x = -1, \frac{7}{3}$

B. $x = \frac{7}{3}$

C. $x = -1$

D. $1, \frac{7}{3}$

Question 26 (1 mark)

The equation $|x^2 - 2x - 3| = k$, where k is a real number has exactly two solutions for:

A. $k = 4$

B. $k > 4$

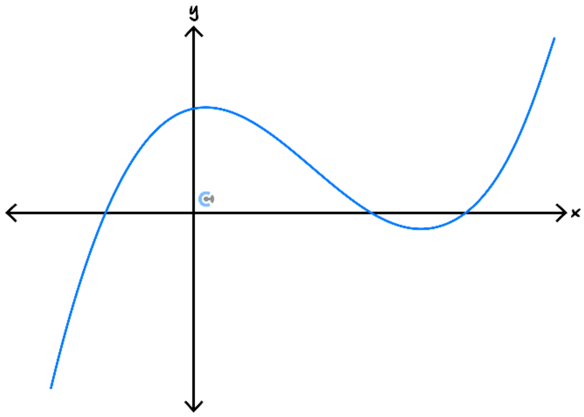
C. $k = 3$

D. $k < 3$

Space for Personal Notes

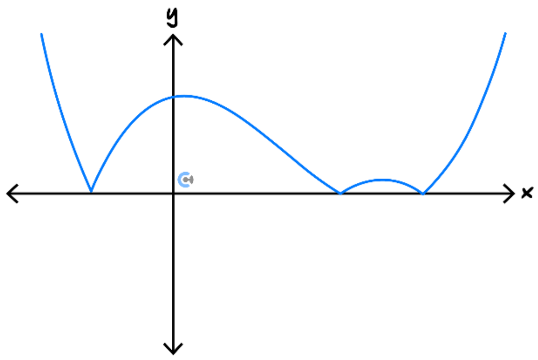
Question 27 (1 mark)

Part of the graph of $y = f(x)$ is shown below.

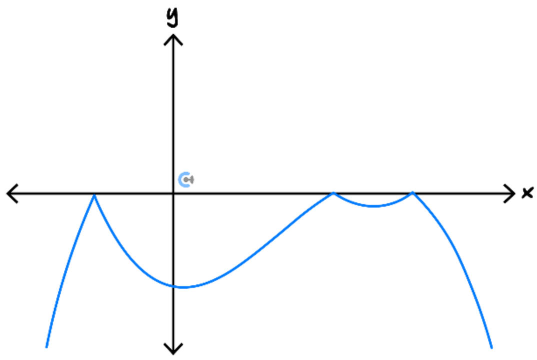


The function $f(-|x|)$ is best represented by:

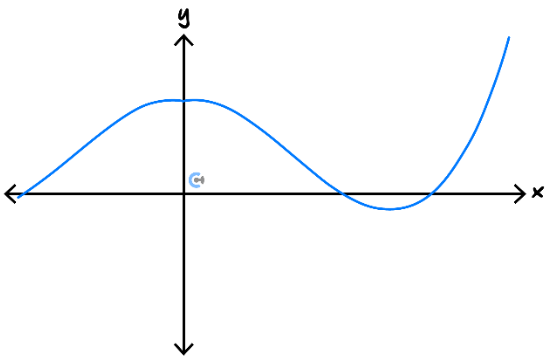
A.



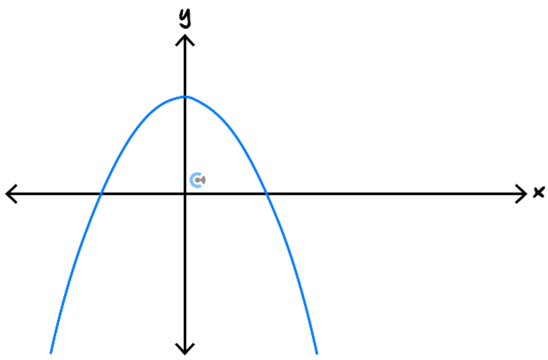
B.



C.



D.



Space for Personal Notes

Question 28 (13 marks)

a. Consider the functions $f(x) = |x - 3| - 1$ and $g(x) = -|x - 3| + 3$.

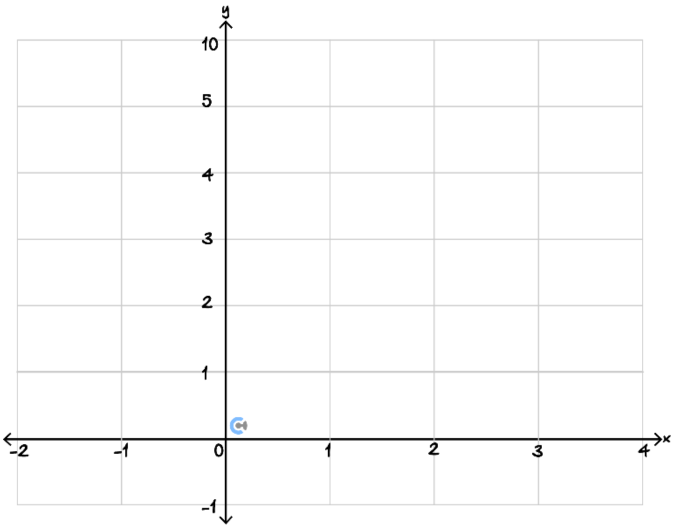
i. Let A be the vertex point of f and let B be the vertex point of g. State the coordinates of A and B. (1 mark)

ii. Let C and D be the points of intersection of $f(x)$ and $g(x)$. State the coordinates of C and D. (2 marks)

iii. Find the area of the square $ABCD$. (2 marks)

Consider the function $h(x) = |(x-3)(x+1)|$.

b. Sketch the graph of $y = h(x)$ on the axis below. Label all axes intercepts and turning points. (3 marks)



c. Solve the inequality $x + 1 > h(x)$ for $x \in \mathbb{R}$. (2 marks)

d. The equation $k - x = h(x)$, where k is a real number, has 4 real solutions. Find the possible value(s) of k. (3 marks)

Contour Check

Learning Objective: [1.1.1]

Study Design

Graphs of sum, difference, product, and composite functions involving functions of the types specified above (not including composite functions that result in reciprocal or quotient functions).

Key Takeaways

- [ ] Modulus finds a ________________ of things.
- [ ] $|a— b|$ is a ________________________ between a and b.
- [ ] $\sqrt{x^2} =$ ________________.
- [ ] For simple modulus equations, remove modulus and put ___________________.

Learning Objective: [1.1.2]

Study Design

Graphs of sum, difference, product, and composite functions involving functions of the types specified above (not including composite functions that result in reciprocal or quotient functions).

Key Takeaways

- [ ] Graph of a simple modulus graph $a|x-h|+k$ is like a straightened _________________________.
- [ ] Wrapping modulus around the function makes the y value always non-______________________.
- [ ] Wrapping the modulus around the x value makes the function symmetrical around the _____ axis.
- [ ] $f(|x|)$ take the RHS and make it symmetrical about the __________ axis.
- [ ] $f(-|x|)$ take the ________ ________ and make it symmetrical about the y-axis.

Learning Objective: [1.1.3]

Key Takeaways

- Partial fractions are the process of ____.
- Must __________________ before doing partial fractions.
- Must do _____________________ before doing partial fractions.
- Linear factors always have a ________________________ at the top.
- Irreducible quadratic factors have a ____________________ function at the top.

Learning Objective: [1.2.1]

Key Takeaways

- To solve modulus inequalities, we should:

1. __________________ either side of the inequality.
2. Find the ________________ when one side is higher than the other.