1.1

CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Specialist Mathematics ½
Modulus & Partial Fractions [1.1]
Workbook

Outline:

Modulus Functions Pg 2-8 ➤ Solving Modulus Equations ➤ Solving Modulus Inequalities **Graphing Modulus Functions** Pg 9-18 ➤ Solving Modulus Functions ➤ Graphing Composite of Modulus Functions	**Partial Fractions** Pg 19-25 ➤ Introduction to Partial Fractions ➤ Case 1 ➤ Case 2 ➤ Case 3

Section A: Modulus Functions

What is a Modulus Function?

Definition

Modulus functions

- **Definition:**

$$f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

- Is a **hybrid function.**
- **Purpose:** Always return a non-negative number.
- **Range:** $[0, \infty)$.

Question 1

Evaluate the following:

a. $|-7|$

$= -(-7)$

$= 7$

b. $|3^2 - 2(3) - 2|$ $= |9-6-2| = |1| = 1$

Discussion: What's the point of modulus?

~~negative~~

size

3

-3 0 3

Exploration: Alternative definition of modulus functions

- What happens when you square -2 and square root it?

$$-2 \rightarrow \underline{\ 4\ } \rightarrow \underline{\ 2\ }$$

- So, what is squaring a number and then square rooting it the same as?

$$\sqrt{x^2} = \underline{\ |x|\ }$$

Alternative definition of Modulus Functions

$$\sqrt{x^2} = |x|$$

NOTE: Important not to forget the modulus in the exams!

Space for Personal Notes

Sub-Section: Solving Modulus Equations

Discussion: How do we solve for modulus equations like $|f(x)| = 2$?

'size'

$f(x) = \pm 2$

Solving equations involving modulus functions

$$|f(x)| = b$$

$$f(x) = \pm b$$

- **Interpretation:**
  - The size of $f(x)$ equals to b.
  - $f(x)$ can be either b or $-b$

Question 2 Walkthrough.

Solve the following equation below.

$$|x - 3| = 2$$

$x - 3 = 2$ or $x - 3 = -2$

$x = 5$ or $x = 1$

Your turn!

Question 3

Solve the following equations for x:

a. $|x + 2| = 5$

$x + 2 = 5$ or $x + 2 = -5$

$x = 3$ or $x = -7$

b. $|3 - \sqrt{x}| = 1$

$3 - \sqrt{x} = 1$ or $3 - \sqrt{x} = -1$

$\sqrt{x} = 2$ or $\sqrt{x} = 4$

$x = 4$ or $x = 16$

TIP: Check your solutions by substituting it back into the equation!

Sub-Section: Solving Modulus Inequalities

How far is the number 5 from 2?

Discussion: What does $|5-2|$ equal to? What does this mean?

$|5-2|=3$

Distance of 3 bt. 5 and 2

$|2-5|=3$

Exploration: Solving modulus inequalities

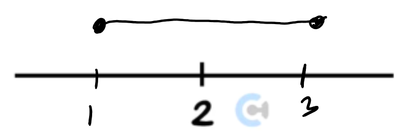
- Understanding:

$$|x-2| = \text{distance between } x \text{ and } 2$$

- Interpretation:

$|x-2| \leq 1 \rightarrow$ distance between x and 2 is [less] / [greater] than or equal to 1

- Solving: Use the number line below to solve $|x-2| \leq 1$.



$x \in [1, 3]$

Space for Personal Notes

Definition

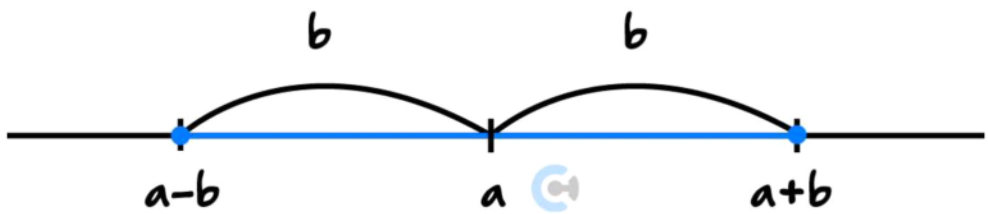
Solving modulus inequalities

$$|x - a| \leq b$$

- Interpretation:

x has a distance from 'a' that is less than or equal to 'b'

- Visualise:



$$a - b \leq x \leq a + b$$

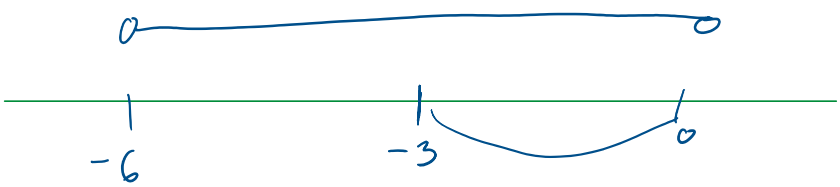
TIP: Always sketch a number line!

Question 4 Walkthrough.

Solve the following inequality.

$$|x + 3| < 3$$

$$|x - (-3)| < 3$$



$$x \in (-6, 0)$$

Question 5

Solve each of the following inequalities for x:

a. $|x| \geq 3$

$|x - 0| \geq 3$

$-3 \quad 0 \quad 3$

$x \leq -3$ or $x \geq 3$

b. $|2x - 4| - 3 \geq 5$

$|2x - 4| \geq 8$

$2|x - 2| \geq 8$

$|x - 2| \geq 4$

$-2 \quad 2 \quad 6$

$x \leq -2$ or $x \geq 6$

Key Takeaways

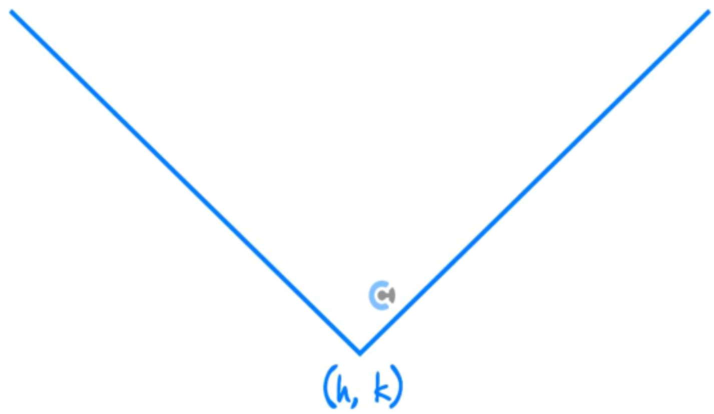
- Modulus finds a size of things.
- $|a - b|$ is a distance between a and b.
- $\sqrt{x^2} = |x|$
- For simple modulus equations, remove modulus and put $\pm$.

Section B: Graphing Modulus Functions

Sub-Section: Sketching Modulus Functions

Let's now consider the graph of Modulus Functions!

Graph of the modulus function



- General form:

$$y = a|x - h| + k$$

Vertex is at (h, k).

- Hybrid form:

$$y = \begin{cases} a(x-h)+k, & x \geq h \\ -a(x-h)+k, & x < h \end{cases}$$

Space for Personal Notes

Question 6 Walkthrough.

Consider the function $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2|x - 1| - 2$.

a. Find the vertex.

$(1, -2)$

b. Find the axes intercepts.

x-int $y = 0$

$0 = 2|x-1| - 2$

$|x-1| = 1$

$x - 1 = \pm 1$, $x = 0, 2$

$(0,0)$
$(2,0)$

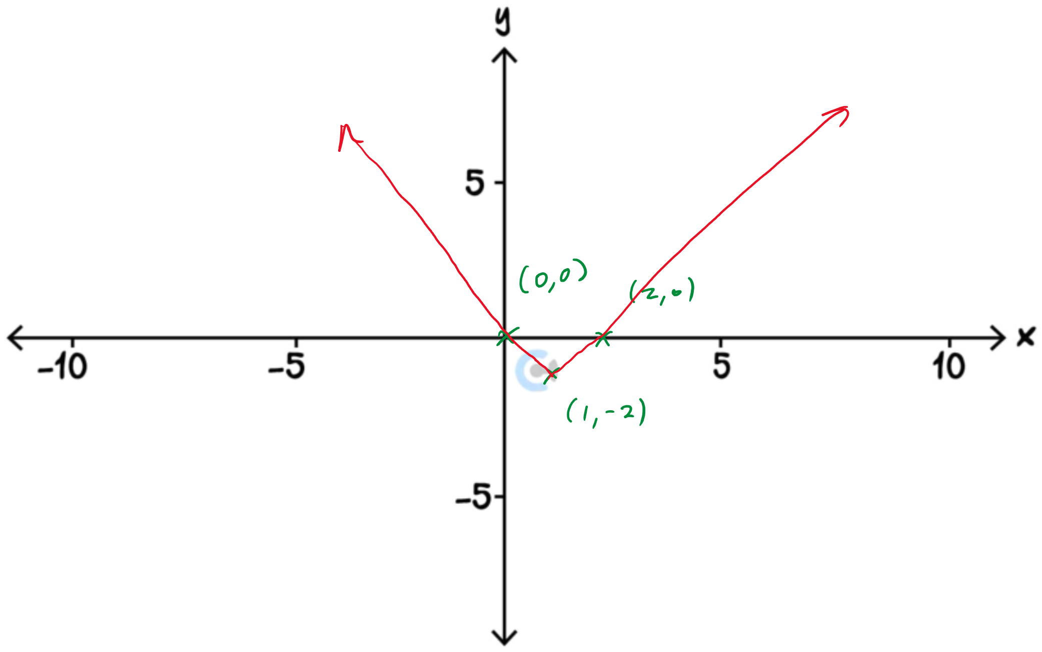
y-int $x = 0$

$f(0) = 2|0-1| - 2$

$= 0$

$(0,0)$

c. Sketch the graph of $y = f(x)$. Label all axes intercepts and the vertex.



TIP: Think of modulus functions as a "**Straightened quadratic**".

Question 7

Consider the function $g: \mathbb{R} \rightarrow \mathbb{R},\ g(x) = -3|x+2| + 6$.

a. Find the vertex.

$(-2, 6)$

b. Find the axes intercept(s).

x-int $\quad y = 0$

$-3|x+2| + 6 = 0$

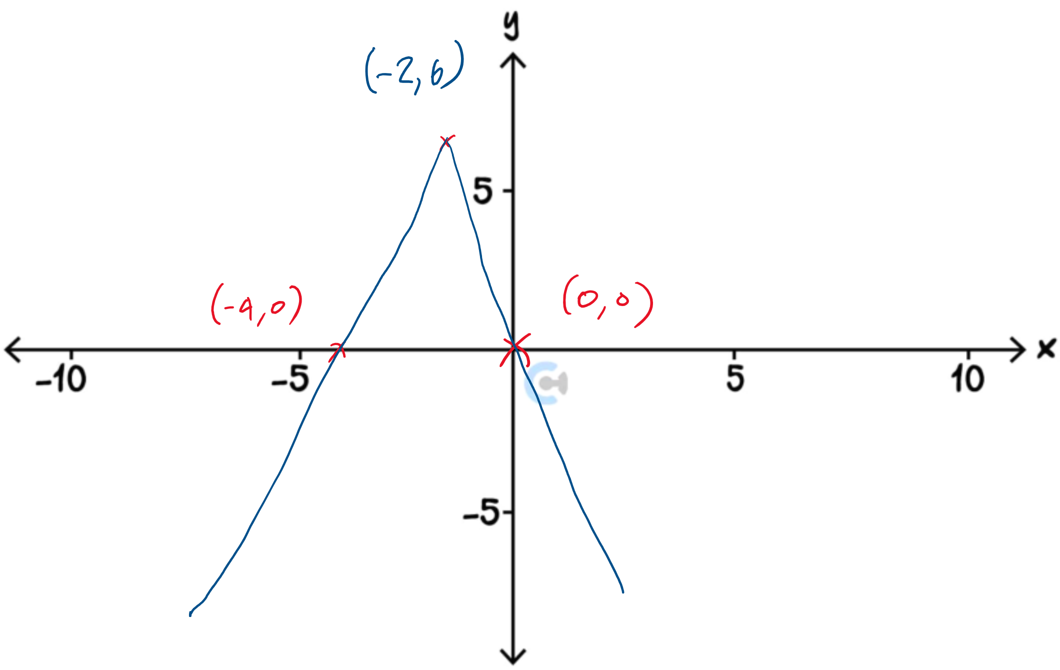
$|x+2| = 2$

$x + 2 = \pm 2, \quad x = -4, 0$

$(0,0)$

$(-4,0)$

c. Sketch the graph of $y = g(x)$. Label all axes intercepts and the vertex.



Sub-Section: Graphing Composite of Modulus Functions

REMINDER: Don't forget!

- $|-6| = 6$ ✓
- $|6| = 6$ ✓

+ve

No

Discussion: Could $|f(x)|$ be negative? Hence what does the graph of $|f(x)|$ look like?

- The output of $y = |f(x)|$ must be: ≥ 0 cant be negative

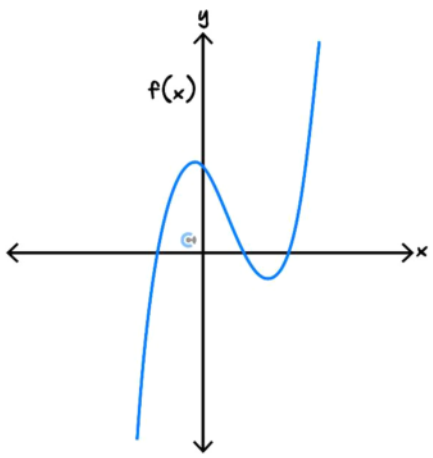
$$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

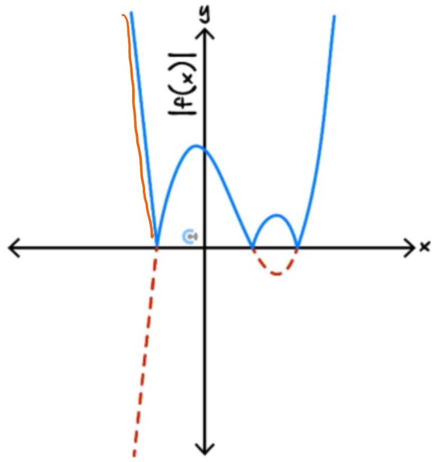
Graphs of composite modulus functions

- Modulus is the Outer Function.

$$y = |f(x)|$$

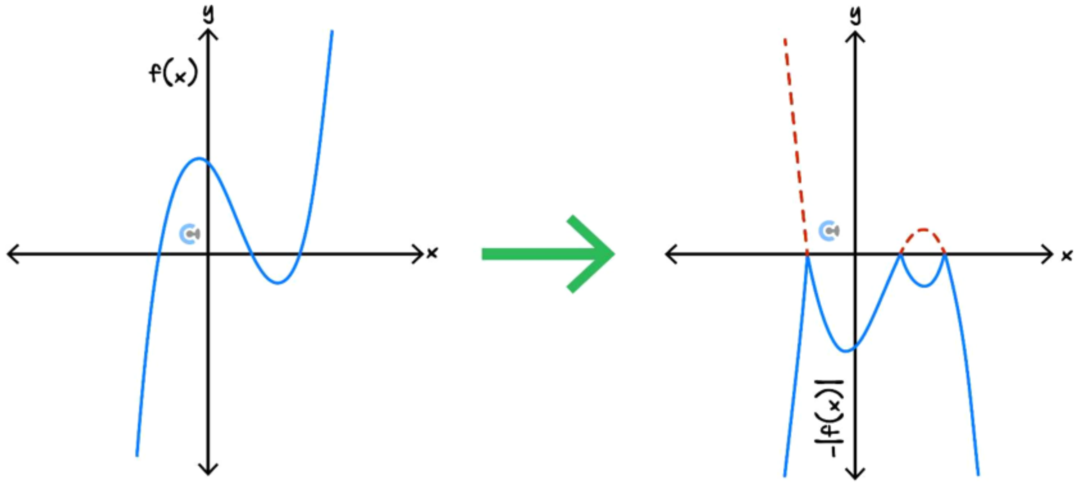
All negative y values are flipped to be positive.





$y = -|f(x)|$

- The graph of $y = |f(x)|$ has undergone a reflection in the [x-axis] / [y-axis]

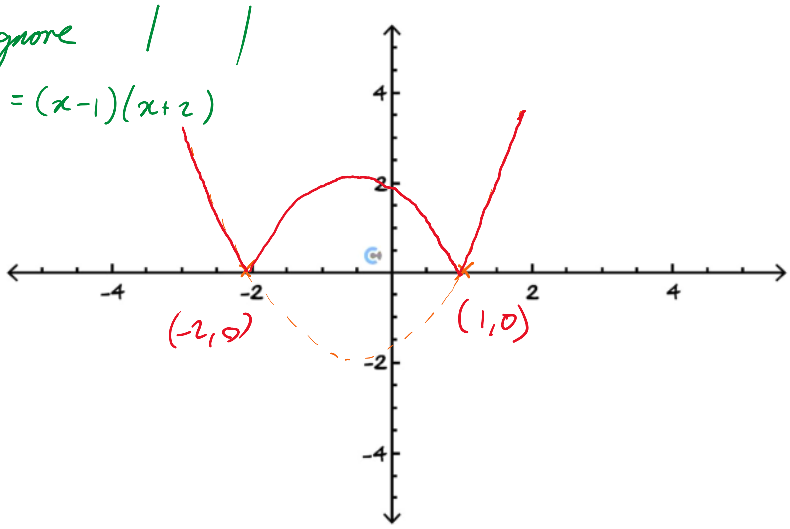


Question 8 Walkthrough.

Sketch the following graph over the specified domain. Label all key intercepts.

$$y = |(x-1)(x+2)|$$

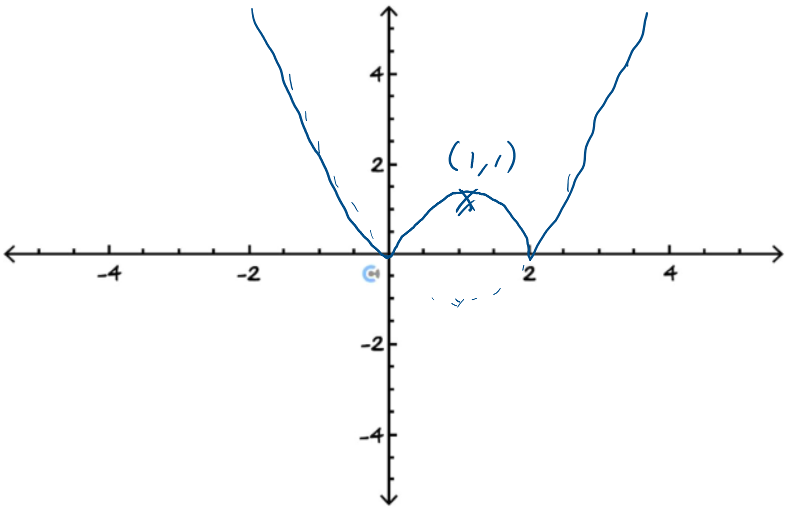
① Ignore | |
$y = (x-1)(x+2)$



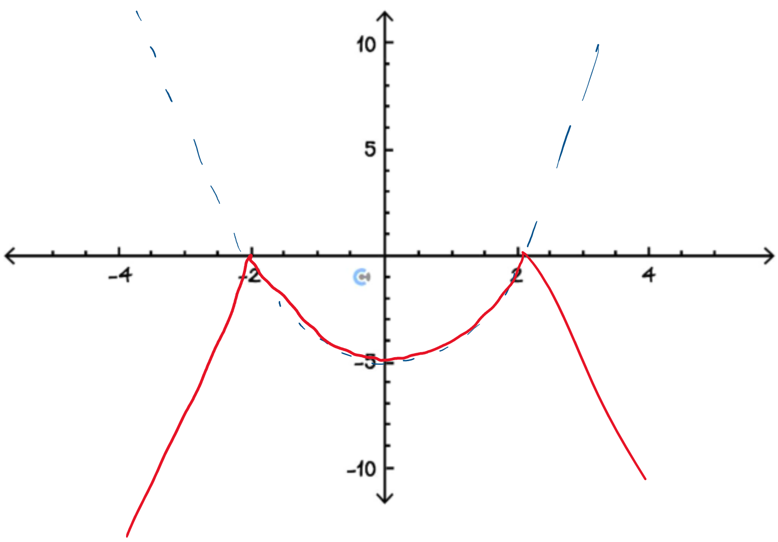
Question 9

Sketch the following graphs over the specified domain. Label all key intercepts.

a. $y = |(x-1)^2 - 1|$



b. $y = -|(x+2)(x-2)|$



Discussion: What would happen if $f(x)$ turned into $f(|x|)$?

- f will always take a [positive] / [negative] value, even if the x value is negative.

$x \rightarrow |x| \rightarrow f(|x|)$

(anything) $[0, \infty)$ $[0, \infty)$

- At:
  - $x = -2$: $f(|-2|) = f(2)$
  - $x = 2$: $f(|2|) = f(2)$

Discussion: Since $f(|-2|) = f(|2|)$, where is $f(|x|)$ symmetrical about?

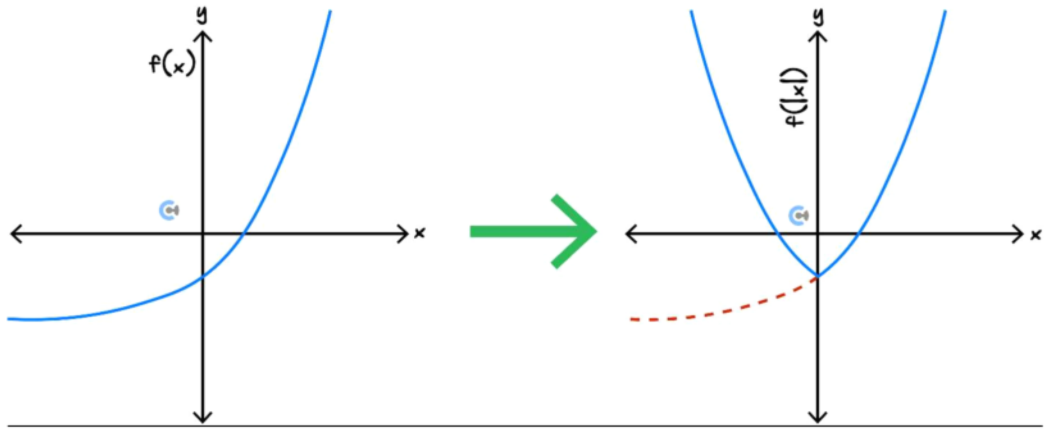
y-axis

Graphs of composite modulus functions

- Modulus is the Inside Function

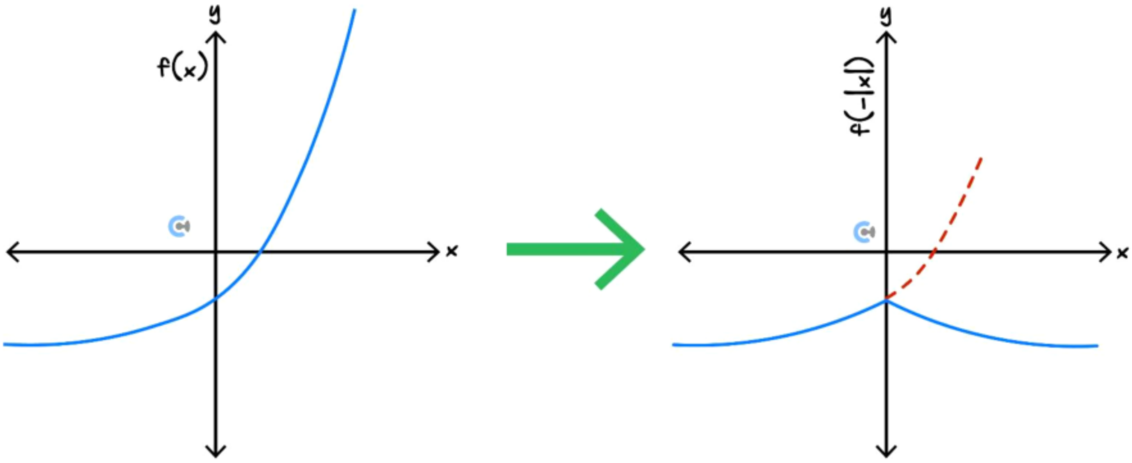
$$y = f(|x|)$$

- Take the positive side and flip it to the other side.



Take the negative side and flip it to the other side.

$$y = f(\cancel{-}|x|)$$

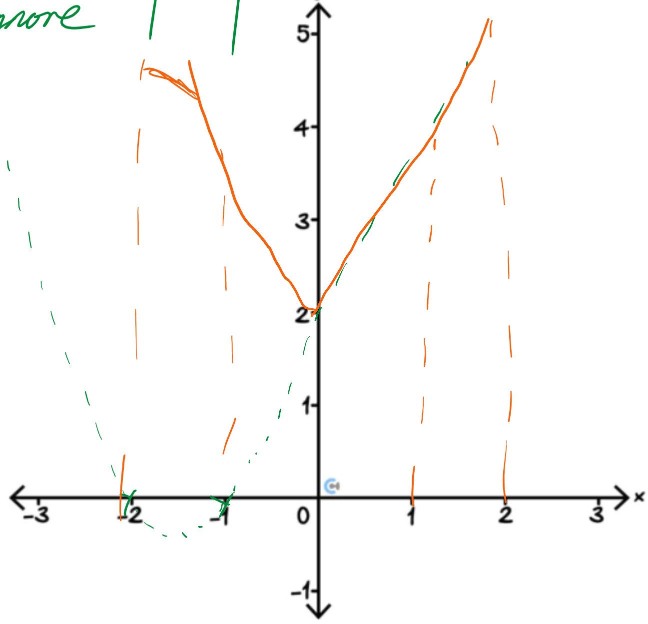


Question 10 Walkthrough.

Sketch the graph below.

$$y = f(|x|), \text{ where } f(x) = (x+1)(x+2)$$

① Ignore | |

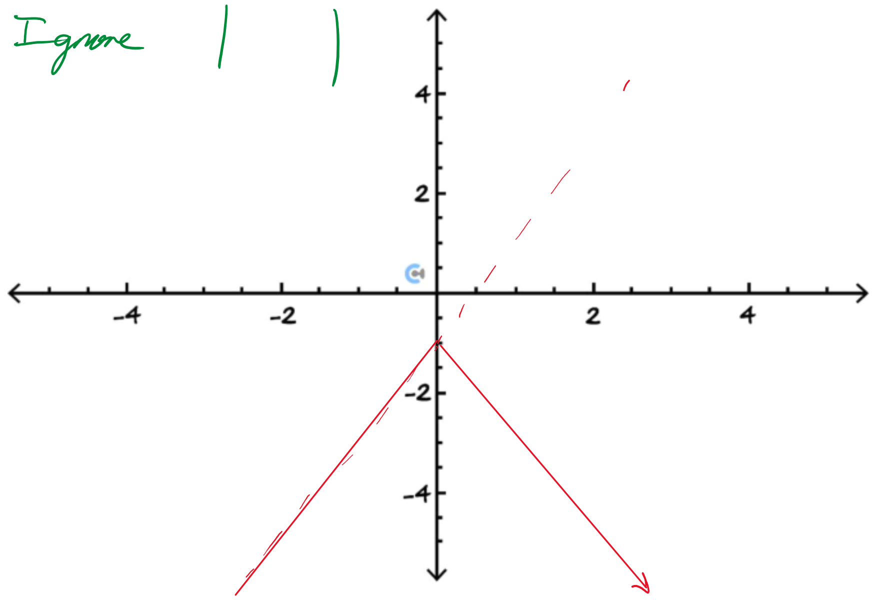


Question 11

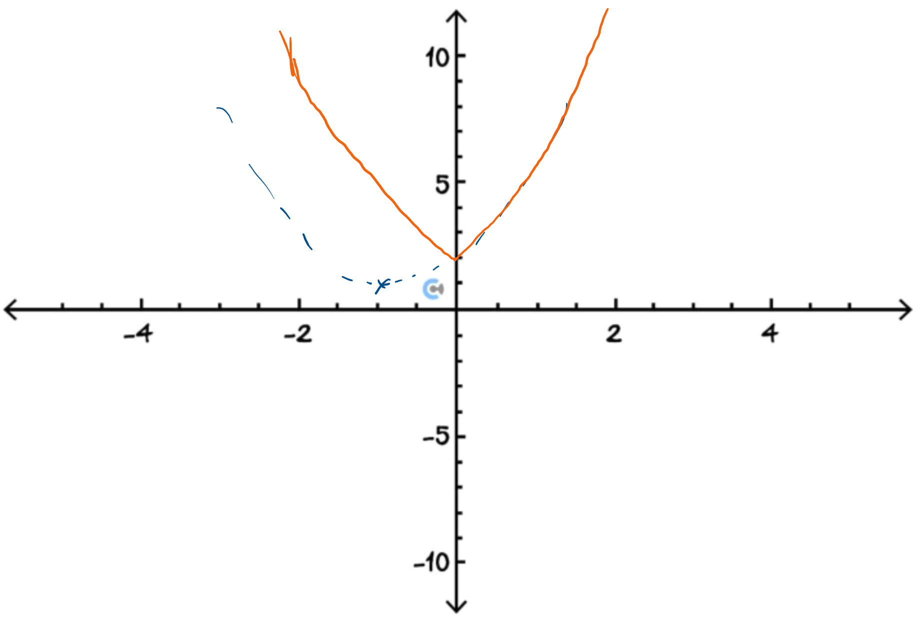
Sketch the graph below.

a. $y = f(-|x|)$, where $f(x) = 2x - 1$.

1 Ignore | |



b. $y = f(|x|)$, where $f(x) = (x+1)^2 + 1$.



Key Takeaways

- ☑ Graph of a simple modulus graph $a|x-h|+k$ is like a straightened quadratic.
- ☑ Wrapping modulus around the function makes the y value always non negative.
- ☑ Wrapping the modulus around the x value makes the function symmetrical around the y-axis.
- ☑ $f(|x|)$ take the RHS and make it symmetrical about the y-axis.
- ☑ $f(-|x|)$ take the LHS and make it symmetrical about the y-axis.

Space for Personal Notes

Section C: Partial Fractions

Sub-Section: Introduction to Partial Fractions

Discussion: What are partial fractions?

Splitting up fractions

Partial fractions

- The rules for partial fractions:

For every factor of this form in the denominator of the function...	There will be partial fraction(s) of this form:
Linear factors: $\frac{1}{(ax+b)(cx+d)}$	$= \frac{A}{ax+b} + \frac{B}{cx+d}$
Repeated linear factor: $\frac{1}{(cx+d)^n}$	$\frac{A}{cx+d} + \frac{B}{(cx+d)^2} + \dots \frac{Z}{(cx+d)^n}$
Irreducible quadratic: $\frac{1}{(ax^2+bx+c)}$	$= \frac{Dx+E}{ax^2+bx+c}$

Must do long division before using any of the rules above.

Sub-Section: Case 1

Let's consider when we have two linear factors in the denominator!

Question 12 Walkthrough.

Perform partial fraction decomposition for $f(x) = \frac{2x-1}{(x-3)(x+2)}$.

$\times(x-3)(x+2)$

$$\left(\frac{2x-1}{(x-3)(x+2)}\right) = \left(\frac{A}{x-3} + \frac{B}{x+2}\right)$$

$$2x-1 = A(x+2) + B(x-3)$$

Let $x=-2$,

$$-5 = A\cancel{(0)} + B(-5)$$

$B = 1$

Let $x = 3$

$$5 = A(5) + \cancel{B(0)}$$

$A = 1$

$$\therefore \frac{2x-1}{(x-3)(x+2)} = \frac{1}{x-3} + \frac{1}{x+2}$$

NOTE: ALWAYS factorise the denominator by its factors first!

Question 13

Perform partial fraction decomposition for the following functions.

a. $\frac{4x+2}{(x-1)(x-5)}$

$$\frac{4x+2}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} = -\frac{3}{2(x-1)} + \frac{11}{2(x-5)}$$

$$4x+2 = A(x-5) + B(x-1)$$

Let $x=5$,
$22 = 4B$, $\boxed{B = \frac{11}{2}}$

Let $x = 1$
$6 = -4A$, $\boxed{A = -3/2}$

b. $\frac{5x+6}{x(x-3)}$

$$\frac{5x+6}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3} = \frac{-2}{x} + \frac{7}{x-3}$$

$$5x+6 = A(x-3) + Bx$$
$$= Ax - 3A + Bx$$
$$5x + 6 = (A+B)x - 3A$$

$A+B=5$

$-3A=6 \implies A=-2$

$B=7$

Sub-Section: Case 2

How about repeated linear factors?

Question 14 Walkthrough.

Perform partial fraction decomposition for $f(x) = \frac{3x^2-7x+4}{x(x-2)^2}$.

$$\times\, x(x-2)^2 \left(\frac{3x^2-7x+4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \right)$$

$$3x^2-7x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

Let $x=0$

$4 = A(-2)^2 + 0 + 0$

$A = 1$

Let $x=2$

$12-14+4 = 0+0+2C$

$C = 1$

Let $A=1,\ C=1,\ x=1$

$3-7+4 = 1(-1)^2 + B(1)(-1) + 1$

$0 = 1 - B + 1$

$B = 2$

$$\therefore \frac{3x^2-7x+4}{x(x-2)} = \frac{1}{x} + \frac{2}{x-2} + \frac{1}{(x-2)^2}$$

NOTE: When a linear factor is repeated, we repeat the splitting by that power.

Question 15

Perform partial fraction decomposition for the following functions.

a. $\frac{4}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$4 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

$= A(x^2+2x+1) + B(x^2-1) + C(x-1)$

$= Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$

$= (A+B)x^2 + (2A+C)x + (A-B-C)$

$A+B=0$

$2A+C=0 \rightarrow C=-2A$

$A-B-C=4$ ← $A-B+2A=4$

$B=3A-4$

$A+3A-4=0$

$A=1$

$C=-2$

$B=-1$

$\boxed{\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{(x+1)^2}}$

b. $\frac{8x+8}{(x-1)(x+3)^2}$

$\frac{8x+8}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2} = \frac{1}{x-1} - \frac{1}{x+3} + \frac{4}{(x+3)^2}$

$8x+8 = A(x+3)^2 + B(x-1)(x+3) + C(x-1)$

Let $x=1$

$16 = 16A$

$A=1$

Let $x=-3$

$-16 = -4C$

$C=4$

Let $x=0$, $A=1$, $C=4$

$8 = 3^2 - 3B - 4$

$8 = 9 - 4 - 3B$

$-3B = 3$

$B=-1$

Sub-Section: Case 3

Finally, non-factorisable quadratic factors!

Question 16 Walkthrough.

Perform partial fraction decomposition for $f(x) = \frac{2x^2}{(x-1)(x^2+1)}$.

$$(x-1)(x^2+1)\left(\frac{2x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}\right)$$

$$= \frac{1}{x-1} + \frac{x+1}{x^2+1}$$

$$2x^2 = A(x^2+1) + (Bx+C)(x-1)$$

$$= Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$2x^2 = (A+B)x^2 + (-B+C)x + (A-C)$$

By equ. coeff.

$A + A = 2 \rightarrow A = 1$

$A + B = 2$

$-B + C = 0 \rightarrow C = B$

$A - C = 0 \rightarrow C = A$

$A = B$

$B = 1$

$C = 1$

NOTE: For quadratic factors that cannot be factorised, we split it as it is.

Question 17

Perform partial fraction decomposition for the following functions.

a. $\frac{2x-2}{(x+1)(x^2+1)}$

$$\frac{2x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{2}{x+1} - \frac{2x}{x^2+1}$$

$$2x-2 = A(x^2+1) + (Bx+C)(x+1)$$
$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$
$$= (A+B)x^2 + (B+C)x + (A+C)$$

$A+B = 0 \longrightarrow B = -A$
$B+C = 2$ $\quad -A+C = 2$
$A+C = -2$ $\quad A+C = -2$ $\quad \ominus$ $\quad 2A = 4,\ A = 2$
$B = -2,$
$C = 0$

b. $\frac{15}{(x+1)(x^2+9)}$

$$\frac{15}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} = \frac{3}{2(x+1)} + \frac{3}{2}\left(\frac{-x+1}{x^2+9}\right)$$

$$15 = A(x^2+9) + (Bx+C)(x+1)$$
$$15 = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$
$$15 = (A+B)x^2 + (B+C)x + (9A+C)$$

$A = 3/2$
$B = -3/2$
$C = 3/2$

Key Takeaways

- ☑ Partial fractions is the process of splitting the factors in the denominator.
- ☑ Must factorise fully before doing partial fractions.
- ☑ Must do long division before doing partial fractions.
- ☑ Linear factors always have a constant at the top.
- ☑ Irreducible quadratic factors have a linear function at the top.

Do feedback form :)
Japan

Contour Check

Learning Objective: [1.1.1]

Study Design

Graphs of sum, difference, product and composite functions involving functions of the types specified above (not including composite functions that result in reciprocal or quotient functions).

Key Takeaways

- Modulus finds a size of things.
- $|a - b|$ is a distance between a and b.
- $\sqrt{x^2} =$ $|x|$.
- For simple modulus equations, remove modulus and put $\pm$.

$|x-2| = 4$, $x-2=\pm 4$

Learning Objective: [1.1.2]

Study Design

Graphs of sum, difference, product and composite functions involving functions of the types specified above (not including composite functions that result in reciprocal or quotient functions).

Key Takeaways

- Graph of a simple modulus graph $a|x-h|+k$ is like a straightened quadratic.
- Wrapping modulus around the function makes the y value always non negative.
  $|f(x)|$
- Wrapping the modulus around the x value makes the function symmetrica l around the y axis.
  $f(|x|)$
- $f(|x|)$ take the RHS and make it symmetrical about the y-axis.
- $f(-|x|)$ take the LHS and make it symmetrical about the y-axis .

Learning Objective: [1.1.3]

Key Takeaways

- Partial fractions is the process of splitting the factors in the denominator.
- Must factorise fully before doing partial fractions.
- Must do long division before doing partial fractions.
- Linear factors always have a constant at the top. $\frac{A}{x+1}$
- Irreducible quadratic factors have a linear function at the top. $\frac{Bx+C}{x^2+1}$

$$\frac{x^4}{x^3+1}$$