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VCE Specialist Mathematics ½

AOS 1 SAC [1.0]

SAC 2 Solutions

40 Marks. 5 Minutes Reading. 35 Minutes Writing.

Section A: SAC Questions (40 Marks)

Question 1 (6 marks)

Only one of the following four sequences is arithmetic, and only one of them is geometric.

$$a_n = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \qquad c_n = 3, 1, \frac{1}{3}, \frac{1}{9}, \ldots$$

$$b_n = 2.5, 5, 7.5, 10, \ldots \qquad d_n = 1, 3, 6, 10, \ldots$$

a. State which sequence is arithmetic and find the common difference of the sequence. (2 marks)

b_n is an arithmetic sequence (1) and common difference is 2.5 (1).

b. State which sequence is geometric and find the common ratio of the sequence. (2 marks)

c_n is a geometric sequence (1) and the common ratio is $\frac{1}{3}$ (1).

c. For the **geometric** sequence, find the **exact** value of the sixth term. Give your answer as a fraction. (2 marks)

$$c_6 = 3 \times \left(\frac{1}{3}\right)^{6-1} = 3 \times \left(\frac{1}{3}\right)^5 = \frac{1}{3^4} = \frac{1}{81} \text{ [1M, 1A]}$$

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Question 2 (3 marks)

Given a geometric sequence as the second term $t_2 = 6$ and $S_2 = 8$, find t_6.

$$ar = 6$$
$$\frac{a(1-r^2)}{1-r} = 8$$

$$\frac{a(1-r)(1+r)}{1-r} = 8 \Rightarrow a(1+r) = 8 \text{ [1M]}$$
$$\Rightarrow a + ar = 8$$
$$\Rightarrow a + 6 = 8$$
$$\Rightarrow a = 2$$

$$r = 3 \text{ [1M]}$$
$$t_6 = 2 \times (3)^5 = 486 \text{ [1A]}$$

Question 3 (6 marks)

An arithmetic sequence has $t_1 = 40$ and $t_5 = 26$.

a. Find the common difference, d. (1 mark)

$$d = \frac{26-40}{4} = -\frac{7}{2} \text{ [1A]}$$

b. Find t_9. (1 mark)

$$t_9 = 40 - \frac{7}{2}(8) = 12 \text{ [1A]}$$

c. Find S_9. (2 marks)

$$S_9 = \frac{9}{2}\left(2(40) + (9-1) \times -\frac{7}{2}\right) \text{ [1M]}$$
$$= \frac{9}{2}(80 - 28) = 9(26) = 234 \text{ [1A]}$$

d. Find the smallest possible value of k, where $k \in N$, such that $S_k < 0$. (2 marks)

$$S_k = \frac{k}{2}\left(2(40) + (k-1) \times -\frac{7}{2}\right)$$
$$= \frac{k}{2}\left(80 + \frac{7}{2} - \frac{7}{2}k\right) = \frac{1}{4}k(167 - 7k) \text{ [1M]}$$

$S_k < 0$ for $k < 0$ or $k > \frac{167}{7}$

$\frac{167}{7} = 23\frac{6}{7}$ therefore, the smallest value is $k = 24$ [1A].

Question 4 (3 marks)

An arithmetic has $u_1 = 41, u_2 = 37, u_3 = 33$ and continues in this way.

a. What is the largest value of n such that $u_n > -200$? (2 marks)

$d = -4$
$t_n = 41 - 4(n-1)$

Solve: $45 - 4n > -200$
$4n < 245$
$n < 61.25$
Largest value of n is 61.

b. For the value of n found in **part a.**, calculate $u_{n+7} - u_n$. (1 mark)

$7(d) = -28$ [1A]

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Question 5 (7 marks)

The first three terms of a geometric sequence are $(k+4)$, k and $(2k-15)$ respectively, where k is a positive constant.

a. Show that $k^2 - 7k - 60 = 0$. (2 marks)

$$\frac{2k-15}{k} = \frac{k}{k+4} (= r)[1M]$$
$$(2k-15)(k+4) = k^2$$
$$2k^2 - 7k - 60 = k^2$$
$$k^2 - 7k - 60 = 0, \text{ as required } [1A].$$

b. Hence, solve for the value of k. (1 mark)

$(k-12)(k+5) = 0$, $k = 12$ as $k > 0$ [1A].

c. Find the common ratio of this geometric sequence. (1 mark)

$$r = \frac{k}{k+4} = \frac{12}{16} = \frac{3}{4} \text{ [1A]}$$

d. The sum of the first four terms in the geometric series can be written as a fully reduced fraction of the form $\frac{p}{q}$ where $p, q \in N$. State the value of p and the value of q. (2 marks)

$$S_4 = \frac{16\left(1 - \left(\frac{3}{4}\right)^4\right)}{1 - \frac{3}{4}} = 64 \times \left(1 - \frac{81}{256}\right) = 64\left(\frac{175}{256}\right) = \frac{175}{4} \text{ [1M]}$$
$$p = 175, q = 4 \text{ [1A]}$$

e. Find the sum to infinity of this series. (1 mark)

$$S_\infty = \frac{a}{1-r} = \frac{16}{1 - \frac{3}{4}} = 64 \text{ [1A]}$$

Question 6 (12 marks)

Let $\{u_1, u_2, \dots, u_n\}$ where $n \in N$ be an arithmetic sequence with the first term equal to a and common difference equal to d.

Let another sequence $\{v_1, v_2, \dots, v_n\}$ where $n \in N$ be defined such that $v_k = 2^{u_k}$, $1 \le k \le n$ and $k \in N$.

a.

i. Show that $\frac{v_{n+1}}{v_n}$ is a constant. (2 marks)

$$\frac{v_{n+1}}{v_n} = \frac{2^{u_{k+1}}}{2^{u_k}} = 2^{u_{k+1} - u_k} = 2^d,$$

which is a constant, since d is a constant [1M, 1A].

ii. Write down the first term of the sequence $\{v_n\}$. (1 mark)

$v_1 = 2^{u_1} = 2^a$ [1A]

iii. Write down the formula for v_n in terms of a, d and n. (1 mark)

$v_n = 2^{u_n} = 2^{a+(n-1)d}$ [1A]

b. Let S_n be the sum of the first n terms in the sequence $\{v_n\}$.

i. Express S_n in terms of a, d and n. (2 marks)

$$S_n = \frac{v_1(1-r^n)}{1-r}$$

$v_1 = 2^a$, $r = 2^d$ from **part a.** [1M]

$$S_n = \frac{2^a(1-2^{d\times n})}{1-2^d} \text{ OR } \frac{2^a(2^{d\times n}-1)}{2^d-1} \text{ [1A]}$$

ii. Find the values of d for which S_∞ exists. (2 marks)

Need $|r| < 1$, so need $2^d < 1$ [1M]

$d < \log_2(1)$

$d < 0$ [1A]

You are now told that S_∞ exists.

iii. Write down S_∞ in terms of a and d. (2 marks)

$$S_n = \frac{2^a\left(1-2^{d\times n}\right)}{1-2^d}$$

$$S_\infty = \lim_{n\to\infty}\left(\frac{2^a(1-2^{d\times n})}{1-2^d}\right) = \frac{2^a}{1-2^d} \text{ as } 2^{dn} \to 0$$

If $d < 0$ and $n \to \infty$ [1M]

$$S_\infty = \frac{2^a}{1-2^d} \text{ [1A]}$$

iv. Given that $S_\infty = 2^{a+1}$, find the value of d. (2 marks)

$$\frac{2^a}{1-2^d} = 2^{a+1}$$

$$1-2^d = \frac{1}{2} \text{ [1M]}$$

$$2^d = \frac{1}{2}$$

$$d = -1 \text{ [1M]}$$

Question 7 (3 marks)

A sequence $t_1, t_2, \ldots, t_n$ is given by:

$$t_{n+1} = (3 - t_n)^2 \text{ with } t_1 = 4$$

a. Find t_2, t_3, and t_4. (1 mark)

$\boxed{t_2 = 1, t_3 = 4, t_4 = 1}$

b. Find t_{10}. (1 mark)

$\boxed{1}$

c. What do you notice about this sequence? (1 mark)

Oscillating, 'even' terms $= 1$, 'odd' terms $= 3$.

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