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VCE Specialist Mathematics ½
AOS 1 Revision [1.0]
SAC 3 Solutions

46 Marks. 10 Minutes Reading. 60 Minutes Writing.

Section A: SAC Questions (46 Marks)

Question 1 (13 marks)

The first few terms of an arithmetic sequence $\{u_n\}$ is given below:

$\boldsymbol{n}$	1	2	3	4	5
$\boldsymbol{u_n}$	5	11	17	23	29

a. Write down an explicit formula for the n^{th} term of the arithmetic sequence $\{u_n\}$. (1 mark)

$t_n = 5 + 4(n - 1)$

b. Express the sequence $\{u_n\}$ as a recurrence relation. (1 mark)

$u_{n+1} = u_n + 4$ and $u_1 = 5$.

c. Find:

i. The eleventh term of the sequence. (1 mark)

$u_{11} = 65$

ii. The sum of the first eleven terms of the sequence. (1 mark)

$S_{11} = 385$

The n^{th} term of the sequence exceeds 200.

d. Find the smallest value of n. (1 mark)

$n = 34$

e. The sum of the first k terms of the sequence is 705. Find the value of k. (1 mark)

$k = 15$

Another arithmetic sequence $\{t_n\}$ is given below:

n	1	2	3	4	5
t_n	151	142	133	124	115

Define $d_n = t_n - u_n$.

f. Find d_7. (2 marks)

$t_7 = 151 + (7-1)(-9) = 97$

$u_7 = 41$

$\therefore d_7 = 56$

g. What happens to d_n as $n \to \infty$? (1 mark)

$d_n \to -\infty$ as the common difference is negative for the sequence d_n.

h. Write down an expression for the sum of the first n terms in the sequence d_n in the form $\hat{S}_n = Pn^2 + Qn$ where P and Q are rational numbers. (2 marks)

$u_n = 5 + 6(n-1)$

$t_n = 151 - 9(n-1)$

$d_n = 146 - 15(n-1)$ [1M]

$\therefore S_n = \frac{n}{2}\left(292 + 15(n-1)\right)$ [1M]

$S_n = -\frac{15}{2}n^2 + \frac{307}{2}n$

i. Hence, find the smallest value of τ for which $\hat{S}_\tau < 0$. (2 marks)

$$\text{solve}\left(\frac{307\cdot n}{2}-\frac{15\cdot n^2}{2}>0,n\right)$$

$$0.<n<20.46667$$

$\tau = 21$

Question 2 (11 marks)

Glenys invests \$60,000 in an account which earns interest at a compound rate of 7% per annum. She wants to make a withdrawal of \$$W$ at the end of each year for part of her living expenses. The withdrawal is made immediately after that year's interest has been paid. She intends to do this for 20 years, so that at the end of this time her account will have a zero balance.

Use the compound interest formula $A_n = P\left(1+\frac{r}{100}\right)^n$, where r = interest rate in %, P = principal amount (initial investment), and A_n = accumulated value of investment after time n.

a. Write an expression for the amount of money in Glenys bank account immediately **after** she has made her first withdrawal. (2 marks)

$$A = P\left(1+\frac{r}{100}\right)^n = P\left(1+\frac{7}{100}\right)^n = P(1\cdot07)^n$$

$$\text{(i)} \quad A_1' = P(1\cdot07) - W = 60\,000(1\cdot07) - W$$

$60000(1.07) - W$

b. Write an expression in terms of W for the amount of money remaining in her account just before her final withdrawal. (3 marks)

$$\begin{aligned} A_2 &= A_1(1.07) - W \\ &= [P(1.07) - W]\,1.07 - W \\ &= P(1.07)^2 - 1.07W - W \\ &= P(1.07)^2 - (1.07 + 1)W \\ A_3 &= A_2(1.07) - W \\ &= \left[P(1.07)^2 - (1.07+1)W\right]1.07 - W \\ &= P(1.07)^3 - \left(1.07^2 + 1.07\right)W - W \\ &= P(1.07)^3 - \left(1.07^2 + 1.07 + 1\right)W \\ &\vdots \\ A_{20} &= P(1.07)^{20} - \left(1.07^{19} + 1.07^{18} + \cdots + 1.07 + 1\right)W \end{aligned}$$

$$\begin{aligned} A_{20} &= P(1.07)^{20} - \left(1.07^{19} + 1.07^{18} + \cdots + 1.07 + 1\right)W \\ &= P(1.07)^{20} - \underbrace{\left(1 + 1.07 + \cdots + 1.07^{19}\right)}_{\text{a GP with } S_n = \frac{a(r^n - 1)}{r - 1}}W \\ &= P(1.07)^{20} - \left[\frac{1\left(1.07^{20} - 1\right)}{1.07 - 1}\right]W \\ &= P(1.07)^{20} - \left(\frac{1.07^{20} - 1}{0.07}\right)W \\ &= 60\,000(1.07)^{20} - \left(\frac{1.07^{20} - 1}{0.07}\right)W \end{aligned}$$

c. Calculate the value of P which leaves the account with a zero balance just after the 20th withdrawal. Give your answer to the nearest cent. (2 marks)

The account is empty when $A_{20} = 0$

$$\begin{aligned} \therefore \quad 0 &= 60\,000(1.07)^{20} - \left(\frac{1.07^{20} - 1}{0.07}\right)W \\ W\left(\frac{1.07^{20} - 1}{0.07}\right) &= 60\,000(1.07)^{20} \\ \therefore \quad W &= \frac{60\,000(1.07)^{20} \times 0.07}{1.07^{20} - 1} \\ &= \$5663.58 \quad \text{(to the nearest cent)} \end{aligned}$$

d. Assume Glenys wishes to be able to withdraw \$7500 per annum for the 20 years. Estimate, to the nearest per cent, the interest rate she would need to earn on her investment. (2 marks)

$$\text{solve}\left(\frac{60000\cdot(1+r)^{20}\cdot r}{(1+r)^{20}-1}=7500, r\right)$$

$r=-1.870646$ or $r=0.1092985$

$r = 11\%$ [nearest %]

e. If the interest rate was still at 7%, how much additional capital (money) would Glenys have to initially invest in the account to be able to withdraw \$7500 per annum for the 20 years. Give your answer to the nearest cent. (2 marks)

$$\text{solve}\left(\frac{(60000+p)\cdot(1+r)^{20}\cdot r}{(1+r)^{20}-1}=7500,p\right)\Big|r=0.07$$

$$p=19455.11$$

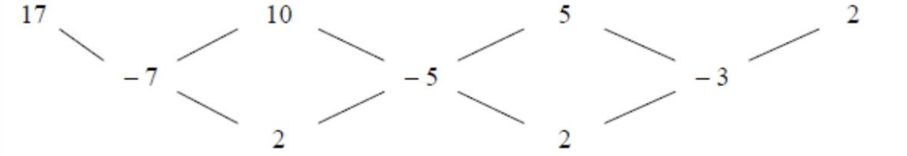
Has to invest an additional \$19,455.11.

Question 3 (10 marks)

An athlete runs along a straight road. His distance d from a fixed point P on the road is measured at different times, n, and has the form $d(n) = an^2 + bn + c$. The distances are recorded in the table below.

Time (in seconds)	1	2	3	4	5	6
Distance (in metres)	17	10	5	2	r	s

a. Determine the values of r and s. (2 marks)



$r = 1$
$s = 2$

✓ complete pattern

✓ $r = 1$

✓ $s = 2$

b. Define the above sequence as a recurrence relation. (2 marks)

$$d_{n+1} = d_n + (-7 + 2n), d_1 = 17$$

c. Determine the values of a, b and c. (2 marks)

$$a+b+c=17$$
$$4a+2b+c=10$$
$$9a+3b+c=5$$
$$3a+b=-7$$
$$5a+b=-5$$
$$2a=2$$

$$a=1$$
$$3(1)+b=-7$$
$$b=-10$$
$$(1)-10+c=17$$
$$c=26$$
$$d(n)=n^2-10n+26$$

d. How far is the athlete from P when $n = 7$ and $n = 8$? (2 marks)

$$d(8)=(8)^2-10(8)+26$$
$$=10 \text{ m}$$

$$d(7) = 5\ m$$

e. Justify the statement: "The athlete is moving towards P when $n < 5$, and away from P when $n > 5$." (2 marks)

Since the distance from P is decreasing for $n < 5$ the athlete is moving towards P. Since the distance from P is increasing for $n > 5$, the athlete is moving away from P.	✓✓ decreasing Moving towards ✓✓ increasing Moving away

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Question 4 (12 marks)

A first-order linear recurrence relation generates a sequence of the form:

$$t_1 = b, t_{n+1} = rt_n + d \quad [1]$$

First order linear recurrence relations with constant coefficients will always have solutions of the form:

$$t_n = c \times r^n + \beta \ [2]$$

Where $c,\ \beta$, are new constants, and r is the same multiplier of t_n as seen in formula [1].

a. Solve the following first order linear recurrence relation, writing an explicit formula for t_n in terms of n in the form of formula [2]. (3 marks)

$$t_{n+1} = -3t_n + 5\,, t_1 = -2$$

1. **Set** $t_n = c \times (-3)^n + \beta$ So $t_{n+1} = c \times (-3)^{n+1} + \beta$

2. Substitute into the recurrence relation to find β

$$t_{n+1} = -3t_n + 5 \rightarrow c \times (-3)^{n+1} + \beta = -3(c \times (-3)^n + \beta) + 5$$

$$c \times (-3)^{n+1} + \beta = c \times (-3)^{n+1} - 3\beta + 5 \rightarrow \beta = -3\beta + 5 \rightarrow \beta = \frac{5}{4}$$

3. Substitute β into the first term to find c

$$u_1 = c \times (-3)^1 + \frac{5}{4}$$

$$c = \frac{13}{12}$$

4. Substitute β and c into t_n $\quad t_n = \frac{13}{12} \times (-3)^n + \frac{5}{4}$

An endangered species of insect has a total population of 400 and lives only in a certain area known as the Old Forest. Conservationists monitoring the situation decide to relocate 40 insects at the beginning of each year from the Old Forest to the New Forest. It is assumed that each population increases by 10% per year (i.e., from the start of the current year to the end of the current year).

b. Find the insect population of each area after one year, just after the second set of 40 insects have been relocated. (2 marks)

$370 \cdot 1.1 - 40$	367.
$30 \cdot 1.1 + 40$	73.

Old forest: $P_1 = 367$
New forest: $Q_1 = 73$

c. The size, P_n, of the insect population at the Old Forest after n years (just after 40 insects have been relocated for that year) is given by the recurrence relation:

$$P_0 = 370, P_n = 1.10P_{n-1} - 40, \text{ for } n \in \mathbb{N}$$

Write down a similar recurrence relation for the size, Q_n, of the insect population at the New Forest after n years. (1 mark)

The New Forest receives 40 insects every year and grows by 10%. The recurrence relation is:

$$Q_0 = 30, \quad Q_n = 1.10Q_{n-1} + 40, \quad \text{for } n \in \mathbb{N}$$

d. Find explicit expressions for P_n and Q_n in terms of n. (3 marks)

$$P_n = -30(1.1)^n + 400$$
$$Q_n = 430(1.1)^n - 400$$

e. Hence, or otherwise, predict:

i. The insect populations of the Old Forest and the New Forest after five years, correct to the nearest whole number. (2 marks)

$q(n):=430\cdot(1.1)^n-400$	*Done*
$p(n):=-30\cdot(1.1)^n+400$	*Done*
$q(5)$	292.5193
$p(5)$	351.6847

$P_5: 352$
$Q_5: 293$

ii. The number of years (correct to one decimal place) that pass before the insect populations of the two areas are most nearly equal. (1 mark)

$\text{solve}(q(n)=p(n),n)$ $\quad n=5.80615$

5.8 years

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