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VCE Specialist Mathematics ½
AOS 1 Revision [1.0]
Contour Check Solutions

Contour Check

Section A: [1.1] - Modulus & Partial Fractions (Checkpoints)

Sub-Section [1.1.1]: Solving Simple Modulus Equations and Inequalities

Question 1

Evaluate:

a. $-|-4|$

$$\boxed{-4}$$

b. $|5|+|-6|$

$$\boxed{11}$$

c. $|-8|^2$

$$\boxed{64}$$

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Question 2

Solve the following equations for x.

a. $|3-2x|=1$

$$\boxed{\begin{array}{l} 3-2x=1 \implies 2x=2 \implies x=1 \\ 3-2x=-1 \implies 2x=4 \implies x=2 \\ \text{Therefore, } x=1 \text{ or } x=2 \end{array}}$$

b. $|x^2|=2$

$$\boxed{x^2=2 \implies x=-\sqrt{2} \text{ or } x=\sqrt{2}}$$

c. $|x^2+1|=2$

$$\boxed{x^2+1=2 \implies x^2=1 \implies x=-1 \text{ or } x=1}$$

Question 3

Solve the following equations for x.

a. $|-x| < 2$

$x < 2$

$x > -2$

Therefore $-2 < x < 2$

b. $|2x - 1| > 5$

$2x - 1 > 5 \implies 2x > 6 \implies x > 3$

$2x - 1 < -5 \implies 2x < -4 \implies x < -2$

Therefore $x < -2$ or $x > 3$

c. $|2x - 5| + 3 < 4$

$|2x - 5| < 1$

$2x - 5 < 1 \implies 2x < 6 \implies x < 3$

$2x - 5 > -1 \implies 2x > 4 \implies x > 2$

Therefore $2 < x < 3$

Question 4

Consider the following equation $|x|^2 - 7|x| + 10 = 0$.

a. Before solving the equation, how many solutions do you expect this equation will have? Why?

There is no correct answer for this part, but one might expect four solutions.

b. Solve the equation.

$x^2 - 7|x| + 10 = 0$

when $x > 0$ we have $|x| = x$

$$\begin{aligned} x^2 - 7x + 10 &= 0 \\ x &= \frac{7 \pm \sqrt{49 - 40}}{2} \\ x &= \frac{7 \pm 3}{2} \\ x &= 2, 5 \end{aligned}$$

when $x < 0$ we have $|x| = -x$

$$\begin{aligned} x^2 + 7x + 10 &= 0 \\ x &= \frac{-7 \pm \sqrt{49 - 40}}{2} \\ x &= \frac{-7 \pm 3}{2} \\ x &= -2, -5 \end{aligned}$$

Therefore $x = \pm 2$ or $x = \pm 5$

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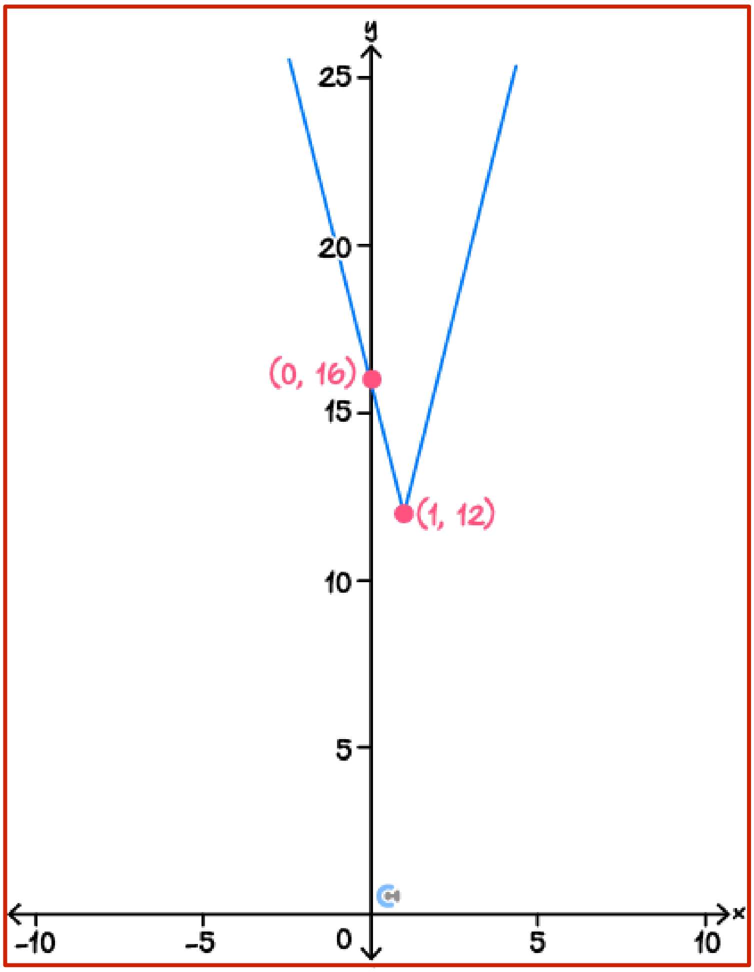
Sub-Section [1.1.2]: Graphing Modulus Functions and Composite Modulus Functions

Question 5

Sketch the graph of the function $f(x) = 4|x - 1| + 12$ on the axes below. Label the axis intercepts and the vertex of the graph.



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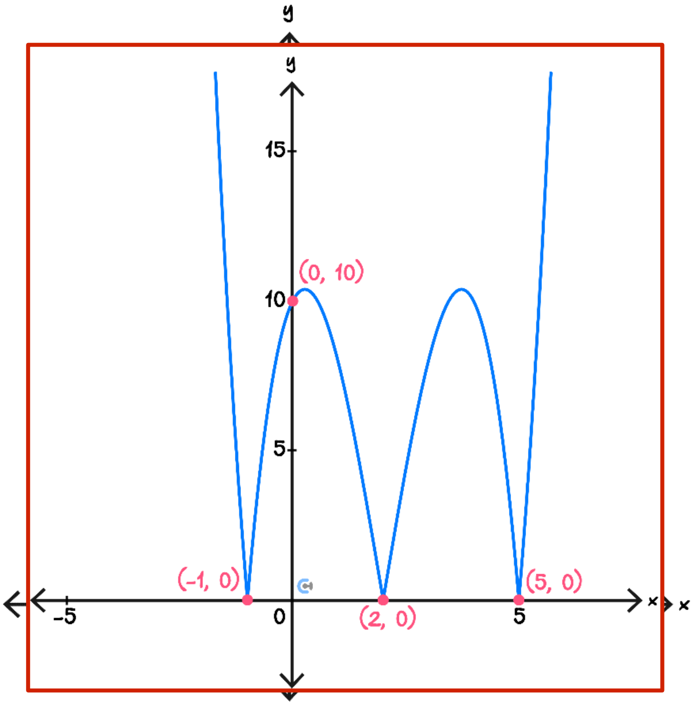
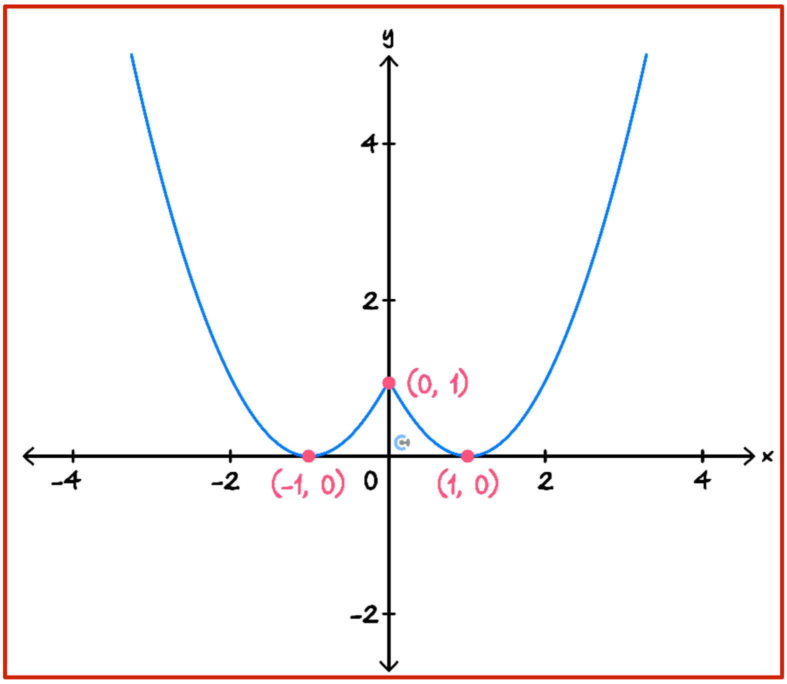
Question 6

Sketch the graph of the function $f(x) = |(x-2)(x+1)(x-5)|$. Label any axis intercepts.



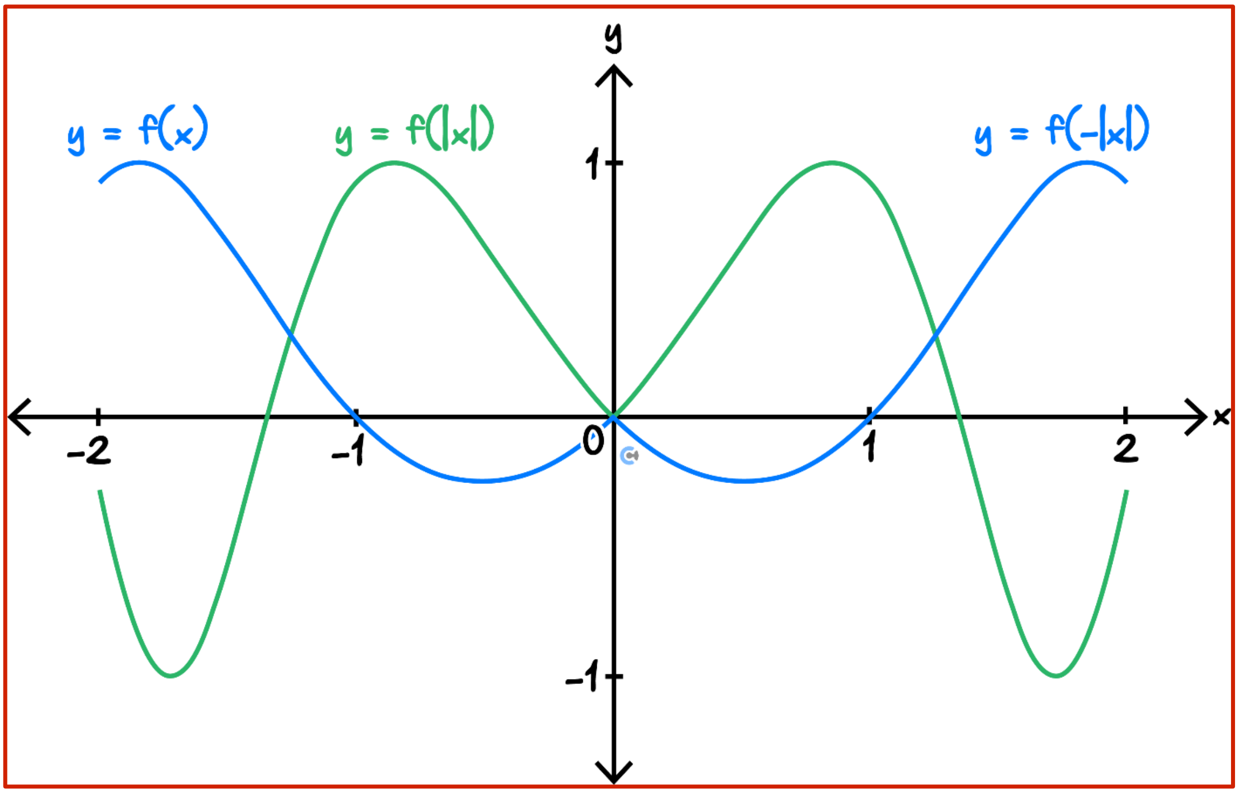
Question 7

Sketch the graph of the function $y = f(|x|)$ where $f(x) = (x-1)^2$. Label any axis intercepts.



Question 8

Sketch the graphs of the functions $y = f(|x|)$ and $y = f(-|x|)$ in the interval $-2 < x < 2$ where the graph of $y = f(x)$ is shown below.



$y = f(|x|)$ is the right (positive) side of the graph reflected in the y-axis
$y = f(-|x|)$ is the left (negative) side of the graph reflected in the y-axis

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Sub-Section [1.1.3]: Apply Partial Fractions to Find a Decomposed Form

Question 9

Perform partial fraction decomposition to write the following functions in the form specified below.

a. $\frac{6x+2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$A(x+1) + B(x-3) = 6x + 2$

$x = 3 \implies 4A = 20 \implies A = 5$

$x = -1 \implies -4B = -4 \implies B = 1$

$$\frac{6x+2}{(x-3)(x+1)} = \frac{5}{x-3} + \frac{1}{x+1}.$$

b. $\frac{5x^2-24x+29}{(x-3)^2(x-2)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x-2}$

$A(x-3)(x-2) + B(x-2) + C(x-3)^2 = 5x^2 - 24x + 29$

$x = 3 \implies B = 2$

$x = 2 \implies C = 1$

$Ax^2 + x^2 = 5x^2 \implies A = 4$

$$\frac{5x^2-24x+29}{(x-3)^2(x-2)} = \frac{4}{x-3} + \frac{2}{(x-3)^2} + \frac{1}{x-2}$$

c. $\frac{7x^2-3x+14}{(x-1)(x^2+3x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3x+5}$

$$A(x^2+3x+5)+(Bx+C)(x-1)=7x^2-3x+14$$
$$x=1 \implies 9A=18 \implies A=2$$
$$10-C=14 \implies C=-4$$
$$2x^2+Bx^2=7x^2 \implies B=5$$
$$\frac{7x^2-3x+14}{(x-1)(x^2+3x+5)}=\frac{2}{x-1}+\frac{5x-4}{x^2+3x+5}$$

Question 10

Perform partial fraction decomposition to the following functions.

a. $\frac{8x-12}{x^2-2x-3}$

$$\frac{8x-12}{x^2-2x-3}=\frac{A}{x+1}+\frac{B}{x-3}$$
$$A(x-3)+B(x+1)=8x-12$$
$$x=3 \implies 4B=12 \implies B=3$$
$$x=-1 \implies -4A=-20 \implies A=5$$
$$\frac{8x-12}{x^2-2x-3}=\frac{5}{x+1}+\frac{3}{x-3}$$

b. $\frac{7x^2+6x-8}{x^3+2x^2}$

$$\frac{7x^2+6x-8}{x^3+2x^2} = \frac{7x^2+6x-8}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$
$$Ax(x+2) + B(x+2) + Cx^2 = 7x^2 + 6x - 8$$
$$x = -2 \implies 4C = 8 \implies C = 2$$
$$x = 0 \implies 2B = -8 \implies B = -4$$
$$Ax^2 + 2x^2 = 7x^2 \implies A = 5$$
$$\frac{7x^2+6x-8}{x^3+2x^2} = \frac{5}{x} - \frac{4}{x^2} + \frac{2}{x+2}$$

c. $\frac{6x^3-x-6}{x^4-2x^3}$

$$\frac{6x^3-x-6}{x^4-2x^3} = \frac{6x^3-x-6}{x^3(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2}$$
$$Ax^2(x-2) + Bx(x-2) + C(x-2) + Dx^3 = 6x^3 - x - 6$$
$$x = 2 \implies 8D = 40 \implies D = 5$$
$$x = 0 \implies -2C = -6 \implies C = 3$$
$$Ax^3 + 5x^3 = 6x^3 \implies A = 1$$
$$-2x^2 + Bx^2 = 0x^2 \implies B = 2$$
$$\frac{6x^3-x-6}{x^4-2x^3} = \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3} + \frac{5}{x-2}$$

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Question 11

Perform partial fraction decomposition to the following functions.

a. $\frac{x^3-4x^2+18}{x^2+x-2}$

Perform polynomial long division/compare coefficients to obtain

$$\begin{aligned}\frac{x^3-4x^2+18}{x^2+x-2} &= \frac{(x^2+x-2)(x-5)+8+7x}{x^2+x-2}\\ &= x-5+\frac{8+7x}{x^2+x-2}\\ &= x-5+\frac{8+7x}{(x+2)(x-1)}\\ &= x-5+\frac{A}{x+2}+\frac{B}{x-1}\end{aligned}$$

$A(x-1)+B(x+2)=7x+8$

$x=1 \implies 3B=15 \implies B=5$

$x=-2 \implies -3A=-6 \implies A=2$

$$\frac{x^3-4x^2+18}{x^2+x-2}=x-5+\frac{2}{x+2}+\frac{5}{x-1}$$

b. $\frac{x^4+x^3-x^2-x-3}{x^2-x-2}$

Perform polynomial long division/compare coefficients to obtain

$$\begin{aligned}\frac{x^4+x^3-x^2-x-3}{x^2-x-2} &= \frac{(x^2-x-2)(x^2+2x+3)+3+6x}{x^2-x-2}\\ &= x^2+2x+3+\frac{3+6x}{(x-2)(x+1)}\\ &= x^2+2x+3+\frac{A}{x-2}+\frac{B}{x+1}\end{aligned}$$

$A(x+1)+B(x-2)=6x+3$

$x=-1 \implies -3B=-3 \implies B=1$

$x=2 \implies 3A=15 \implies A=5$

$$\frac{x^4+x^3-x^2-x-3}{x^2-x-2}=x^2+2x+3+\frac{5}{x-2}+\frac{1}{x+1}$$

c. $\frac{7x^4+10x^3+24x^2-38x-35}{(x-1)(x^2+2x+5)}$

Perform polynomial long division/compare coefficients to obtain

$$\begin{aligned}\frac{7x^4+10x^3+24x^2-38x-35}{(x-1)(x^2+2x+5)} &= \frac{(x^3+x^2+3x-5)(7x+3)-20-12x}{x^3+x^2+3x-5}\\ &= 7x+3+\frac{-20-12x}{(x-1)(x^2+2x+5)}\\ &= 7x+3+\frac{A}{x-1}+\frac{Bx+C}{x^2+2x+5}\end{aligned}$$

$A(x^2+2x+5)+(Bx+C)(x-1) = -12x-20$

$x=1 \implies 8A=-32 \implies A=-4$

$-4x^2+Bx^2=0x^2 \implies B=4$

$-20-C=-20 \implies C=0$

$$\frac{7x^4+10x^3+24x^2-38x-35}{(x-1)(x^2+2x+5)} = 7x+3-\frac{4}{x-1}+\frac{4x}{x^2+2x+5}$$

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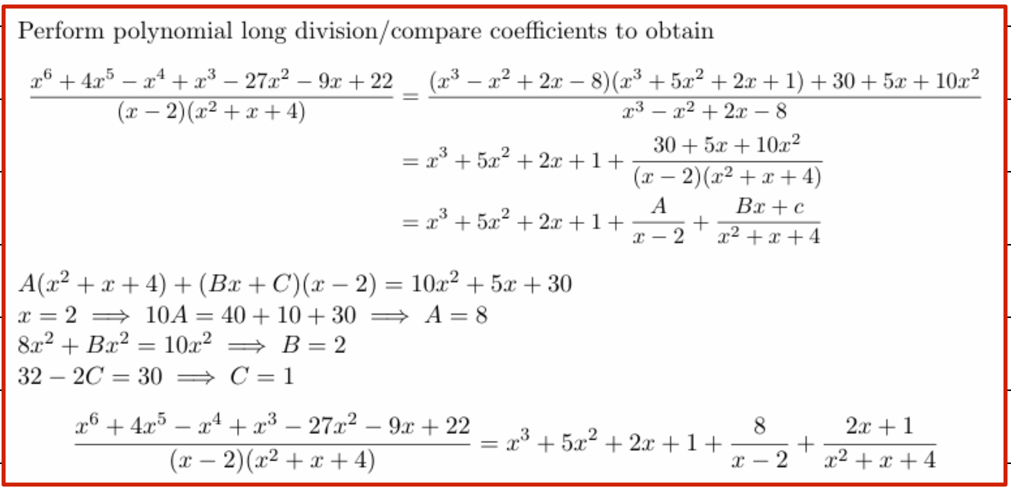
Question 12

Perform partial fraction decomposition to the function $f(x) = \frac{x^6+4x^5-x^4+x^3-27x^2-9x+22}{(x-2)(x^2+x+4)}$.



Perform polynomial long division/compare coefficients to obtain

$$\frac{x^6+4x^5-x^4+x^3-27x^2-9x+22}{(x-2)(x^2+x+4)} = \frac{(x^3-x^2+2x-8)(x^3+5x^2+2x+1)+30+5x+10x^2}{x^3-x^2+2x-8}$$

$$= x^3+5x^2+2x+1+\frac{30+5x+10x^2}{(x-2)(x^2+x+4)}$$

$$= x^3+5x^2+2x+1+\frac{A}{x-2}+\frac{Bx+c}{x^2+x+4}$$

$A(x^2+x+4)+(Bx+C)(x-2) = 10x^2+5x+30$

$x = 2 \implies 10A = 40+10+30 \implies A = 8$

$8x^2+Bx^2 = 10x^2 \implies B = 2$

$32-2C = 30 \implies C = 1$

$$\frac{x^6+4x^5-x^4+x^3-27x^2-9x+22}{(x-2)(x^2+x+4)} = x^3+5x^2+2x+1+\frac{8}{x-2}+\frac{2x+1}{x^2+x+4}$$

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Section B: [1.2] - Modulus & Partial Fractions Exam Skills (Checkpoints)

Sub-Section [1.2.1]: Solving Advanced Algebra and Inequalities

Question 13

Solve the equation $|x-1|+3=|3x+1|-2$ for $x \in \mathbb{R}$.

$$\boxed{x=-\frac{7}{2},\frac{3}{2}}$$

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Question 14

Solve the equation $|2x-3| = -2|x+1| + 5$ for $x \in \mathbb{R}$.

$$\boxed{-1 \leq x \leq \frac{3}{2}}$$

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Question 15

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Solve the inequality $\frac{1}{|x-4|}+2<x+6$ for $x \in \mathbb{R}$.

$$\boxed{-\sqrt{15}<x<\sqrt{15} \text{ or } x>\sqrt{17}}$$

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Question 16

Solve the inequality $\left|\frac{x-4}{x+1}\right| - 3 > |x+2|$ for $x \in \mathbb{R}$.

$$\boxed{x \in \left(\frac{1}{2}(-1-\sqrt{21}), \frac{1}{2}(3\sqrt{5}-7)\right) \setminus \{-1\}}$$

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Section C: [1.3] – Sequences & Series (Checkpoints)

Sub-Section [1.3.1]: Finding Sequence from Recurrence Relations

Question 17

Given $t_n = 6 + 4 \cdot t_{n-1}$ and $t_1 = 3$, find the value of t_3. Is the sequence an arithmetic sequence, geometric sequence, or neither?

$$\boxed{t_3 = 78}$$

Question 18

Given $t_n = t_{n-1}^{t_{n-1}}$ and $t_1 = 2$, find the value of n so that, $t_n = 256$.

$$\boxed{n = 3}$$

Question 19

Given $t_n = t_{n-1}^2$ and $t_1 = 3$, find the smallest n so that, $t_n > 100$.

$$\boxed{n = 4}$$

Question 20

Given $t_n = -t_{n-1}$ and $t_1 = 2$. Write down the first few terms in the sequence and hence, write down a formula for the general term t_n.

$$\boxed{t_n = 2(-1)^{n-1}}$$

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Sub-Section [1.3.2]: Finding Arithmetic Sequence, Mean and Series

Question 21

Consider the arithmetic sequence, $t_n = t_{n-1} + 5$ and $t_1 = 2$.

a. Find t_{10}.

$$\boxed{t_n = 2 + 5(n-1) \implies t_{10} = 47}$$

b. Find the arithmetic mean of t_3 and t_{10}.

$$\boxed{\frac{t_3 + t_{10}}{2} = \frac{59}{2}}$$

c. Evaluate S_4.

$$\boxed{S_4 = 38}$$

Question 22 🌶🌶

Find the value of x so that, the arithmetic mean of 8 and $2x + 6$ is 17.

$$\boxed{x = 10}$$

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Question 23

Let $t_n = 5 + dn$. Find the value of d if, $S_4 = 50$.

$$\boxed{d = 3}$$

Question 24

Given that $t_4 = 16$ and $S_8 = 136$, find the values of a (the first term) and d (the common difference) and hence, write down the general term t_n of the sequence.

$$\boxed{a = 10,\ d = 2,\ t_n = 10 + 2(n - 1)}$$

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Sub-Section [1.3.3]: Finding Geometric Sequence, Mean and Series

Question 25

Given $t_n = 4t_{n-1}$ and $t_1 = 3$.

a. Find t_3.

$$\boxed{t_3 = 48}$$

b. Find the geometric mean of t_2 and t_5.

$$\boxed{\sqrt{t_2 \cdot t_5} = 96}$$

c. Evaluate S_5.

$$\boxed{S_5 = \frac{3 \cdot (4^5 - 1)}{4 - 1} = 1023}$$

Question 26 🌶🌶

Suppose that t_n is a geometric series such that, $t_5 = 40.5$ and $t_9 = 3280.5$. Find the common ratio of the geometric series.

$$\boxed{r = 3}$$

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Question 27

Let $t_n = 4 \cdot r^n$. Find the value(s) of r given that, the geometric mean between t_4 and t_8 is 256.

$$\boxed{r = \pm 2}$$

Question 28

Given $t_n = 6 \cdot t_{n-1}$ and $t_1 = 7$. Find the smallest value of n so that, S_n first exceeds 1000.

$$\boxed{n = 4}$$

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Sub-Section [1.3.4]: Infinite Geometric Series

Question 29

Find the value of the infinite series:

$$\frac{7}{2} - \frac{7}{4} + \frac{7}{8} - \frac{7}{16} + \cdots$$

$$\boxed{S_\infty = \frac{7}{3}}$$

Question 30

Find the value of the infinite series:

$$2 + \frac{2}{7} + \frac{2}{49} + \frac{2}{343} + \cdots$$

$$\boxed{S_\infty = \frac{7}{3} \text{ (again!)}}$$

Question 31

Find the value of r, given that:

$$5 + 5r + 5r^2 + 5r^3 + \cdots = \frac{45}{8}$$

$$\boxed{r = \frac{1}{9}}$$

Question 32

Find the value of a, given that:

$$a - \frac{a}{6} + \frac{a}{36} - \frac{a}{216} + \cdots = \frac{54}{5}$$

$$\boxed{a = 9}$$

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Section D: [1.4] -Sequences & Series Exam Skills (Checkpoints)

Sub-Section [1.4.1]: Find Sequences from Two Terms

Question 33

Define the arithmetic sequence in terms of n if $t_3 = -10$ and $t_{13} = 10$.

The common difference d is equal to $\frac{10-(-10)}{13-3} = 2$.

Thus, $t_n = a + 2(n-1)$

Since $t_3 = a + 6 = -10$ we see that $a = -14$ Hence $t_n = -16 + 2n$.

Question 34

Define possible geometric sequences in terms of n if $t_4 = \frac{1}{4}$ and $t_7 = \frac{27}{4}$.

The common ratio r satisfies $r^3 = t_7/t_4 = 27$. Thus $r = 3$.

Since, $t_n = a \times 3^{(n-1)}$ and $t_4 = 27a = \frac{1}{4}$ we see that $a = \frac{1}{108}$ and $t_n = \frac{3^n}{324}$.

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Question 35

Consider the arithmetic, a_n sequence with the following properties, $a_3 = 8$, $a_6 = -\frac{5}{2}$.

g_n is a geometric sequence with the property that $g_3 = a_3$ and $g_5 = a_5$.

Find g_n in terms of n.

We know that the common difference for a_n is $d = \dfrac{-\frac{5}{2} - 8}{6 - 3} = -\dfrac{7}{2}$. Thus $a_5 = a_3 + 2d = 8 - 7 = 1$.

Now since $g_3 = 8$ and $g_5 = 1$ we know that the common ratio for g_n, r satisfies $r^2 = \dfrac{1}{8}$, hence $r = \pm\dfrac{1}{2\sqrt{2}}$.

If $r = \dfrac{1}{2\sqrt{2}}$ we can see that $g_n = \boxed{128\sqrt{2}}\left(\dfrac{1}{2\sqrt{2}}\right)^n$.

If $r = -\dfrac{1}{2\sqrt{2}}$ we can see that $g_n = \boxed{-128\sqrt{2}}\left(-\dfrac{1}{2\sqrt{2}}\right)^n$.

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Question 36

Consider the following sequence, $a_n = b^n + c + dn$.

It is known that $a_1 = 0$, $a_2 = 1$ and $a_3 = 4$.

Find the values of b, c and d.

We construct a system of equations from the given information.

$$\begin{aligned} a_1 &= b + c + d = 0 \qquad (1) \\ a_2 &= b^2 + c + 2d = 1 \qquad (2) \\ a_3 &= b^3 + c + 3d = 4 \qquad (3) \\ & \qquad (4) \end{aligned}$$

By subtracting (1) from (2) we get the equation $b^2 - b + d = 1$.
By subtracting (1) from (3) we get the equation $b^3 - b + 2d = 4$.
By subtracting 2 $\times$ the former equation from the latter equation we get $b^3 - 2b^2 + b = 2$.
We can factorise the above cubic to get $b^2(b-2) + b - 2 = (b-2)(b^2+1) = 0$ to see that $b = 2$.
Substituting that value of b into one of the new equations we created yields $4 - 2 + d = 1 \implies d = -1$.
Substituting both of those values into (1) yields $2 + c - 1 = 0 \implies c = -1$.

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Sub-Section [1.4.2]: Apply Recurrence Relation To Different Types of Sequences

Question 37

Consider the sequence a_n, with the property that $a_3 = -5$ and $a_n = 2a_{n-1} + 1$.

a. Find a_1.

We know that $-5 = a_3 = 2a_2 + 1$. Thus $2a_2 = -6$ and $a_2 = -3$.
Thus $-3 = a_2 = 2a_1 + 1$ hence $2a_1 = -4$ and $a_1 = -2$.

b. Now assume that $a_1 = b$ and $a_n = 2a_{n-1} + 1$. Find a value of b such that $a_n = b$ for all values of n.

It is sufficient to have $a_2 = a_1 = b$.
Thus we solve $b = 2b + 1$ for b to get $b = -1$.

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Question 38

Consider the sequence defined by the following recursive relationship:

$$f_{n+1} = \frac{f_n + f_{n-1}}{4}$$

The sequence can be expressed in the form $f_n = a^n$. Find all possible values of a.

$$\boxed{\text{If } n = 1, \text{ we know that } 4a^2 = a + 1 \text{ thus } a = \frac{-1 \pm \sqrt{1+16}}{8} = \frac{-1 \pm \sqrt{17}}{8}}$$

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Question 39

Consider the Fibonacci Sequence, f_n defined as such:

$$f_1 = f_2 = 1$$

$$f_{n+1} = f_n + f_{n-1} \text{ for } n \geq 2$$

Now consider the sequence $a_n = a2^n$.

Show that for a suitable value of a, $a_n > f_n$ for all values of n.

If $a = 1$ we see that $a_1 = 2 > f_1 = 1$ and $a_2 = 4 > f_1 = 1$.
Now assume that $a_{n-1} > f_{n-1}$. Since the Fibonacci sequence is increasing, we also know that $a_{n-1} > f_{n-2}$.
Thus $a_n = 2a_{n-1} = a_{n-1} + a_{n-1} > f_{n-1} + f_{n-2} = f_n$.
Since our original statement is true for $n = 1$ and $n = 2$, the above logic shows that it will be true for $n = 3$ and hence $n = 4$ and hence any value of n.

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Question 40

Find a sequence, a_n that satisfies the recursive relationship, $a_n = 4a_{n-1} + 2a_{n-2} - 12a_{n-3} - 9a_{n-4}$, as well as the conditions:

$$a_2 = 2 \text{ and } a_3 = 4$$

Hint: $((x-1)^2 - 4)^2 = x^4 - 4x^3 - 2x^2 + 12x + 9$

If θ and ϕ are roots of $x^4 - 4x^3 - 2x^2 + 12x + 9$ we know that a sequence $a_n = a\theta^n + b\phi^n$ will satisfy our recursive relationship for any real a and b.
We solve the equation $((x-1)^2 - 4)^2 = 0$.

$$\begin{aligned} ((x-1)^2 - 4)^2 = 0 &\implies (x-1)^2 = 4 \\ &\implies x = 1 \pm 2 \\ &\implies x = -1, 3 \end{aligned}$$

Thus we consider a sequence $a_n = a(-1)^n + b(3)^n$ and solve for a and b.
Since $2 = a + 9b$ and $4 = -a + 27b$ we see that $6 = 36b \implies b = \frac{1}{6}$. Substituting this back into an equation we see that $a = \frac{1}{2}$. Hence

$$a_n = \frac{(-1)^n}{2} + \frac{3^n}{6}$$

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Section E: [1.1-1.4] – Overall Exam 1 Questions

Question 41

Solve the equation $|x - 4| = 2|x + 8|$.

$\boxed{x = -20, -4}$

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Question 42

Solve the inequality $x + 2 > \frac{1}{\sqrt{x^2-4x+4}}$ for $x \in \mathbb{R}$.

Equivalent to $x + 2 > \frac{1}{|x-2|}$

$-\sqrt{3} < x < \sqrt{3}$ or $x > \sqrt{5}$

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Question 43

a. Perform partial fraction decomposition for $f(x) = \frac{6x}{(x-1)(x+2)}$.

$$\boxed{\frac{2}{x-1} + \frac{4}{x+2}}$$

b. Express $g(x) = \frac{x^3+8}{(x+2)(x^2+4x+4)}$ in the form $\frac{A}{(x+2)^2} + \frac{B}{x+2} + C$ for real numbers A, B and C.

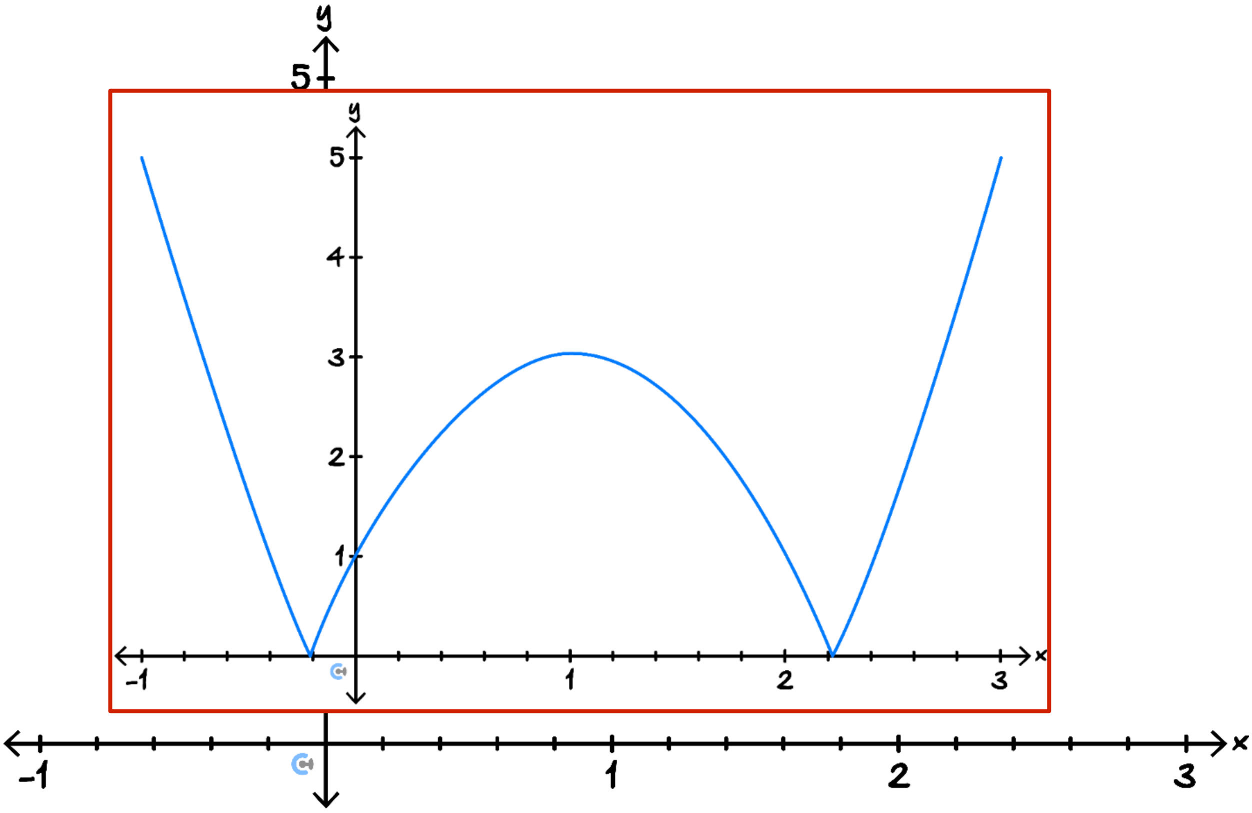
$$\boxed{\frac{12}{(x+2)^2} - \frac{6}{x+2} + 1}$$

Space for Personal Notes

Question 44

Let $f(x) = 2x^2 - 4x - 1$.

Sketch the graph of $y = |f(x)|$ on the axis below. Label all axes intercepts and turning points.



$x-$intercepts : $(1-\frac{\sqrt{6}}{2},0),(1+\frac{\sqrt{6}}{2},0)$ and y-intercept $(0,1)$ and TP $(1,3)$

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Question 45

Consider the function f with rule $f(x) = \frac{x^2+x+4}{x+1}$.

a. Show that the rule for the function f can be written as $f(x) = x + \frac{4}{x+1}$.

$$f(x) = \frac{x(x+1)+4}{x+1} = x + \frac{4}{x+1}$$

Long division :

$$\begin{array}{r|l} & x \\ \hline x+1 & x^2 + x + 4 \\ & -(x^2 + x) \\ \hline & 0 + 4 \end{array}$$

$$f(x) = x + \frac{4}{x+1}$$

b. Solve the inequality $f(x) > x + 5$ for $x \in \mathbb{R}$.

$$-1 < x < -\frac{1}{5}$$

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Question 46 (3 marks)

Consider the arithmetic sequence, a_n with the following properties:

$$a_5 = 7 \text{ and } a_8 = 19$$

a. Find $a_2 - a_1$. (1 mark)

$$\boxed{a_2 - a_1 = \frac{a_8 - a_5}{3} = \frac{19 - 7}{3} = 4}$$

b. Find a_1. (1 mark)

$$\boxed{a_1 = a_5 - 4 \times 4 = 7 - 16 = -9.}$$

c. Hence, find a_n for any natural number n. (1 mark)

$$\boxed{a_n = -13 + 4n.}$$

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Question 47 (4 marks)

Consider the following geometric progression, $b_n = 2 \times \left(-\frac{2}{3}\right)^{n-3}$.

a. Find the geometric mean of $b_1, b_2, \ldots, b_5$. (2 marks)

$$2 \times \left(-\frac{2}{3}\right)^{-3} \times \sqrt[5]{\left(-\frac{2}{3}\right)^{\sum_{n=1}^{5} n}}$$

$$= -\frac{27}{4} \times \sqrt[5]{\left(-\frac{2}{3}\right)^{15}}$$

$$= -\frac{27}{4} \times \left(-\frac{2}{3}\right)^{3} = 2$$

b. Evaluate $5b_1 - 5b_2 + 5b_3 + \ldots$ (2 marks)

$$5b_1 - 5b_2 + 5b_3 + \ldots = 5 \times 2 \times \left(\frac{9}{4}\right)\left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \ldots\right)$$

$$= \frac{45}{2} \times \frac{1}{1 - \frac{2}{3}} = \frac{135}{2}$$

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Question 48 (4 marks)

Consider a positive sequence a_n with $a_n > 0$ for all natural numbers n.

a. If $a_1 + a_2 + \ldots + a_n < 5$ for all values of n, show that there exists an integer k, such that for all $n > k$, $a_n < 1$. (3 marks)

Assume such an integer k does not exist. Then there is some n_1 such that $a_{n_1} \geq 1$. And there will be an $n_2 > n_1$ such that $a_{n_2} \geq 1$, and an $n_3 > n_2$ such that $a_{n_3} \geq 1$, and an $n_4 > n_3$ such that $a_{n_4} \geq 1$, and finally an $n_5 > n_4$ such that $a_{n_5} \geq 1$.
From here we see that since $a_n > 0$ for all n, $a_1 + a_2 + \ldots a_{n_5} \geq a_{n_1} + a_{n_2} + a_{n_3} + a_{n_4} + a_{n_5} \geq 5$ a contradiction.
Hence the statement in the question must be true.

b. Explain why a_{1000} is not necessarily less than 0.1. (1 mark)

We can set $a_n = 0$ for all $n \neq 1000$ and $a_{1000} = 1 > 0.1$. This satisties the statement $a_1 + a_2 + \cdots + a_n < 5$ for all values of n as well as the statement $a_{1000} > 0.1$.

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Question 49 (4 marks)

Consider a sequence, $\phi_n = ab^n + cd^n$, defined by the following recursive relationship:

$$\phi_{n+1} = 5\phi_n - 6\phi_{n-1}$$

If $\phi_2 = 7$ and $\phi_3 = 17$, find possible values of a, b, c and d.

We note that the sequences $(b^n)_n$ and $(d^n)_n$ must also satisfy the recursive relationship.
Hence b and d both satisfy the polynomial equation, $x^2 - 5x + 6 = (x-3)(x-2) = 0$.
We can then set $b = 3$ and $d = 2$.
Now we simply solve for a and c.
Since $\phi_2 = 7$ we get $7 = 9a + 4c$, and since $\phi_3 = 17$ we get $17 = 27a + 8c$.
From here we see that $3 = 9a$ thus $a = \frac{1}{3}$, and hence $c = 1$.

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Question 50 (5 marks)

Consider the following two sequences:

$$a_n = 3n - 1 \text{ and } b_n = 3 \times 2^{-n}$$

a. Express the sequence $c_n = b_{a_n}$ in terms of n. (1 mark)

$$\boxed{c_n = 3 \times 2^{-(3n-1)} = 6 \times 2^{-3n}}$$

b. The arithmetic mean of $a_1, \dots a_p$ is 17.

i. Find the value of p. (1 mark)

By the arithmetic mean formula we know that $a_1 + a_p = 2 + 3p - 1 = 34$. Hence $3p = 33 \implies p = 11$.

ii. Hence, or otherwise find the geometric mean of $c_1, \dots c_p$. (2 marks)

The geometric mean of $c_1, \dots, c_{11}$ is

$$\sqrt[11]{3 \times 2^{-a_1} \times \cdots \times 3 \times 2^{-a_{11}}} = 3 \times \boxed{2^{-17}}$$

c. Evaluate $c_1 + c_2 + \dots$. (1 mark)

We know that $c_1 = \frac{6}{8}$ and the common ratio is $\frac{1}{8}$, hence our sum is equal to,

$$\frac{6}{8} \times \frac{1}{1 - \frac{1}{8}} = \frac{6}{7}$$

Section F: [1.1 - 1.4] - Overall Exam 2 Questions

Question 51 (1 mark)

The equation $|2x - 3| = -|x + 2| + 6$, where $x \in \mathbb{R}$, has solution(s):

A. $x = -1, \frac{7}{3}$

B. $x = \frac{5}{3}$

C. $x = -1$

D. $x = 7, \frac{5}{3}$

Question 52 (1 mark)

The graph of $y = |2x - 1| - |x - 3|$ is the same as the graph of $y = -2 - x$ for which of the following ranges of x values:

A. $x > \frac{1}{2}$

B. $x \leq \frac{1}{2}$

C. $\frac{1}{2} \leq x \leq 3$

D. $x \geq 3$

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Question 53 (1 mark)

Which one of the following, where A, B, C, and D are non-zero real numbers, is a partial fraction form for the expression?

$$\frac{x-3}{(x^2-1)(x-2)}$$

A. $\frac{A}{x^2-1} - \frac{B}{(x-2)^2}$

B. $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2}$

C. $\frac{Ax+B}{x^2-1} + \frac{C}{x-2} + \frac{Dx}{x-2}$

D. $\frac{A}{x^2-1} + \frac{C}{x-2} + \frac{D}{x-4}$

Question 54 (1 mark)

The equation $|x^2 + 2x - 8| = k$, where k is a real number has exactly four solutions for:

A. $k = 9$

B. $0 < k < 9$

C. $k > 9$

D. $k > 0$

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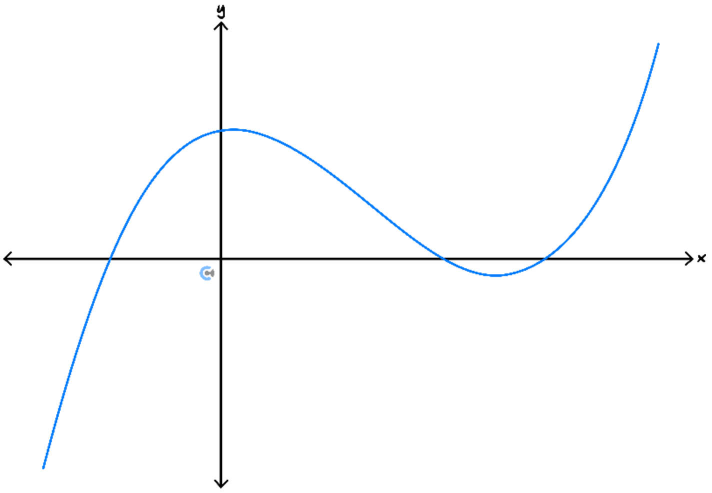
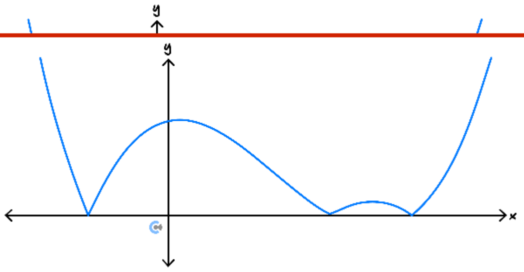
Question 55 (1 mark)

Part of the graph of $y = f(x)$ is shown below.

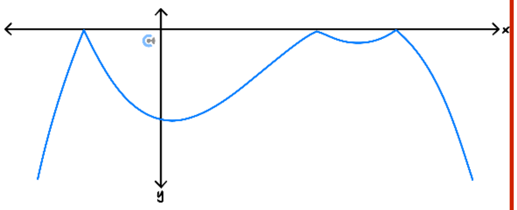


The function $f(|x|)$ is best represented by:

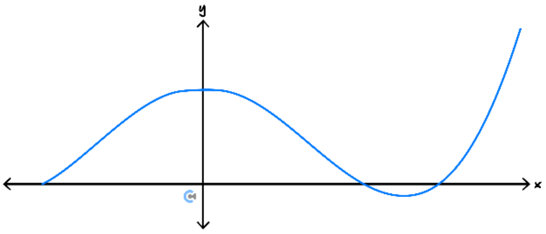
A.



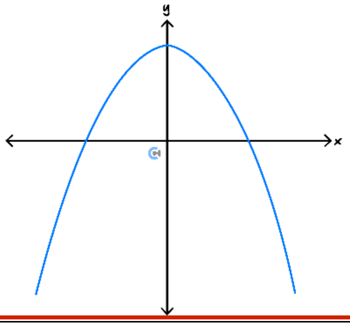
B.



C.



D.



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Question 56 (1 mark)

Consider the following sequence a_n defined recursively.

$$a_1 = 2$$

$$a_2 = 3$$

$$a_n = a_{n-1} + a_{n-2}$$

Evaluate a_{10}.

A. 55

B. 89

C. 144

D. 233

Question 57 (1 mark)

Consider the geometric sequence, a_n.

It is known that $a_1 + a_2 + a_3 + \ldots = 4$ and that $a_1 = 2$.

The geometric mean of $a_1, a_2 \ldots a_8$ is:

A. $\frac{1}{4\sqrt{2}}$

B. $\frac{1}{32}$

C. $\frac{1}{1048576}$

D. $\frac{1}{2}$

Space for Personal Notes

Question 58 (1 mark)

The sequence with consecutive entries $1, -3, 5, -7$ could be:

A. An arithmetic sequence.

B. A geometric sequence.

C. Either an arithmetic or a geometric sequence.

D. Neither an arithmetic nor a geometric sequence.

Question 59 (1 mark)

How many entries are sufficient to uniquely determine all the entries in an arithmetic progression?

A. 1

B. 2

C. 3

D. 4

Question 60 (1 mark)

Let a_n be an arithmetic sequence and let $b_n = 2^n$ be a geometric sequence.

Define the sequence $c_n = b_{a_n}$.

The arithmetic mean of $a_1, a_2, \dots a_p$ is 3.

The geometric mean of $c_1, c_2, \dots, c_p$ is:

A. 9

B. 5

C. 8

D. Impossible to tell with the current information.

Question 61 (1 mark)

Which one of the following, where A, B, C and D are non-zero real numbers, is the partial fraction form for the expression $\frac{2x^2+3x+1}{(2x+1)^2(x-1)}$?

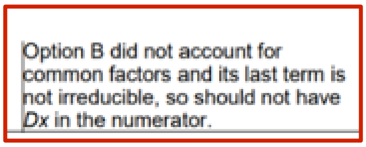
A. $\frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x+1}$

B. $\frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3} + \frac{Dx}{x^2-1}$

C. $\frac{A}{2x+1} + \frac{Bx+C}{x^2-1}$

D. $\frac{A}{2x+1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}$

E. $\frac{A}{2x+1} + \frac{Bx+C}{(2x+1)^2} + \frac{D}{x-1}$



Option B did not account for common factors and its last term is not irreducible, so should not have Dx in the numerator.

Question 62 (1 mark)

For non-zero real constants a and b, where $b < 0$, the expression $\frac{1}{ax(x^2+b)}$ in partial fraction form with linear denominators, where A, B and C are real constants, is:

A. $\frac{A}{ax} + \frac{Bx+C}{x^2+b}$

B. $\frac{A}{ax} + \frac{B}{x+\sqrt{b}} + \frac{C}{x-\sqrt{b}}$

C. $\frac{A}{ax} + \frac{B}{ax+\sqrt{|b|}} + \frac{C}{ax-\sqrt{|b|}}$

D. $\frac{A}{x} + \frac{B}{x+\sqrt{|b|}} + \frac{C}{x-\sqrt{|b|}}$

E. $\frac{A}{ax} + \frac{B}{(x+\sqrt{b})^2} + \frac{C}{x+b}$

Option A results from not considering that $b < 0$.

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Question 63 (1 mark)

For the interval $\frac{1}{2} \leq x \leq 3$, the graph of $y = |2x - 1| - |x - 3|$ is the same as the graph of:

A. $y = -x - 2$

B. $y = 3x - 4$

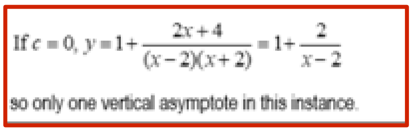
C. $y = x + 2$

D. $y = 3x + 2$

E. $y = x - 4$

Question 64 (1 mark)

The graph of $y = \frac{x^2+2x+c}{x^2-4}$ where $c \in R$, will **always** have:



If $c = 0$, $y = 1 + \frac{2x+4}{(x-2)(x+2)} = 1 + \frac{2}{x-2}$

so only one vertical asymptote in this instance.

A. Two vertical asymptotes and one horizontal asymptote.

B. Two horizontal asymptotes and one vertical asymptote.

C. A vertical asymptote with equation $x = -2$ and one horizontal asymptote with equation $y = 1$.

D. One horizontal asymptote with equation $y = 1$ and only one vertical asymptote with equation $x = 2$.

E. A horizontal asymptote with equation $y = 1$ and at least one vertical asymptote.

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Question 65 (1 mark)

Given that A, B, C and D are non-zero rational numbers, the expression $\frac{3x+1}{x(x-2)^2}$ can be represented in partial fraction form as:

A. $\frac{A}{x}+\frac{B}{(x-2)}$

B. $\frac{A}{x}+\frac{B}{(x-2)^2}$

C. $\frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}$

D. $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-2)}$

E. $\frac{A}{x}+\frac{Bx}{(x-2)}+\frac{Cx+D}{(x-2)^2}$

Question 66 (1 mark)

Consider the function with rule $f(x) = |x-3|+|x+3| - a$, where a is a real constant. The graph of $\frac{1}{f(x)}$ will have three asymptotes if the set of values of a is:

A. $\{-3,3\}$

B. $\{ \}$

C. $[6,\infty)$

D. $(-\infty,6)$

E. $[-3,3]$

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Question 67 (1 mark)

The expression $\frac{ax+b}{(2x-1)^2(x-1)}$ has partial fraction form $\frac{1}{(x+1)} - \frac{2}{(2x-1)} - \frac{1}{(2x-1)^2}$.

The values of a and b, where a and b are non-zero real constants, are respectively:

A. 12 and 21

B. 7 and 16

C. -5 and 4

D. -7 and 2

E. 3 and 6

Question 68 (1 mark)

Which one of the following, where A, B, C and D are non-zero real numbers, is a partial fraction form for the expression $\frac{x}{(x^2+1)(x-4)^2}$?

A. $\frac{A}{x^2+1} - \frac{B}{(x-4)^2}$

B. $\frac{Ax+B}{x^2+1} - \frac{C}{(x-4)^2} + \frac{Dx}{x-4}$

C. $\frac{Ax+B}{x^2+1} + \frac{C}{(x-4)^2} + \frac{Dx}{x-4}$

D. $\frac{A}{x^2+1} + \frac{C}{(x-4)^2} + \frac{D}{x-4}$

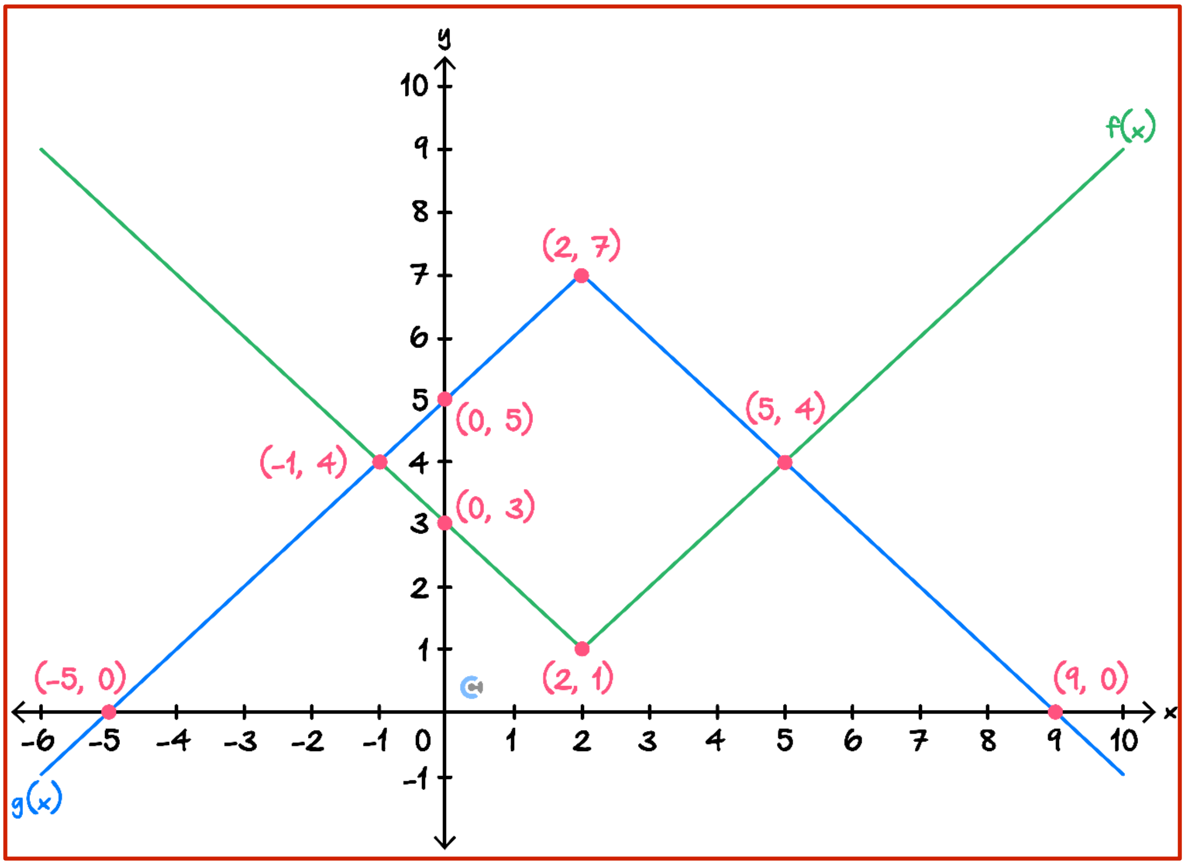
E. $\frac{Ax+B}{x^2+1} + \frac{C}{(x-4)^2}$

Space for Personal Notes

Question 69

Consider the functions $f(x) = |x - 2| + 1$ and $g(x) = -|x - 2| + 7$.

a. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the axes below. Label all points of intersection, axes intercepts, and vertex points with coordinates.



b. Solve the inequality $f(x) < g(x)$.

$\boxed{-1 < x < 5}$

c.

i. Find the value(s) of k for which the line $y = k - x$ never intersects the graph of $y = g(x)$.

$$\boxed{k > 9}$$

ii. Find the value(s) of k for which $k - x = g(x)$ has infinitely many solutions.

$$\boxed{k = 9}$$

d. Find the area of the region bounded between the graphs of $y = f(x)$ and $y = g(x)$.

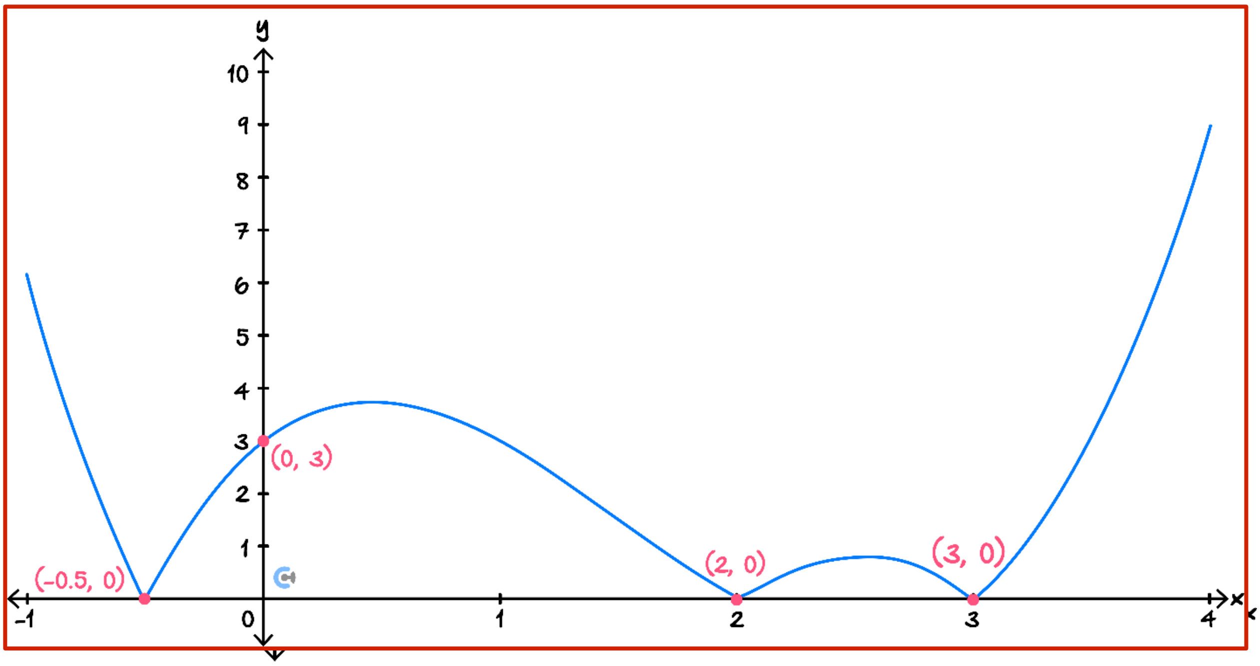
$$\boxed{\text{Area} = 2 \times \frac{1}{2} \times 6 \times 3 = 18}$$

Space for Personal Notes

Question 70

Consider the function $h(x) = \left|x^3 - \frac{9x^2}{2} + \frac{7x}{2} + 3\right|$.

a. Sketch the graph of $y = h(x)$ on the axis below. Label all axes intercepts.



b. Solve the inequality $x + 5 > h(x)$ for $x \in \mathbb{R}$. Give your answer correct to two decimal places.

$$\boxed{-0.87 < x < 4}$$

c. The equation $h(x) = k$, where k is a real number, has 6 real solutions. Find the possible value(s) of k. Give your answer correct to three decimal places.

The maximum value of $h(x)$ for $2 < x < 3$ is ≈ 0.755

Therefore, $0 < k < 0.755$.

Question 71 (9 marks)

An island has 10 fertile immortal monkeys. Every year, each pair of two fertile monkeys produces another monkey.

Let m_n denote the population of monkeys on the Island at the start of the year n.

a. Show that $m_n = 10 \times \left(\frac{3}{2}\right)^{n-1}$. (1 mark)

Hint: For this question simply approximate all answers using series and round at the end of calculations.

Every year, 2 monkeys turn into 3 monkeys. Hence the ratio of increase is $\frac{3}{2}$.

Hence $m_n = a \times \left(\frac{3}{2}\right)^n$

At the start of the first year there are 10 monkeys, i.e. $m_1 = 10$.

We solve for a to get $a = 10 \times \frac{2}{3}$.

Hence $m_n = 10 \times \left(\frac{3}{2}\right)^{n-1}$.

b. At the end of every year, 20 additional sterile immortal monkeys (they can't reproduce) are introduced.

i. Find the number of monkeys on the Island by the start of the 5^{th} year. (1 mark)

We know that 4 groups of sterile monkeys will be introduced. Hence the number of monkeys will be

$$m_5 + 80 \approx 131$$

ii. After how many years will there be more fertile monkeys than sterile monkeys? (2 marks)

The number of fertile monkeys at the start of year n is m_n. The number of sterile monkeys at the start of year n is $-20 + 20n$.
The number of fertile monkeys should eventually eclipse the number of sterile monkeys. Thus we solve $m_n = -20 + 20n$ for n. This yields $n = 7.21$ which we round up to $n = 8$.

c. At the end of each year, monkeys who have been on the Island for at least a year pay their taxes to the Jade Emperor (the initial monkeys pay tax at the end of the first year). At the end of 10 years how many times has the Jade Emperor received a tax form? (3 marks)

At the end of the n'th year there will be m_n fertile monkeys submitting their taxes. Thus the number of tax forms submitted by the fertile monkeys will be,

$$10 \times \left(1 + \frac{3}{2} + \cdots + \left(\frac{3}{2}\right)^9\right) = 10 \times \frac{1 - \left(\frac{3}{2}\right)^9}{1 - \frac{3}{2}} \approx 749$$

At the end of the n'th year there will be $-20 + 20n$ infertile monkey's submitting their taxes.
Thus $5 \times (0 + 180) = 900$ tax forms will be submitted by infertile monkeys.
Overall the Jade emperor will get 1649 tax forms.

d. After p years the infertile monkeys start attacking the fertile monkeys, killing 1000 monkeys a year. State the possible values of p, such that the population of fertile monkeys does not decrease. (2 marks)

The population m_p must be such that $\frac{3}{2}(m_p - 1000) \geq m_p$.
This means that $m_p \geq 3000$.
This will occur if $p \geq 16$.

Space for Personal Notes

Question 72 (9 marks)

Consider the harmonic sequence, $h_n = \frac{1}{n}$ and its associated series $H_n = \sum_{i=1}^{n} h_i$.

a. Find H_5. (1 mark)

$$\boxed{H_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}}$$

b.

i. Show that $h_{2^n+1} + h_{2^n+2} + \ldots + h_{2^{n+1}} > \frac{1}{2}$. (2 marks)

Observe that there are 2^n integers between 2^{n+1} and $\boxed{2^n+1}$ inclusive). Thus

$$\begin{aligned} h_{2^n+1} + h_{2^n+2} + \cdots + h_{2^{n+1}} &= \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &> \underbrace{\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}}_{2^n \text{ times}} \\ &= 2^n \times \frac{1}{2^{n+1}} \\ &= \frac{1}{2} \end{aligned}$$

ii. Hence, or otherwise find the smallest value of n such that $H_n > 3$. (1 mark)

We see that $h_1 = 1$ and $h_2 = \frac{1}{2}$. Then $h_3 + h_4 \geq \frac{1}{2}$, and $h_5 + \cdots + h_8 \geq \frac{1}{2}$ and lastly $h_9 + h_{16} \geq \frac{1}{2}$.

Thus $H_{16} \geq \frac{1}{2}$.

We will then go back from 16 to find the largest value of n such that $H_n < 3$.

This value turns out to be 10, with $H_{10} = 2.93$.

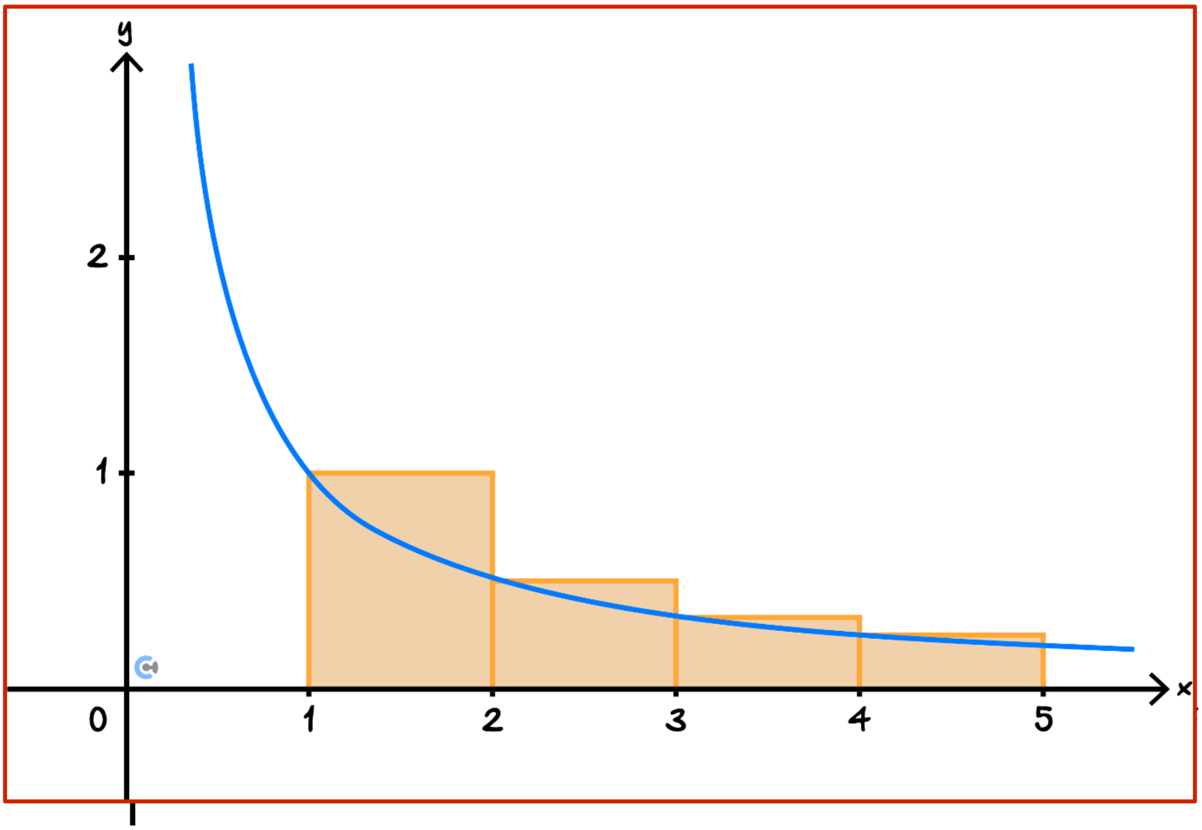
Thus our smallest value of n is 11.

iii. Argue why for all real m there exists some n such that $H_n > m$. (1 mark)

From part **i** we know that $H_{2^n} > \dfrac{1}{2}n$. Thus for any m we know that $H_{2^{2\lceil m \rceil}} > \lceil m \rceil > m$.

c. The area bounded by the graph $y = \frac{1}{x}$, the x-axis, and the lines $x = 1, x = a$ for $a > 1$ is equal to $\log_e(a)$.

The graph of $y = \frac{1}{x}$ is shown below.



Draw a region with an area H_5 and use that region to argue why for all $m \in R$ there exists an n such that $H_n > m$. (4 marks)

From the graph above we see that H_n is greater than the area bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = n + 1$.
This area is equal to $\log_e(n + 1)$.
Hence we see that $H_{\lceil e^m \rceil - 1} > m$.

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