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VCE Mathematical Methods ¾

Discrete Random Variables II [5.2]

Homework Solutions

Admin Info & Homework Outline:

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Section A: Compulsory Questions

Sub-Section [5.2.1]: Basics - Distribution of Discrete Random Variables

Question 1

Could the following be a valid probability mass function? Justify your answer in one clear sentence.

x	1	2	3
$\text{P}(X = x)$	0.3	0.3	0.4

Yes, it is a valid probability mass function because all probabilities are between 0 and 1, and they add up to 1.

$$0.3 + 0.3 + 0.4 = 1.0$$

Question 2

A discrete random variable, X, takes the values shown in the table, along with its probabilities.

x	1	2	3	4	5
$\Pr(X = x)$	$4k$	$4k$	$2k$	k	k

Find:

a. The constant, k.

$$4k + 4k + 2k + k + k = 12k = 1 \Rightarrow k = \frac{1}{12}$$

b. $\Pr(X = 1 | X < 2)$

$$\Pr(X = 1|X < 2) = \frac{\Pr(X = 1)}{\Pr(X < 2)} = \frac{4k}{4k} = 1$$

c. $\Pr(X < 5 | X > 2)$

$$\Pr(X < 5 \mid X > 2) = \frac{\Pr(3 \text{ or } 4)}{\Pr(3 \text{ or } 4 \text{ or } 5)} = \frac{3k}{4k} = \frac{3}{4}$$

d. $\Pr(X > 4 | X > 3)$

$$\Pr(X > 4|X > 3) = \frac{\Pr(X = 5)}{\Pr(X = 4) + \Pr(X = 5)} = \frac{k}{k + k} = \frac{1}{2}$$

Question 3

Given the probability distribution.

X	0	1	2
$\Pr(X = x)$	$\frac{1}{2k}$	$\frac{k}{5}$	$\frac{7}{10k}$

Find the value(s) of k.

$$\text{solve}\left(\frac{1}{2\cdot k}+\frac{k}{5}+\frac{7}{10\cdot k}=1,k\right) \quad k=2 \text{ or } k=3$$

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Sub-Section [5.2.2]: Basics – Mean, Median and Mode

Question 4

Let the random variable X have the following distribution:

x	1	2	3	4	5
$\text{P}(X = x)$	0.1	0.2	0.3	0.2	0.2

a. What is the mode of X?

3

b. What is the median of X?

We find the value where the **cumulative probability** reaches 0.5.

Cumulative probabilities:

- $P(X = 1) = 0.1$
- $P(X \leq 2) = 0.1 + 0.2 = 0.3$
- $P(X \leq 3) = 0.3 + 0.3 = 0.6$

So the cumulative probability crosses 0.5 at **X = 3**

Answer: Median = **3**

c. Calculate the mean of X.

$$\text{Mean} = \sum x \cdot P(X = x)$$

$$= 1(0.1) + 2(0.2) + 3(0.3) + 4(0.2) + 5(0.2)$$
$$= 0.1 + 0.4 + 0.9 + 0.8 + 1.0 = \boxed{3.2}$$

Question 5

State True or False for each of the following, and explain briefly:

a. A data set can have more than one mode.

True – If two or more values have the highest (and equal) frequency, the dataset is multimodal.

b. The median is always one of the values in the data set.

False – In even-sized datasets, the median is often the average of the two middle values, which may not be in the dataset.

c. The mean is unaffected by extreme values (outliers).

False – The mean is sensitive to outliers and can be pulled in their direction.

Question 6

The random variable X has the following probability distribution:

x	-3	0	3
$\Pr(X = x)$	$2(0.1-p)$	$5p$	$p-0.2$

Find the mean of X.

$$\begin{aligned} E(X) &= -3\times 2(0.1-p)+0\times 5p+3\times(p-0.2) \\ &= -0.6+6p+3p-0.6 \\ &= 9p-1.2 \end{aligned}$$

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Sub-Section [5.2.3]: Basics - Measures of Spread

Question 7

Let random variable X represent the number of coding tasks completed by a student in one week. Its probability distribution is as follows:

x	0	1	2	3
$P(X = x)$	a	0.3	0.4	b

- The mean of X is 1.8.

a. Using the total probability rule and the formula for the expected value, form a system of equations to solve for a and b.

$a = \frac{1}{15}$

$b = \frac{7}{30}$

b. Find variance and standard deviation of X.

$\text{Var}(X) = E(X^2) - [E(X)]^2 = 4 - \frac{81}{25} = \frac{100-81}{25} = \frac{19}{25}$

$\text{Standard Deviation} = \sqrt{\frac{19}{25}} = \frac{\sqrt{19}}{5}$

Now, consider another student's performance, represented by random variable Y, with the following distribution:

y	0	1	2	3
$P(Y = y)$	0.1	0.4	0.3	0.2

c. Compare the spread of X and Y.

Which student demonstrates more consistent task completion? Justify using standard deviations.

Step 1: Compute $E(Y)$

$$E(Y) = 0 \cdot \frac{1}{10} + 1 \cdot \frac{4}{10} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{2}{10} = \frac{4}{10} + \frac{6}{10} + \frac{6}{10} = \frac{16}{10} = \frac{8}{5}$$

Step 2: Compute $E(Y^2)$

$$E(Y^2) = 0^2 \cdot \frac{1}{10} + 1^2 \cdot \frac{4}{10} + 2^2 \cdot \frac{3}{10} + 3^2 \cdot \frac{2}{10}$$

$$= \frac{4}{10} + \frac{12}{10} + \frac{18}{10} = \frac{34}{10}$$

$$\text{Var}(Y) = \frac{34}{10} - \left(\frac{8}{5}\right)^2 = \frac{34}{10} - \frac{64}{25} = \frac{85 - 64}{25} = \frac{21}{25}$$

$$\text{Standard Deviation of Y} = \frac{\sqrt{21}}{5}$$

- $\text{SD}(X) = \frac{\sqrt{19}}{5}$
- $\text{SD}(Y) = \frac{\sqrt{21}}{5}$

Since $\sqrt{19} < \sqrt{21}$, **X has less spread**, i.e., **X is more consistent**.

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Sub-Section [5.2.4]: Basics – Linear Transformations of Random Variables

Question 8

If $\text{Var}(X) = 16, \text{E}[X] = 7$, find:

a. $\text{Var}(3X)$

$$\text{Var}(aX) = a^2 \cdot \text{Var}(X) \Rightarrow \text{Var}(3X) = 3^2 \cdot 16 = 9 \cdot 16 = \boxed{144}$$

b. $\text{Var}(2 - X)$

$$\text{Var}(a - X) = \text{Var}(-X) = (-1)^2 \cdot \text{Var}(X) = 1 \cdot 16 = \boxed{16}$$

c. $\text{Sd}(2X)$

$$\text{Sd}(2X) = \sqrt{\text{Var}(2X)} = \sqrt{4 \cdot 16} = \sqrt{64} = \boxed{8}$$

d. $\text{E}(5X - 3)$

$$\mathbb{E}[aX + b] = a \cdot \mathbb{E}[X] + b \Rightarrow \mathbb{E}[5X - 3] = 5 \cdot 7 - 3 = 35 - 3 = \boxed{32}$$

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Sub-Section: Problem Solving

Question 9 Tech-Active.

A company is trialling a new performance-based bonus scheme for its sales interns. The random variable X represents the number of successful product pitches an intern makes in one day (from 0 to 3).

The probability distribution is:

x	0	1	2	3
$p(x)$	0.12	0.36	0.38	0.14

The intern's daily bonus (in points), denoted by S, is calculated using a non-linear rule:

$$S = 3X - X^2 + 2$$

This formula was designed to reward moderate success but discourage aggressive over-selling, which may result in poor customer experience.

a. Calculate the expected value of X.

$$E(X) = \sum x \cdot p(x) = 0(0.12) + 1(0.36) + 2(0.38) + 3(0.14)$$
$$= 0 + 0.36 + 0.76 + 0.42 = \boxed{1.54}$$

b. Calculate the variance and standard deviation of X, correct to 4 decimal places.

First, compute $E(X^2)$:

$$E(X^2) = 0^2(0.12) + 1^2(0.36) + 2^2(0.38) + 3^2(0.14)$$
$$= 0 + 0.36 + 4(0.38) + 9(0.14) = 0.36 + 1.52 + 1.26 = \boxed{3.14}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 3.14 - (1.54)^2 = 3.14 - 2.3716 = \boxed{0.7684}$$

$\text{SD(X)} = \sqrt{0.7684} = 0.8766$

c. Find median and mode of X.

- **Median** is the value where cumulative probability crosses 0.5 → **Median = 2**
- **Mode** is the value with highest probability → **Mode = 2**

d. Use the transformation $S = 3X - X^2 + 2$ to compute the probability distribution of S.

s	2	4
p(s)	0.26	0.74

e. Calculate the expected value $\mathrm{E}(S)$ using $\mathrm{E}(S) = \mathrm{E}[3X - X^2 + 2]$.

$$E(S) = \sum s \cdot p(s) = 2(0.26) + 4(0.74) = 0.52 + 2.96 = \boxed{3.48}$$

Or using:

$$E(S) = E(3X - X^2 + 2) = 3E(X) - E(X^2) + 2 = 3(1.54) - 3.14 + 2 = 4.62 - 3.14 + 2 = \boxed{3.48}$$

f. Calculate the variance and standard deviation of S.

$$E(S^2) = 2^2(0.26) + 4^2(0.74) = 4(0.26) + 16(0.74) = 1.04 + 11.84 = \boxed{12.88}$$

$$\mathrm{Var}(S) = E(S^2) - [E(S)]^2 = 12.88 - (3.48)^2 = 12.88 - 12.1104 = \boxed{0.7696}$$

$$\mathrm{SD(S)}=\sqrt{0.7696} = 0.8773$$

g. Interpret your result: Does this bonus scheme reward interns fairly, or does it disproportionately reward or punish certain performance levels?

- Interns making **1 or 2 sales** get the same bonus (S = 4), despite different efforts.
- Interns making **0 or 3 sales** get the same lower bonus (S = 2), though one put in high effort.
- This **discourages both underperformance and over-aggressiveness.**
- But may feel **unfair** to top performers (3 sales).

The scheme may disproportionately **punish high performers** and **favor moderate ones**.

h. Given that an intern scored at least 1 sale, what is the probability that they earned the maximum bonus?

We want:

$$P(S=4 \mid X \geq 1)$$

From earlier:

- $P(S=4 \wedge X \geq 1) = P(X=1) + P(X=2) = 0.36 + 0.38 = 0.74$
- $P(X \geq 1) = 1 - P(X=0) = 1 - 0.12 = 0.88$

$$P(S=4 \mid X \geq 1) = \frac{0.74}{0.88} = \boxed{\frac{37}{44} \approx 0.84}$$

Question 10

A discrete random variable, X, takes the values shown in the table, along with its probabilities.

x	1	2	3	4
$\Pr(X = x)$	0	$6k^2 - 3$	$2 + 2k$	$2k^2 - 1$

Find:

a. The constant, k.

Use the total probability rule:

$$\Pr(X=1) + \Pr(X=2) + \Pr(X=3) + \Pr(X=4) = 1$$

Substitute values:

$$0 + (6k^2 - 3) + (2 + 2k) + (2k^2 - 1) = 1$$

Combine like terms:

$$(6k^2 + 2k^2) + 2k + (-3 + 2 - 1) = 1 \Rightarrow 8k^2 + 2k - 2 = 1$$

$$8k^2 + 2k - 3 = 0$$

Solve the quadratic:

Use the quadratic formula:

$$k = \frac{-2 \pm \sqrt{2^2 - 4(8)(-3)}}{2 \cdot 8} = \frac{-2 \pm \sqrt{4 + 96}}{16} = \frac{-2 \pm \sqrt{100}}{16} = \frac{-2 \pm 10}{16}$$

$$k = \frac{8}{16} = \frac{1}{2} \quad \text{or} \quad k = \frac{-12}{16} = -\frac{3}{4}$$

Since probabilities must be non-negative, test both:

Try $k = \frac{1}{2}$:

- $\Pr(X=2) = 6(\frac{1}{4}) - 3 = \frac{6}{4} - 3 = \frac{3}{2} - 3 = -\frac{3}{2}$ ❌ (negative)

Try $k = -\frac{3}{4}$:

- $k^2 = \frac{9}{16}$
- $\Pr(X=2) = 6(\frac{9}{16}) - 3 = \frac{54}{16} - 3 = \frac{54-48}{16} = \frac{6}{16} = \frac{3}{8}$
- $\Pr(X=3) = 2 + 2(-\frac{3}{4}) = 2 - \frac{3}{2} = \frac{1}{2}$
- $\Pr(X=4) = 2(\frac{9}{16}) - 1 = \frac{18}{16} - 1 = \frac{2}{16} = \frac{1}{8}$

Add them:

$\frac{3}{8} + \frac{1}{2} + \frac{1}{8} = \frac{3+4+1}{8} = \frac{8}{8} = 1$ ✅

So the valid value is:

✅ a. $\boxed{k = -\frac{3}{4}}$

b. $\text{E}(X)$, the expected value of X.

$$E(X) = \sum x \cdot \Pr(X=x) = 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{8} = \frac{6}{8} + \frac{12}{8} + \frac{4}{8} = \frac{22}{8} = \boxed{\frac{11}{4} = 2.75}$$

c. Var(X), the variance of X.

$$E[X^2] = 8, Var[X] = 8 - \frac{11^2}{4} = \frac{7}{16}$$

Question 11

Let X be a discrete random variable representing the number of successful experiments a student completes in a lab session. The distribution is:

x	0	1	2	3	4
$p(x)$	a	b	c	0.25	0.15

- The distribution is valid (i.e. total probability is 1).
- The expected number of successes is $\mathrm{E}(X) = 2.05$.
- The variance of X is 1.5475.

a. Find the relationship between a, b and c using the total probability rule.

$$a + b + c + 0.25 + 0.15 = 1 \Rightarrow a + b + c = 0.6$$

b. Find the values of a, b and c.

$$\text{solve}\left(\begin{cases} a+b+c=0.6 \\ b+4\cdot c+9\cdot 0.25+16\cdot 0.15-(2.05)^2=1.5475 \\ b+2\cdot c+3\cdot 0.25+4\cdot 0.15=2.05 \end{cases},\{a,b,c\}\right)$$

$$a=0.1 \text{ and } b=0.3 \text{ and } c=0.2$$

Define a new bonus function: $S = 5X - X^2$, representing a score system where extra success starts to penalise accuracy.

c. Compute the new distribution of S.

s	[illegible]	[illegible]	[illegible]	[illegible]	4
$p(s)$	0.1	0.3	0.2	0.25	0.15

s	0	4	6
p(s)	0.1	0.45	0.45

d. Given that a student had at least 2 successes, what is the probability they scored more than 4?

$$\Pr(S > 4 | X \geq 2) = \frac{0.45}{0.6} = 0.75$$

e. Do you think the score function $S = 5X - X^2$, rewards balanced performance or peak performance? Justify your answer.

The function rewards **balanced performance**. It discourages underperformance (0– 1) and penalises high values (like 4) to prevent aggressive overselling.

Sub-Section: The Tech-Free "Final Boss" [VCAA Level]

Question 12

A random variable X has unknown mean $\mu = \mathrm{E}(X)$ and unknown variance $\sigma^2 = \mathrm{Var}(X)$.

Two statistics professors define linear transformations of X as follows:

- Professor $\boldsymbol{A}$ defines $A = 3X + 4$ and tells you:

$$\mathrm{E}(A) = 25, \mathrm{Var}(A) = 36$$

- Professor $\boldsymbol{B}$ defines $B = 2X - 5$.

a. Find $\mathrm{E}(B)$ and $\mathrm{Var}(B)$.

From $A = 3X + 4$:

- $E(A) = 3E(X) + 4 = 25 \Rightarrow 3E(X) = 21 \Rightarrow E(X) = 7$
- $\mathrm{Var}(A) = 9 \cdot \mathrm{Var}(X) = 36 \Rightarrow \mathrm{Var}(X) = 4$

Now use these to find $E(B)$ and $\mathrm{Var}(B)$:

From $B = 2X - 5$:

- $E(B) = 2E(X) - 5 = 2(7) - 5 = 14 - 5 = 9$
- $\mathrm{Var}(B) = 4 \cdot \mathrm{Var}(X) = 4 \cdot 4 = 16$

b. Define a new transformation $Y = X^2 - 2X + 1$.

Without knowing the full distribution of X, find $\mathrm{E}(Y)$.

$$E(X^2) = \mathrm{Var}(X) + [E(X)]^2 = 4 + 7^2 = 4 + 49 = 53$$

$$E(Y) = E(X^2 - 2X + 1) = E(X^2) - 2E(X) + 1 = 53 - 2(7) + 1 = 53 - 14 + 1 = 40$$

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Sub-Section: The Tech-Active "Final Boss" [VCAA Level]

Question 13

Let X be the number of specialty smoothies machine sold by Fresh Blend Café on any given day. The probability distribution for this discrete random variable is as follows:

x	1	2	3	4
$\Pr(X = x)$	0.15	0.45	0.2	0.2

Fresh Blend Café receives a profit of \$275 for every smoothie machine sold. The cost of operating the café is \$190 per day. The profit (in dollars) per day is a function of the random variable X such that:
$Y = 275X - 190$.

a. Set up the probability distribution for the profit, \$$Y$, per day.

y	85	360	635	910
$\Pr(Y = y)$	0.15	0.45	0.2	0.2

b. Find the expected daily profit for Fresh Blend Café.

$$\mathrm{E}(Y) = (85 \times 0.15) + (360 \times 0.45) + (635 \times 0.2) + (910 \times 0.2) = \$483.75$$

c. What is the most likely daily profit?

\$360

d. If the company increases the operating cost from \$190 to \$220, what is the new expected daily profit?

If the operating cost increases to \$220, the new expected profit becomes \$**453.75**.

e. Determine $\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.

$\mathrm{E}(Y^2) = 316540$

$\mathrm{SD}(Y) = \sqrt{316540 - (538)^2}$

$= 164.6086$

$\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma)$

$= \Pr\begin{pmatrix} 538 - 2(164.609) \le Y \\ \le 538 + 2(164.609) \end{pmatrix}$

$= \Pr(208.7827 \le Y \le 867.2173)$

$= \Pr(330 \le Y \le 830)$

$= 0.90$

Let the selling profit per smoothie machine be changed from \$275 to an unknown value pp, keeping cost fixed at \$190. Define profit as $Y_p = pX - 190$.

f. Find the minimum value of p (rounded to the nearest dollar) such that there is less than 10% chance of making a loss.

$pX < 190 \Rightarrow p < \dfrac{190}{X}$

- If $p < 190$, then loss only occurs when $X = 1$ (because 190/1 = 190)
  - Probability of loss = Pr(X=1) = 0.15 → **too high** (> 10%)
- Try $p = 190$:
  - For $X = 1$, $Y_p = 190 \times 1 - 190 = 0$ → **no loss**
  - For all $X \ge 2$, profit is positive
    → So, **no loss occurs**, so $\Pr(Y_p < 0) = 0$

Hence, **minimum value of** p such that there's **<10% chance of making a loss** is:

$\boxed{190}$

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Section B: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 14

A discrete random variable, X, has the following probability distribution:

x	1	2	3	4	5	6	7
$\Pr(X = x)$	t	t	$2t$	$3t$	$2t^2$	$4t^2$	$4t^2 + 2t$

a. Find the value of t.

$\text{solve}(t+t+2\cdot t+3\cdot t+2\cdot t^2+4\cdot t^2+4\cdot t^2+2\cdot t=1,t)$ $\quad t=-1 \text{ or } t=\frac{1}{10}$

$t=-1 \text{ or } t=\frac{1}{10} | t>0$ $\quad t=\frac{1}{10}$

b. Evaluate $\Pr(X \geq 6)$.

$4\cdot t^2+4\cdot t^2+2\cdot t | t=\frac{1}{10}$ $\quad \frac{7}{25}$

c. If $\Pr(X \leq p) > 0.5$, find the minimum value of p.

$t+t+2\cdot t+3\cdot t | t=\frac{1}{10}$ $\quad \frac{7}{10}$

$t+t+2\cdot t | t=\frac{1}{10}$ $\quad \frac{2}{5}$

hence $p=4$

Question 15

The discrete random variable X can take only the values 0, 1, 2, 3, 4, 5. The probability distribution of X is given by the following:

$$\Pr(X = 0) = \Pr(X = 1) = \Pr(X = 2) = a$$
$$\Pr(X = 3) = \Pr(X = 4) = \Pr(X = 5) = b$$
$$\Pr(X \geq 2) = 3\Pr(X < 2)$$

Where a and b are constants.

a. Determine the values of a and b.

$$\text{solve}\left(\begin{cases} 3\cdot a + 3\cdot b = 1 \\ 3\cdot b + a = 2\cdot a\cdot 3 \end{cases}, \{a, b\}\right) \qquad a = \frac{1}{8} \text{ and } b = \frac{5}{24}$$

b. Show that the expectation of X is $\frac{23}{8}$.

$$\text{E}(X) = 3\left(\frac{1}{8}\right) + 12\left(\frac{5}{24}\right)$$
$$= \frac{23}{8}$$

c. Determine the variance of X.

$$\text{E}(X^2) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{1}{8} + 2^2 \times \frac{1}{8}$$
$$+3^2 \times \frac{5}{24} + 4^2 \times \frac{5}{24} + 5^2 \times \frac{5}{24}$$
$$= \frac{265}{24}$$
$$\text{Var}(X) = \frac{256}{24} - \left(\frac{23}{8}\right)^2$$
$$= \frac{533}{192}$$

Question 16

The random variable X represents the number of custom coffee orders that Barista Serra prepares in a day. The probability distribution is as follows:

x	8	9	10	11	12	13
$\Pr(X = x)$	0.10	0.25	0.30	0.20	0.10	0.05

a. Calculate the mean number of custom coffee orders that Serra prepares in a day.

$$\mu = (8\times0.1)+(9\times0.25)+(10\times0.3)+(11\times0.2)+(12\times0.1)+(13\times0.05) = 10.1$$

Serra receives bonuses depending on how many custom coffees she serves each day.

- If Serra prepares fewer than 9 custom orders, she receives no bonus.
- If she serves between 9 and 11 customers (inclusive), she receives a $\$150$ bonus.
- If she serves 12 or more customers, she receives a $\$270$ bonus.

Let Y be the bonus amount Serra receives each day.

b. Construct the probability distribution for Y.

y	0	150	270
$\Pr(Y=y)$	0.1	0.75	0.15

c. Find the expected value for Y.

$$E(Y) = (0 \times 0.1) + (150 \times 0.75) + (270 \times 0.15)$$
$$= 153$$

Question 17

A discrete random variable, Y, has the probability distribution shown:

y	0	1	2	3	4	5
$p(y)$	0.42	0.25	0.14	0.14	0.04	0.01

If $\mu = 1.4$ and $\sigma = 1.2$, show that $\Pr(\mu - 2\sigma \leq Y \leq \mu + 2\sigma) \approx 0.95$.

$$\Pr(\mu - 2\sigma \leq Y \leq \mu + 2\sigma) = \Pr(0 \leq Y \leq 3) = 0.42 + 0.25 + 0.14 + 0.14 = 0.95$$

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Sub-Section: Exam 2 (Tech-Free)

Question 18

Consider the following table:

x	-4	-2	0	2	4
$p(x)$	$2k$	$2k$	$3k$	$3k$	0.2

For the table to represent a probability function, the value of k is:

A. 0.08

B. 0.09

C. 0.1

D. 0.2

E. 0.5

Question 19

x	1	2	3	4	5
$p(x)$	0.05	0.07	0.12	0.35	0.41

For this probability distribution, the median of X is:

A. 1

B. 2

C. 3

D. 4

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Question 20

For this probability distribution, the mode of X is:

A. 1

B. 3

C. 4

D. 5

Question 21

For this probability distribution, the expected value, $\text{E}(X)$, is:

A. 1.95

B. 3.86

C. 4.00

D. 4.50

Question 22

If $\text{E}(X) = 6.5$, then $\text{E}(2X - 1)$ is:

A. 6.5

B. 12

C. 13

D. 25

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Question 23

For a random variable, X, $\text{E}(X) = 1.45$ and $\text{E}(X^2) = 2.60$. The standard deviation of X is closest to:

A. 1.15

B. 1.07

C. 5.31

D. 0.71

Question 24

The random variable X has the following probability distribution:

x	-1	0	1
$\Pr(X = x)$	0.2	m	n

If the mean of X is 0.3, then:

A. $m = 0.5, n = 0.5$

B. $m = 0.2, n = 0.5$

C. $m = 0.3, n = 0.5$

D. $m = 0.5, n = 0.3$

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Question 25

The random variable X has the following probability distribution:

x	-3	0	3
$\Pr(X = x)$	$0.5 - 2p$	p	$3p$

The mean of X is:

A. $6p - 3$

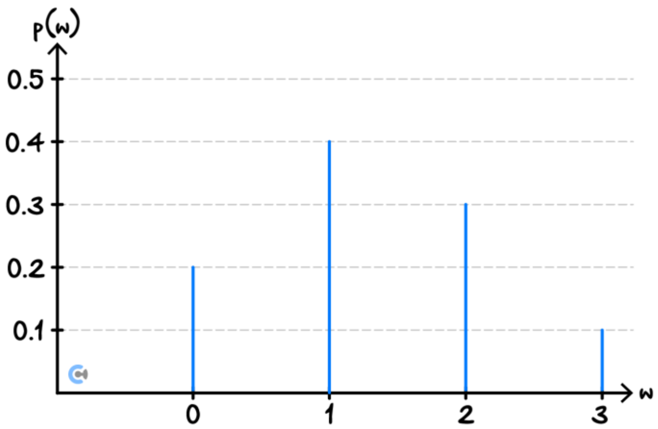
B. $-9p$

C. $15p - 1.5$

D. $1.5 - 9p$

Question 26

The graph of a probability distribution for the random variable w is shown below.



The expected value of w is equal to:

A. 0.25

B. 1

C. 1.3

D. 1.5

E. 2.5

Question 27

The random variable X has the following probability distribution. If the mean of X is 1.2, then the value of a is:

x	0	1	2
$\Pr(X = x)$	a	b	0.5

A. 0

B. 0.2

C. 0.25

D. 0.3

E. 0.5

Question 28

Let X be the number of specialty coffee machines sold by Brew Buzz on any given day. The probability distribution for this discrete random variable is as follows:

x	1	2	3	4
$\Pr(X = x)$	0.1	0.4	0.3	0.2

Brew Buzz makes a profit of $**250** on every coffee machine sold. The cost of operating the kiosk is $**170** per day. The profit (in dollars) per day is a function of the random variable X such that:
$Y = 250X - 170$.

a. Set up the probability distribution for the profit, $$Y$, per day.

Y	80	330	580	830
Pr(Y=y)	0.1	0.4	0.3	0.2

b. Find the expected daily profit for Brew Buzz.

$$\mu = \sum(y \cdot P(Y = y)) = (80)(0.1) + (330)(0.4) + (580)(0.3) + (830)(0.2)$$
$$= 8 + 132 + 174 + 166 = \boxed{480}$$

c. Determine $\Pr(\mu - \sigma \leq y \leq \mu + \sigma)$.

- Mean μ=480
- Standard deviation σ≈229.13

$$\Pr(250.87 \leq Y \leq 709.13) = 0.4 + 0.3 = \boxed{0.7}$$

Question 29

Two teams use different bonus formulas for their interns:

- Team A: $S_A = 3X + 2$
- Team B: $S_B = 5X - X^2$

Both teams have the same distribution for X:

x	0	1	2	3
$\Pr(X = x)$	0.1	0.3	0.4	0.2

a. Determine the mode and median of X. Explain your method clearly.

- **Mode** = value of X with highest probability → $x = 2$ (since 0.4 is max)
- **Median**: The smallest x where cumulative probability ≥ 0.5

$$P(X \leq 0) = 0.1, \quad P(X \leq 1) = 0.4, \quad P(X \leq 2) = 0.8 \Rightarrow \boxed{\text{Median} = 2}$$

b. Compute the mean and standard deviation for team A's scheme without recalculating the distribution.

We use linear transformation rules:

$$S_A = 3X + 2 \Rightarrow \mathbb{E}(S_A) = 3\mathbb{E}(X) + 2, \quad \mathrm{SD}(S_A) = 3 \cdot \mathrm{SD}(X)$$

Let's find $\mathbb{E}(X)$:

$$\mathbb{E}(X) = 0(0.1) + 1(0.3) + 2(0.4) + 3(0.2) = 0 + 0.3 + 0.8 + 0.6 = 1.7$$

$$\mathbb{E}(S_A) = 3(1.7) + 2 = 5.1 + 2 = \boxed{7.1}$$

Find $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$:

$$\mathbb{E}(X^2) = 0^2(0.1) + 1^2(0.3) + 2^2(0.4) + 3^2(0.2) = 0 + 0.3 + 1.6 + 1.8 = 3.7$$

$$\mathrm{Var}(X) = 3.7 - (1.7)^2 = 3.7 - 2.89 = 0.81 \Rightarrow \mathrm{SD}(X) = \sqrt{0.81} = 0.9$$

So:

$$\mathrm{SD}(S_A) = 3 \cdot 0.9 = \boxed{2.7}$$

c. Set up the probability distribution for team B's scheme.

Use $S_B = 5X - X^2$ for each value of X:

X	0	1	2	3
S_B	0	4	6	6

So:

S_B	0	4	6
P	0.1	0.3	0.4 + 0.2 = 0.6

d. Find the variance for team B's scheme.

$$\mathbb{E}(S_B) = 0(0.1) + 4(0.3) + 6(0.6) = 0 + 1.2 + 3.6 = \boxed{4.8}$$

$$\mathbb{E}(S_B^2) = 0^2(0.1) + 4^2(0.3) + 6^2(0.6) = 0 + 4.8 + 21.6 = \boxed{26.4}$$

$$\mathrm{Var}(S_B) = 26.4 - (4.8)^2 = 26.4 - 23.04 = \boxed{3.36}$$

e. Compare the spread (standard deviation) of Team A and Team B's schemes.

Which one has greater variability? What does this say about the consistency of rewards?

$\mathrm{SD}(S_A) = 2.7$

$\mathrm{SD}(S_B) = \sqrt{3.36} \approx 1.83$

Answer:

Team A has greater variability → **less consistent** rewards

Team B is more consistent

f. An intern is considered for promotion if their bonus on a given day is greater than or equal to the median of their team's bonus scheme.

Calculate the median of both bonus schemes and determine which team has more interns qualifying for promotion.

Team A:

$S_A = 3X + 2$	2	5	8	11
P	0.1	0.3	0.4	0.2

Cumulative:

- $P(S_A \leq 5) = 0.4$
- $P(S_A \leq 8) = 0.8$

➡️ **Median is 8** → Interns with $S_A \geq 8 = 0.4 + 0.2 = 0.6$

Team B:

S_B	0	4	6
P	0.1	0.3	0.6

Cumulative:

- $P(S_B \leq 4) = 0.4$
- $P(S_B \leq 6) = 1.0$

➡️ **Median is 6** → Interns with $S_B \geq 6$ = 0.6

✅ **Answer:**

Both teams: 60% of interns qualify for promotion.

g. Team B considers changing their rule to $S'_B = 6X - X^2 - 1$ to improve fairness.

Do you expect the spread to increase or decrease compared to the current S_B? Explain without recalculating the full distribution.

Compare:

- Current $S_B = 5X - X^2$
- New $S'_B = 6X - X^2 - 1$

This is just:

$$S'_B = S_B + X - 1$$

Since you're **adding a linear term** $(X - 1)$, and X is a variable, the **spread will increase**, because adding a variable term increases variance.

Question 30

Let X be a discrete random variable representing the number of tasks a student completes in a collaborative project. The distribution is partially known:

x	0	1	2	3	4
$\Pr(X = x)$	a	b	0.3	0.25	c

You are told:

- The distribution is valid (i.e., total probability $= 1$).
- The expected number of tasks is $\text{E}(X) = 2.3$.
- The variance of X is 1.21.

a. Use the total probability rule to write a relationship between $a,\ b$, and c.

$$a + b + 0.3 + 0.25 + c = 1 \Rightarrow a + b + c = \boxed{0.45}$$

b. Use the expected value and variance information to form two more equations in $a,\ b$, and c. Then, solve for the values of $a,\ b$, and c.

Step 1: Write $\mathbb{E}(X)$:

$$\mathbb{E}(X) = 0a + 1b + 2(0.3) + 3(0.25) + 4c = b + 0.6 + 0.75 + 4c = b + 4c + 1.35$$

Set this equal to 2.3:

$$b + 4c + 1.35 = 2.3 \Rightarrow \boxed{b + 4c = 0.95} \quad (1)$$

Step 2: Write $\mathbb{E}(X^2)$:

$$\mathbb{E}(X^2) = 0^2a + 1^2b + 2^2(0.3) + 3^2(0.25) + 4^2c = b + 1.2 + 2.25 + 16c = b + 16c + 3.45$$

Then use $\text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$:

$$1.21 = b + 16c + 3.45 - (2.3)^2 = b + 16c + 3.45 - 5.29 = b + 16c - 1.84 \Rightarrow \boxed{b + 16c = 3.05} \quad (2)$$

Step 3: Solve the system of 3 equations

We now have:

1. $a + b + c = 0.45$
2. $b + 4c = 0.95$
3. $b + 16c = 3.05$

$a = 0.025$
$b = 0.25$
$c = 0.175$

Define a scoring function:

$$S = 6X - X^2 + 1$$

This rule aims to reward productivity but control for overcommitment.

c. Compute the probability distribution of S using the solved values of $a,\ b$, and c.

x	0	1	2	3	4
s	1	6	9	10	9
Pr(S=s)	0.025	0.25	0.3	0.25	0.175

A bonus of at least **8** is required for a student to earn a research opportunity.

d. Find the probability they earn it, given that they completed at least 2 projects.

$$\Pr(S \geq 8 | X \geq 2) = 1$$

e. Explain how this relates to the function's goal of "rewarding productivity but controlling for overcommitment."

The function achieves its goal by increasing the score for completing more tasks, but penalizing very high task counts (x=4) to discourage overcommitment. This balances productivity with sustainability.

Space for Personal Notes

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