CONTOUREDUCTION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Discrete Random Variables I [5.1]
Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Compulsory Questions	Pg 2-Pg 22
Supplementary Questions	Pg 23-Pg 40

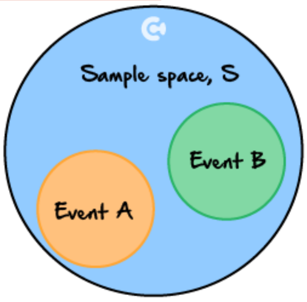
Section A: Compulsory Questions

Sub-Section: Recap

Cheat Sheet

[5.1.1] – Basics of Probability

- **Probability**
  - In probability, we are aiming to quantify uncertainty.



  - Sample space (ε) = set of all possible outcomes.
  - Probability of an event = proportion of the sample space that the event takes up.
  - The probability of an event must **always** take a value between 0 and 1 inclusive.
- **Sample Space (ε)**
  - The set of all possible outcomes in an experiment.
  - For tossing two coins in a row, the sample space is:

$$\varepsilon = \{HH, HT, TH, TT\}$$

  - For rolling a standard 6-sided dice, the sample space is:

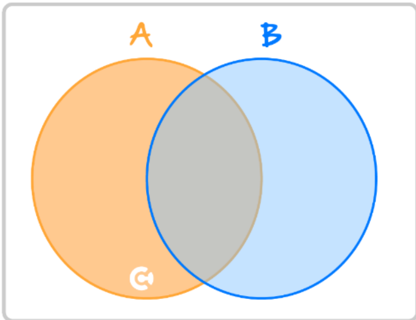
$$\varepsilon = \{1, 2, 3, 4, 5, 6\}$$

  - Total probability adds up to 1.

- **Calculating Probabilities for Equally Likely Outcomes**
  - When there is some number of equally likely outcomes, the probability of a "successful" outcome can be calculated as:

$$\Pr(success) = \frac{number\ of\ successful\ outcomes}{number\ of\ total\ outcomes}$$

- **Union, Intersection, and Complement**



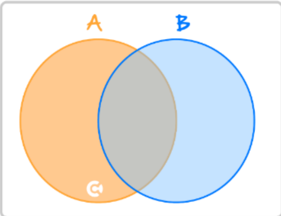
  - $A \cup B$ =
    - Union of two events aka "or".
    - Equivalent to either event A **OR** event B **OR** BOTH occurring.
  - $A \cap B$ =
    - Intersection of two events aka "and".
    - Equivalent to both event A **AND** event B occurring.
  - A' =
    - A complement aka "not".
    - e.g., if A = dice rolled a 6, then A' = dice rolled anything except a 6.

$$\Pr(A') = 1 - \Pr(A)$$

Cheat Sheet

[5.1.2] - Use Venn diagrams and/or Karnaugh Maps and Apply the Addition Rule

Venn Diagram



- Venn diagrams are useful to visualise the two events.

Karnaugh Tables

- We can also represent probability problems using a Karnaugh map.

	B	B'	
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
	$\Pr(B)$	$\Pr(B')$	1

  - The rows and columns add up to the last cell value.

- Remember the **total** probability must always add to Solution Pending

The Addition Rule

- When we add the probabilities of A and B, we count the outcomes contained in $A \cap B$ **twice.**
- So, we must subtract one of them to get the probability of $A \cup B$.

$$\Pr(A \cup B) = \boxed{\Pr(A) + \Pr(B) - \Pr(A \cap B)}$$

[5.1.3] - Mutually Exclusive and Independent Events

Mutually Exclusive Events

$$\Pr(A \cap B) = \boxed{0}$$

- **Mutually Exclusive Events:** Events that cannot occur simultaneously.

Independent Events

- When A and B are independent:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

- **Independent Events:** Do not affect the likelihood of each other.

Cheat Sheet

[5.1.4] - Conditional Probability and Tree Diagrams.

Conditional Probability

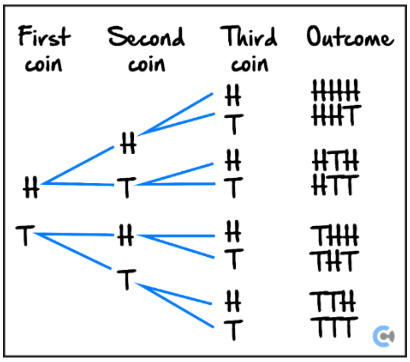
$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

"Likelihood of A given that B has happened"

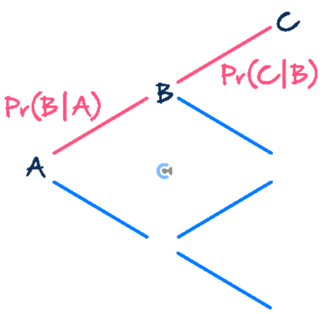
- **Conditional Probability**: Calculating the probability of an event given that another event has occurred.

Tree diagrams

- Useful for multiple sequence events.
- To calculate the probability of a sequence, we multiply **the probabilities along the relevant branches.**
- For instance, the following tree diagram shows the outcomes of three successive coin tosses.



Tree Diagram for Condition Probability



- Tree diagram is perfect for conditional probability as each branch is conditional probability.

$$\textit{Each branch} = \Pr(\text{Leaf}|\text{Root})$$

Conditional Probability with Independent Events

$$\Pr(A|B) = \Pr(A)$$

- If A and B are independent, the given condition does **not** affect the probability of the event.

Sub-Section [5.1.1]: Basics of Probability

Question 1

a. List the **sample space** for the **sum of the two dice**.

$\{2,3,4,5,6,7,8,9,10,11,12\}$

b. How many outcomes are there in the sample space? (1 mark)

There are **11 outcomes** in the sample space for the **sum of the dice**.

c. Are all outcomes **equally likely**? Briefly justify your answer.

No, the outcomes are **not equally likely**.
Example:
- There is **only 1 way** to get a sum of 2: (1,1).
- There are **6 ways** to get a sum of 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1).

So, different sums have **different numbers of combinations**, and are **not equally likely**.

d. Let event A be "the sum is greater than 9", and event B be "the sum is an even number". Write the outcomes for $A, B, A \cap B$, and $A \cup B$.

- $A = \{10,11,12\}$
- $B = \{2,4,6,8,10,12\}$
- $A \cap B = \{10,12\}$
- $A \cup B = \{2,4,6,8,10,11,12\}$

e. Find the **complement** of event A, and explain what it means in context.

- $A' = \{2,3,4,5,6,7,8,9\}$

Context: The complement of A represents the event that the sum is 9 or less.

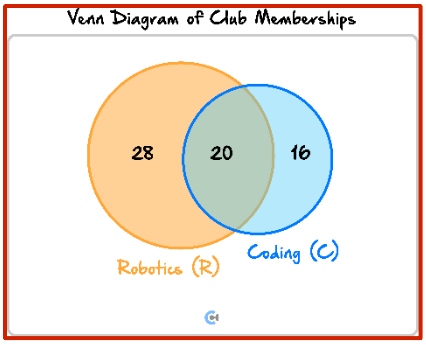
Sub-Section [5.1.2]: Use Venn diagrams and/or Karnaugh Maps and Apply the Addition Rule

Question 2

In a group of 80 students surveyed about their extracurricular activities:

- 48 are in the **Robotics Club** (R).
- 36 are in the **Coding Club** (C).
- 20 are in **both** clubs.

a. Draw a Venn diagram representing this information.



b. How many students are in **only the Robotics Club**?

Only Robotics $= 48 - 20 = 28$ students

c. How many students are in **at least one** of the two clubs?

$28 + 16 + 20 = 64$ students

d. Use the **Addition Rule** to verify your answer in **part c.**

(***Reminder***: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$)

$\mathrm{P}(R \cup C) = \mathrm{P}(R) + \mathrm{P}(C) - \mathrm{P}(R \cap C)$
$\mathrm{P}(R \cup C) = 48 + 36 - 20 = 64$

e. Construct a **Karnaugh table** to find the number of students in each category:

- Robotics Club vs Not in Robotics Club.
- Coding Club vs Not in Coding Club.

	In Coding Club	Not in Coding Club
In Robotics Club	20	16
Not in Robotics Club	28	16

Space for Personal Notes

Sub-Section [5.1.3]: Mutually Exclusive and Independent Events

Question 3

- The probability that **Student A** passes a quiz is $\mathrm{P}(A) = 0.6$.
- The probability that **Student B** passes the quiz is $\mathrm{P}(B) = 0.4$.

Assume events A and B are Mutually Exclusive.

a. Find $\mathrm{P}(A \cap B)$.

$\mathrm{P}(A \cap B) = 0$
(Mutually exclusive events cannot happen together.)

b. Find $\mathrm{P}(A \cup B)$.

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) = 0.6 + 0.4 = 1$

c. Find P(neither A nor B).

P(neither A nor B) $= 1 - \mathrm{P}(A \cup B) = 1 - 1 = 0$

Assume events A and B are Independent.

d. Find $\mathrm{P}(A \cap B)$.

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) = 0.6 \times 0.4 = 0.24$

e. Find $\mathrm{P}(A \cup B)$.

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.6 + 0.4 - 0.24 = 0.76$

f. Find P(only A passes).

$\boxed{\mathrm{P}(\text{only } A) = \mathrm{P}(A \cap B') = \mathrm{P}(A) \times (1 - \mathrm{P}(B)) = 0.6 \times 0.6 = 0.36}$

Space for Personal Notes

Sub-Section [5.1.4]: Conditional Probability and Tree Diagrams

Question 4

A medical test is used to detect a rare virus. In a particular population:

- 2% of people actually have the virus → $\mathrm{P}(V) = 0.02$.
- If a person **has** the virus, the test correctly returns **positive** 95% of the time → $\mathrm{P}(T^+ \mid V) = 0.95$.
- If a person **does not have** the virus, the test still returns **false positive** 5% of the time → $\mathrm{P}(T^+ \mid V') = 0.05$.

a. Find the probability that a randomly selected person **tests positive**, $\mathrm{P}(T^+)$.

$$P(T^+) = P(V) \cdot P(T^+ \mid V) + P(V') \cdot P(T^+ \mid V')$$
$$= (0.02)(0.95) + (0.98)(0.05) = 0.019 + 0.049 = \boxed{0.068}$$

b. Use your result from **part a.** to find the probability that a person **actually has the virus given they tested positive**, $\mathrm{P}(V \mid T^+)$, correct to 4 decimal places.

$$P(V \mid T^+) = \frac{P(V \cap T^+)}{P(T^+)} = \frac{(0.02)(0.95)}{0.068} = \frac{0.019}{0.068} \approx \boxed{0.2794}$$

c. Find the probability that a randomly selected person **tests negative**, $\mathrm{P}(T^-)$.

$$P(T^-) = 1 - P(T^+) = 1 - 0.068 = \boxed{0.932}$$

d. Are the events “havin[illegible]er.

Check:

If independent, then $\mathrm{P}(V \cap T^+) = \mathrm{P}(V) \cdot \mathrm{P}(T^+)$

From earlier:

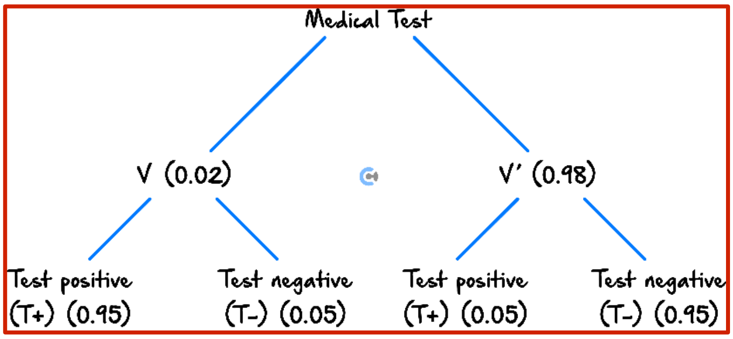
- $\mathrm{P}(V \cap T^+) = (0.02)(0.95) = 0.019$
- $\mathrm{P}(V) \cdot \mathrm{P}(T^+) = (0.02)(0.068) = 0.00136$
- Since $0.019 \neq 0.00136$, they are **not independent**.

e. Draw a tree diagram representing the full situation:

First branch: Virus status (V / V').

Second branch: Test result (T^+/T^-).

Label all branches with correct probabilities.



f. Use the diagram to calculate $\mathrm{P}(V' \cap T^+)$.

$\mathrm{P}(V' \cap T^+) = (0.98)(0.05) = 0.049$

Space for Personal Notes

Sub-Section: Problem Solving

Question 5

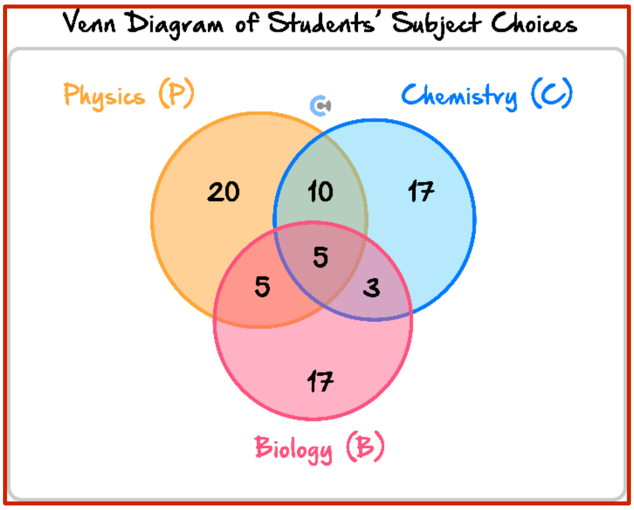
At a school science fair, a survey was conducted among 100 students about their project topics. Each student could choose to present on any combination of the following:

- **Physics** (P)
- **Chemistry** (C)
- **Biology** (B)

The results showed:

- 40 students chose **Physics.**
- 35 students chose **Chemistry.**
- 30 students chose **Biology.**
- 15 chose **both Physics and Chemistry.**
- 10 chose **both Physics and Biology.**
- 8 chose **both Chemistry and Biology.**
- 5 chose **all three subjects.**

a. Draw a Venn diagram to represent this information. Label each region clearly with the number of students.



b. How many students chose **only Physics**?

20 students

c. How many students did **not choose any of the three subjects**?

$20 + 17 + 17 + 10 + 5 + 3 + 5 = 77$
$100 - 77 = 23$ students

d. What is the probability that a randomly selected student chose **at least one** of the three subjects?

$\text{P(atleast one)} = \frac{77}{100} = 0.77$

Space for Personal Notes

Question 6

In a card game, a player draws a card from one of **two decks**, Deck A and Deck B.

- The player chooses Deck A with probability $\boldsymbol{p}$, and Deck B with probability $\mathbf{1-\boldsymbol{p}}$.
- In Deck A, the probability of drawing a red card is $\boldsymbol{r}$.
- In Deck B, the probability of drawing a red card is $\boldsymbol{s}$.

The player draws one card (without revealing the deck) and it is **red**.

a. In terms of p, r, s, what is the probability that the card came from Deck A, given that it was red?

$$P(A \mid \text{Red}) = \frac{P(A \cap \text{Red})}{P(\text{Red})} = \frac{P(A) \cdot P(\text{Red} \mid A)}{P(A) \cdot P(\text{Red} \mid A) + P(B) \cdot P(\text{Red} \mid B)}$$

$$P(A \mid \text{Red}) = \frac{pr}{pr + (1-p)s}$$

b. Explain what would happen to this probability if $r = s$. Interpret your result in words.

If $r = s$, then:

$$P(A \mid \text{Red}) = \frac{pr}{pr + (1-p)r} = \frac{pr}{r[p + (1-p)]} = \frac{pr}{r} = \boxed{p}$$

If both decks are equally likely to produce a red card, then observing a red card gives no extra information about which deck was chosen.

Space for Personal Notes

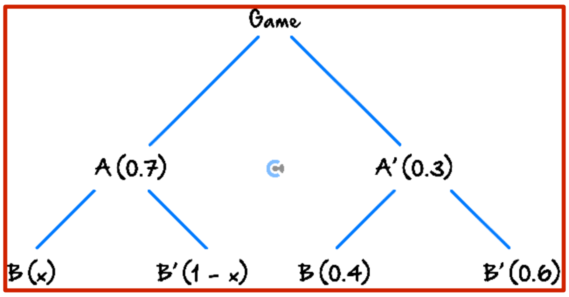
Question 7

In a certain game, a player rolls a weighted die and may then draw a card depending on the result.

- With probability **0.7**, the player rolls a **number greater than 3** (Event A).
- The player then draws a **winning card (Event B)** based on the result of the die:
  - If the number was **greater than 3**, the probability of drawing a winning card is **unknown**, say x.
  - If the number was **3 or less**, the probability of drawing a winning card is **0.4**.

It is known that the overall probability that the player **draws a winning card** is **0.61**.

a. Draw a tree diagram showing the sequence of events and known probabilities.



b. Use the diagram to form an equation and solve for x, the probability of drawing a winning card given the number rolled is greater than 3.

$$P(B) = P(A) \cdot P(B \mid A) + P(A') \cdot P(B \mid A')$$
$$0.61 = (0.7)(x) + (0.3)(0.4)$$
$$0.61 = 0.7x + 0.12$$
$$0.49 = 0.7x$$
$$x = \frac{0.49}{0.7} = \boxed{0.7}$$

c. Interpret your answer: Is it likely the card draw is fair for players who roll higher numbers?

Since $x = 0.7$, the probability of drawing a winning card is **higher** when the number is greater than 3.
Interpretation:
Yes, the card draw is **favourable** to players who roll higher numbers. It is **not fair**, because players who roll higher than 3 have a **better chance** of drawing a winning card (0.7 vs 0.4).

Space for Personal Notes

Question 8 Tech-Active.

A factory produces a large number of batteries. A test is used to detect whether a battery is faulty. The following probabilities are known:

- 3% of batteries are faulty.
- If a battery is faulty, the test correctly identifies it as faulty 92% of the time.
- If a battery is not faulty, the test still incorrectly identifies it as faulty 7% of the time.

Let $\boldsymbol{T^+}$ be the event that a battery **tests positive** for being faulty.

a. Find the probability that a randomly selected battery tests positive, $\mathrm{P}(T^+)$.

$$\begin{aligned} P(T^+) &= P(F)\cdot P(T^+|F) + P(F')\cdot P(T^+|F') \\ &= (0.03)(0.92) + (0.97)(0.07) \\ &= 0.0276 + 0.0679 \\ &= \boxed{0.0955} \end{aligned}$$

b. Find the probability that a battery is actually faulty given that it tested positive, $\mathrm{P}(F\ |T^+)$, correct to 4 decimal places.

$$P(F|T^+) = \frac{P(F\cap T^+)}{P(T^+)} = \frac{(0.03)(0.92)}{0.0955} = \frac{0.0276}{0.0955} \approx \boxed{0.2889}$$

c. Are the events "being faulty" and "testing positive" independent? Justify your answer mathematically.

Check whether:

$$P(F\cap T^+) \stackrel{?}{=} P(F)\cdot P(T^+)$$

$$P(F\cap T^+) = (0.03)(0.92) = 0.0276$$

$$P(F)\cdot P(T^+) = (0.03)(0.0955) = 0.002865$$

Since **0.0276 ≠ 0.002865**, they are **not independent**.

d. Draw a Karnaugh table showing the joint probabilities of being faulty (F), not faulty (F'), testing positive (T^+), and testing negative (T^-). Fill in all 4 probabilities in the table.

	T+	T−
F	0.0276	0.0024
F'	0.0679	0.9021

Space for Personal Notes

Question 9

A **healthcare provider** is reviewing the performance of its **annual health check-up package for patients**. It is known that at any given yearly check-up:

- The probability that a patient requires a **cholesterol test** is $7/10$,
- The probability that a patient requires a **blood pressure check** is $3/10$,
- The probability that a patient requires **both** is $1/10$.

a. State the probability that at any given yearly check-up, a patient will require a **blood pressure check without** a **cholesterol test**.

$\boxed{0.2}$

b. ~~The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^+$.~~

The healthcare provider is planning a new **check-up model**. The production goals are:

- Probability of requiring a cholesterol test $= \frac{m}{m+n}$,
- Probability of requiring a blood pressure check $= \frac{n}{m+n}$,
- Probability of requiring both $= \frac{1}{m+n}$,

Where $m, n \in Z^+$.

Determine m in terms of n, if the probability that a patient will require a blood pressure check without a cholesterol test is $\mathbf{0.05}$ at any given yearly check-up.

$$\boxed{\frac{n-1}{m+n} = 0.05}$$

$$\boxed{\begin{aligned} n - 1 &= 0.05(m+n) \\ m &= 19n - 20 \end{aligned}}$$

Question 10

To win a **mini basketball arcade game**, Emma must shoot the ball into the hoop. Unfortunately, Emma is not very confident in her shooting. If Emma misses, she can try again. The **probability of scoring on any given shot is** $\frac{1}{5}$. Assume that the outcome of each shot is **independent** of the others. A **maximum of five attempts** can be made.

a. What is the probability that Emma does **not** make a successful shot in any of her five attempts?

$$P(\text{no success}) = \left(\frac{4}{5}\right)^5 = \boxed{\frac{1024}{3125}}$$

b. Calculate the probability that Emma makes **at least one successful shot** out of the five attempts. Express your answer in the zform $\frac{a}{b}$, where a and b are positive integers.

$$P(\text{at least one success}) = 1 - \left(\frac{4}{5}\right)^5 = 1 - \frac{1024}{3125} = \frac{3125 - 1024}{3125} = \boxed{\frac{2101}{3125}}$$

c. Calculate the probability that Emma makes her **first successful shot on the fourth or fifth attempt**. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers.

$$P(\text{success on 4th}) = \left(\frac{4}{5}\right)^3 \cdot \frac{1}{5} = \frac{64}{125} \cdot \frac{1}{5} = \frac{64}{625}$$

$$P(\text{success on 5th}) = \left(\frac{4}{5}\right)^4 \cdot \frac{1}{5} = \frac{256}{625} \cdot \frac{1}{5} = \frac{256}{3125}$$

$$\frac{64}{625} + \frac{256}{3125} = \frac{320}{3125} + \frac{256}{3125} = \boxed{\frac{576}{3125}}$$

Space for Personal Notes

Sub-Section: The Tech-Free "Final Boss" [VCAA Level]

Question 11

For events A and B from a sample space, $\Pr(A \mid B) = \frac{1}{7}$ and $\Pr(B \mid A) = \frac{1}{6}$. Let $\Pr(A \cap B) = p$.

a. Find $\Pr(A)$ in terms of p.

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{p}{\Pr(A)} \Rightarrow \Pr(A) = \frac{p}{\Pr(B \mid A)} = \frac{p}{1/6} = \boxed{6p}$$

b. Find $\Pr(A \cap B')$ in terms of p.

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B') = p + \Pr(A \cap B') \Rightarrow \Pr(A \cap B') = \Pr(A) - p = 6p - p = \boxed{5p}$$

c. Given that $\Pr(A \cup B) \leq \frac{1}{7}$, state the largest possible value for p.

We use:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

We already know:

- $\Pr(A) = 6p$
- Use $\Pr(A \mid B) = \frac{1}{7} = \frac{p}{\Pr(B)} \Rightarrow \Pr(B) = \frac{p}{1/7} = 7p$

So:

$$\Pr(A \cup B) = 6p + 7p - p = 12p$$

Given:

$$12p \leq \frac{1}{7} \Rightarrow p \leq \frac{1}{84}$$

So, largest possible value for p is $\frac{1}{84}$.

Space for Personal Notes

Sub-Section: The Tech-Active "Final Boss" [VCAA Level]

Question 12

A school is reviewing the punctuality of its students arriving to class. The school claims that **each student has a 90% chance of arriving on time** to any given class. Assume that all students' arrival times are independent.

a. On a day when 3 students are expected in class, what is the probability that **all 3 students arrive on time**?

$$\boxed{P(\text{all 3 on time}) = 0.9 \times 0.9 \times 0.9 = 0.9^3 = 0.729}$$

b. On another day, there are n **students** scheduled for class.

i. Express, in terms of n, the probability that **at least one student does not arrive on time**.

$$\boxed{P(\text{at least one late}) = 1 - P(\text{all on time}) = 1 - 0.9^n}$$

ii. Hence, or otherwise, find the **minimum value of** n such that there is at least a **0.95 probability** that **one or more students do not arrive on time**.

$\text{solve}\left((0.9)^x \le 0.05, x\right)$ $\quad x \ge 28.4332$

$\boxed{n = 29}$

c. A school analyst believes that the 90% punctuality claim is only accurate for students attending **morning classes**. For students attending **afternoon classes**, the analyst believes the probability of arriving on time is x, where $0.4 \leq x \leq 0.8$.
After examining a large sample of students, the analyst estimates that the **overall probability of a student arriving on time is 0.88**.
Let the probability that a student attends an afternoon class be y.
Find the minimum and maximum values of y, assuming the analyst's belief is correct.

We're solving:

$$\text{Overall probability} = (1-y)(0.9) + yx = 0.88$$

Minimum value of y (use best-case for afternoon students → $x = 0.8$)

$$0.9(1-y) + 0.8y = 0.88 \Rightarrow 0.9 - 0.9y + 0.8y = 0.88 \Rightarrow 0.9 - 0.1y = 0.88 \Rightarrow y = 0.2$$

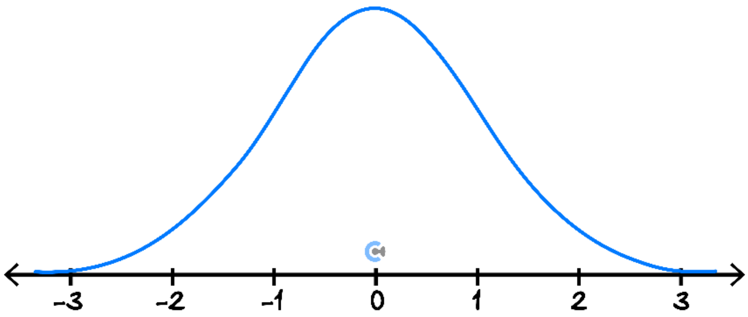
Maximum value of y (use worst-case for afternoon students → $x = 0.4$)

$$0.9 - 0.9y + 0.4y = 0.88 \Rightarrow 0.9 - 0.5y = 0.88 \Rightarrow y = 0.04$$

So:

$$\boxed{0.04 \leq y \leq 0.2}$$

d. The school used to ring the morning bell at **9: 00 am**. A past study showed that **60% of students arrived between 8: 57 am and 9: 02 am**.
The school believes that student arrival times follow a **symmetrical pattern**, like the one shown below, where the **highest point** of the curve represents 9: 00 **am**.



The principal wants to ring a second bell where 60% of the students arrive 3.5 minutes before to 1.5 minutes after the bell.
Find the possible time(s) the new bell might be, give both answers to the **nearest half minute**.

By property of symmetry, there should also be **60% of students arrived between 8: 58 am and 9: 03 am**.
So, the second bell should ring at 8: 58.5 am and 8: 59.5 am.

Section B: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 13

For events A and B from a sample space, $\Pr(A \mid B) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{4}$.

a. Calculate $\Pr(A \cap B)$.

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\frac{3}{5} = \frac{\Pr(A \cap B)}{1/4} \Rightarrow \Pr(A \cap B) = \frac{3}{5} \times \frac{1}{4} = \frac{3}{20}$$

b. Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A.

$$\Pr(A' \cap B) = \Pr(B) - \Pr(A \cap B) = \frac{1}{4} - \frac{3}{20} = \frac{5}{20} - \frac{3}{20} = \frac{2}{20} = \frac{1}{10}$$

c. If events A and B are independent, calculate $\Pr(A \cup B)$.

If A and B are independent:

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

Let's find $\Pr(A)$ using:

$$\Pr(A \cap B) = \frac{3}{20}, \quad \Pr(B) = \frac{1}{4} \Rightarrow \Pr(A) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{3/20}{1/4} = \frac{3}{5}$$

Now compute $\Pr(A \cup B)$:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{3}{5} + \frac{1}{4} - \frac{3}{20}$$

Convert all to a common denominator (20):

$$\frac{3}{5} = \frac{12}{20}, \quad \frac{1}{4} = \frac{5}{20}, \quad \frac{3}{20} \Rightarrow \Pr(A \cup B) = \frac{12 + 5 - 3}{20} = \frac{14}{20} = \frac{7}{10}$$

Space for Personal N

Question 14

Alex has **four coins** in her bag:

- Two of them are **unbiased**, where $\Pr(\text{head}) = \frac{1}{2}$.
- One is **biased towards tails**, where $\Pr(\text{head}) = \frac{1}{4}$.
- One is **very biased towards heads**, where $\Pr(\text{head}) = \frac{3}{4}$.

She picks one coin at **random**, then tosses it **once**.

a. What is the **overall probability** that Alex tosses a head?

(**Hint:** Use a tree diagram or a table if it helps.)

Coins:

- 2 unbiased: Pr(head) = ½
- 1 biased towards tails: Pr(head) = ¼
- 1 very biased towards heads: Pr(head) = ¾

Each coin is equally likely to be picked:
→ Pr(pick any coin) = ¼

Now compute the total probability of getting a head:

$$\Pr(\text{head}) = 2 \times \left(\frac{1}{4} \times \frac{1}{2}\right) + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4}$$

$$= \frac{2}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{1}{4} + \frac{1}{16} + \frac{3}{16} = \frac{4}{16} + \frac{1}{16} + \frac{3}{16} = \frac{8}{16} = \boxed{\frac{1}{2}}$$

b. Given that she tossed a head, what is the **probability that the coin she picked was one of the unbiased coins**?

(**Hint:** Think about all the ways a head could have happened.)

Let:

- U: chose an unbiased coin
- H: tossed a head

We want:

$$\Pr(U \mid H) = \frac{\Pr(H \mid U) \cdot \Pr(U)}{\Pr(H)}$$

We already have:

- $\Pr(H) = \frac{1}{2}$
- $\Pr(U) = \frac{2}{4} = \frac{1}{2}$
- $\Pr(H \mid U) = \frac{1}{2}$

Then:

$$\Pr(U \mid H) = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{1}{4} \div \frac{1}{2} = \boxed{\frac{1}{2}}$$

c. Suppose Alex repeats this experiment a second time with a new random coin draw.
Are the events:

- "She gets a head on the first toss" and,
- "She gets a head on the second toss",

independent? Justify your answer.

Yes, the two events are **independent**, because:

- A new random coin is drawn for each toss.
- The outcome of the first toss does not influence the outcome of the second toss.
- The probability of getting a head on either toss remains the same and is unaffected by what happened in the other toss.

Space for Personal Notes

Question 15

A tech company runs two types of recruitment ads:

- **Ad Type A** is shown with probability x,
- **Ad Type B** is shown with probability $1 - x$,
  where $0 < x < 1$.

If shown **Ad Type A**, a candidate applies with probability $0.6 + 0.1x$.

If shown **Ad Type B**, a candidate applies with probability $0.9 - 0.2x$.

Let P(Apply) be the overall probability that a candidate applies after seeing an ad.

a. Write an expression for P(Apply) in terms of x.

Let's expand:

$$P(x) = x(0.6 + 0.1x) + (1 - x)(0.9 - 0.2x)$$

First term:

$$x(0.6 + 0.1x) = 0.6x + 0.1x^2$$

Second term:

$$(1 - x)(0.9 - 0.2x) = 0.9 - 0.9x - 0.2x + 0.2x^2 = 0.9 - 1.1x + 0.2x^2$$

Now add:

$$P(x) = (0.6x + 0.1x^2) + (0.9 - 1.1x + 0.2x^2) = 0.9 - 0.5x + 0.3x^2$$

b. Use calculus to find the value of x that **minimises** the overall probability of a candidate applying.

This is a **quadratic** of the form $ax^2 + bx + c$, with:

- $a = 0.3$
- $b = -0.5$
- $c = 0.9$

Minimum occurs at:

$$x = \frac{-b}{2a} = \frac{-(-0.5)}{2 \cdot 0.3} = \frac{0.5}{0.6} = \frac{5}{6}$$

c. What is the **maximum** value of P(Apply), and when does it occur?

Since this is a **concave upward parabola** (positive coefficient on x^2), the maximum occurs at **endpoints**.

Evaluate at:

- $x = 0$:

$$P(0) = 0.3(0)^2 - 0.5(0) + 0.9 = \boxed{0.9}$$

- $x = 1$:

$$P(1) = 0.3(1)^2 - 0.5(1) + 0.9 = 0.3 - 0.5 + 0.9 = \boxed{0.7}$$

✅ **Answer (c):**

- Maximum value is $\boxed{0.9}$, occurs at $\boxed{x = 0}$

Space for Personal Notes

Question 16

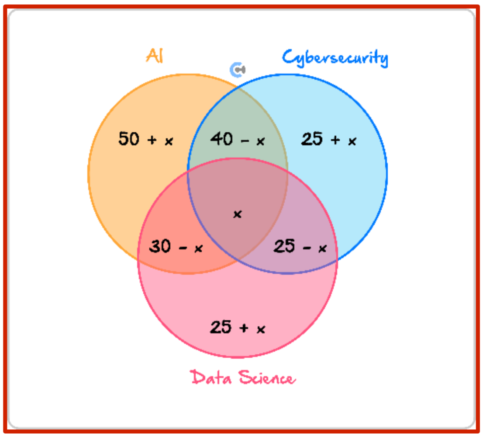
At a tech conference, 230 attendees were surveyed about the workshops they attended:

- **120 attended the AI workshop (A).**
- **90 attended the Cybersecurity workshop (C).**
- **80 attended the Data Science workshop (D).**
- **40 attended both AI and Cybersecurity.**
- **30 attended both AI and Data Science.**
- **25 attended both Cybersecurity and Data Science.**
- **x attendees attended all three workshops.**

It is also known that **5 people did not attend any of the workshops**.

a. Draw a Venn diagram with three overlapping circles. Label all regions in terms of x.

(Start from the intersection of all three events.)



b. Express the total number of attendees who attended **at least one** workshop in terms of x.

$$(50+x)+(25+x)+(25+x)+(40-x)+(30-x)+(25-x)+x=195+x$$

c. Use this information to form and solve an equation to find the value of x.

Given:

- Total surveyed = 230
- 25 did **not attend** any → 205 attended **at least one**

Set:

$$195 + x = 205 \Rightarrow x = \boxed{10}$$

d. Given that an attendee went to both the Cybersecurity and Data Science workshops, what is the probability that they also attended the AI workshop?

$$P(\text{AI} \mid \text{Cybersecurity} \cap \text{Data Science}) = \frac{P(\text{AI} \cap \text{Cybersecurity} \cap \text{Data Science})}{P(\text{Cybersecurity} \cap \text{Data Science})}$$

$$\frac{10}{25}$$

Space for Personal Notes

Sub-Section: Exam 2 (Tech-Active)

Question 17

A box contains a marbles that are identical in every way except colour, of which b marbles are coloured red and the remainder of the marbles are coloured white. Two marbles are drawn randomly from the box.

If the first marble is not replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is:

A. $\frac{b^2+(a-b)^2}{a^2}$

B. $\frac{a^2+a(2b+1)+2b^2}{a(a-1)}$

C. $\frac{2b(a-b-1)}{a(a-1)}$

D. $\frac{a^2-a(2b+1)+2b^2}{a(a-1)}$

Question 18

Two events, A and B, are independent, where $\Pr(B) = 2\Pr(A)$ and $\Pr(A \cup B) = 0.48$. $\Pr(A)$ is approximately equal to:

A. 0.1

B. 0.2

C. 0.3

D. 0.4

E. 0.5

Space for Personal Notes

Question 19

In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles.

Each white marble scores -5 points and each red marble scores $+4$ points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score.

What is the probability that the final score will equal -1?

A. $\frac{2}{3}$

B. $\frac{1}{5}$

C. $\frac{2}{5}$

D. $\frac{8}{15}$

Question 20

A box contains 10 red marbles and 6 yellow marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are of different colours is:

A. $\frac{5}{8}$

B. $\frac{3}{5}$

C. $\frac{1}{2}$

D. $\frac{15}{56}$

Space for Personal Notes

Question 21

A box contains 3 red marbles and 2 blue marbles. Two marbles are drawn from the box, without replacement.

The probability that they are the same colour is:

A. $\frac{1}{2}$

B. $\frac{2}{5}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

Question 22

A and B are events of a sample space. Given that $\Pr(A \mid B) = p, \Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}, \Pr(B \mid A)$, is equal to:

A. p^6

B. $p^{\frac{4}{3}}$

C. $p^{\frac{7}{3}}$

D. $p^{\frac{8}{3}}$

Question 23

For events A and $B, \Pr(A \cap B) = p, \Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is:

A. 0

B. $\frac{1}{4}$

C. $\frac{3}{8}$

D. $\frac{1}{2}$

Question 24

Consider two events A and B, $\Pr(A) = \frac{1}{5}$ and $\Pr(B) = \frac{1}{6}$.

Which of the following statements is true?

A. If A and B are independent events, then $\Pr(A \cup B) = \frac{2}{5}$.

B. If A and B are mutually exclusive events, then $\Pr(A \cup B) = \frac{2}{9}$.

C. If A and B are independent events, then $\Pr(A \cap B) = \frac{9}{20}$.

D. If A and B are mutually exclusive events, then $\Pr(A \cup B) = \frac{11}{30}$.

Question 25

Let A and B be two independent events from a sample space.

If $\Pr(A) = p, \Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A \cup B')$ is equal to:

A. $1 - p - p^2$

B. $p^2 - p^3$

C. $p - p^3$

D. $1 - p + p^3$

Space for Personal Notes

Question 26

In a certain random experiment, the events V and W are independent events.

a. If $\Pr(V \cup W) = 0.78$ and $\Pr(V \cap W) = 0.18$, calculate $\Pr(V)$ and $\Pr(W)$, given Pr $\Pr(V) < \Pr(W)$, correct to 2 decimal places.

We know:

$$\Pr(V \cup W) = \Pr(V) + \Pr(W) - \Pr(V \cap W) = 0.78 \quad (1)$$
$$\Pr(V \cap W) = \Pr(V)\Pr(W) = 0.18 \quad (2)$$

Let $\Pr(V) = v$, then $\Pr(W) = 0.78 - v + 0.18 = 0.96 - v$

Substitute into equation (2):

$$v(0.96 - v) = 0.18 \Rightarrow 0.96v - v^2 = 0.18 \Rightarrow v^2 - 0.96v + 0.18 = 0$$

Solve with the quadratic formula:

$$v = \frac{0.96 \pm \sqrt{(-0.96)^2 - 4(1)(0.18)}}{2} = \frac{0.96 \pm \sqrt{0.9216 - 0.72}}{2} = \frac{0.96 \pm \sqrt{0.2016}}{2} = \frac{0.96 \pm 0.448999}{2}$$

$$v_1 = \frac{0.96 - 0.449}{2} = 0.2555 \approx 0.26 \quad \text{(smaller one)}$$

$$v_2 = \frac{0.96 + 0.449}{2} = 0.7045 \approx 0.70$$

Since $\Pr(V) < \Pr(W)$, choose $v = 0.26$, so:

- $\Pr(V) = 0.26$
- $\Pr(W) = 0.96 - 0.26 = 0.70$

b. Determine the probability that neither V nor W occur.

$$\boxed{1 - \Pr(V \cup W) = 1 - 0.78 = 0.22}$$

Space for Personal Notes

Question 27

A game uses a digital spinner with three colour sections: **red, blue, and yellow**. The spinner is programmed differently for two difficulty levels:

- On **Level 1**, the probabilities are:
  Red $= 0.4$, Blue $= 0.4$, Yellow $= 0.2$

- On **Level 2**, the probability of landing on red is modelled by a quadratic function:

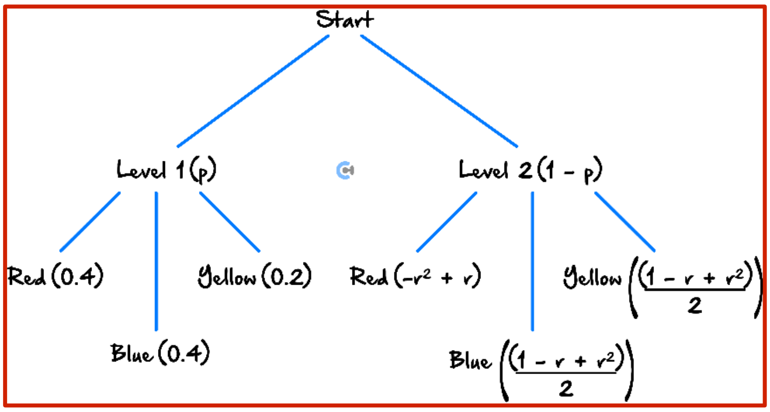
$$\Pr \Pr(\text{Level2}) = -r^2 + r$$

The rest of the probability is equally split between blue and yellow.
Each game round consists of:

1. A level is chosen at random with probability p for Level 1 and $1 - p$ for Level 2, $p \neq 0$.

2. The spinner is spun once.

a. Construct a tree diagram showing all outcomes in terms of p and r. Label each branch with the appropriate probability.

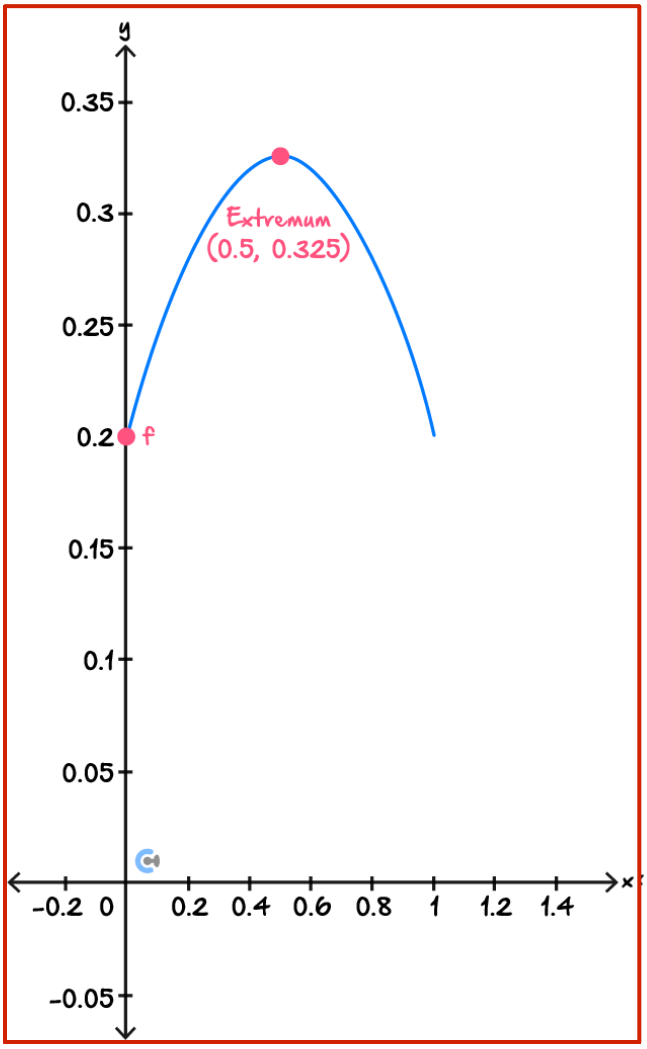


b. Write an expression for the total probability of landing on red, in terms of p and r.

$$\boxed{\Pr(\text{Red}) = p \cdot 0.4 + (1 - p)(-r^2 + r)}$$

c. Let $p = 0.5$.

Graph the total probability of landing on red for $r \in [0,1]$.



Substitute $p = 0.5$:

$$\Pr(\text{Red}) = 0.4(0.5) + 0.5(-r^2 + r) = 0.2 + 0.5(-r^2 + r) = -0.5r^2 + 0.5r + 0.2$$

d. What are the maximum and minimum total probabilities of landing on red over the domain $r \in [0,1]$? At what values of r do they occur?

Max: 0.325 at $r = 0.5$
Min: 0.2 at $r = 0$ or $r = 1$

e. Given that red was the result, write an expression for the probability that Level 1 was used in terms of p and r.

$$\Pr(\text{Level 1} \mid \text{Red}) = \frac{\Pr(\text{Red} \mid \text{Level 1}) \cdot \Pr(\text{Level 1})}{\Pr(\text{Red})} = \frac{0.4p}{0.4p + (1-p)(-r^2+r)}$$

f. Evaluate the expression from **part e.** for $p = 0.5$ and $r = 0.8$. Round to 4 decimal places.

$$\Pr(\text{Level 1} \mid \text{Red}) = \frac{0.2}{0.28} = \boxed{0.7143}$$

g.

i. Are the events "Spinner landed on yellow" and "Level 1 was used" **mutually exclusive**? Explain.

No - Yellow can occur in **both** levels.

ii. Find the expression for p in terms of r such that the events "Spinner landed on yellow" and "Level 1 was used" are **independent**.

For independence:

$$\Pr(\text{Yellow} \cap \text{Level 1}) = \Pr(\text{Yellow}) \cdot \Pr(\text{Level 1})$$

- $\Pr(\text{Yellow} \cap \text{Level 1}) = 0.2p$
- $\Pr(\text{Level 1}) = p$
- $\Pr(\text{Yellow}) = 0.2p + \frac{1-p}{2}(1-r+r^2)$

So:

$$0.2p = p \cdot \left[0.2p + \frac{1-p}{2}(1-r+r^2)\right]$$

Cancel $p \neq 0$:

$$0.2 = 0.2p + \frac{1-p}{2}(1-r+r^2)$$

Multiply both sides by 2:

$$0.4 = 0.4p + (1-p)(1-r+r^2)$$

Now solve for p:

$$\boxed{p = 1 - \frac{0.4}{1-r+r^2}}$$

Space for Personal Notes

Question 28

John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw. The probability that John hits the bullseye with a single throw is $\frac{1}{5}$. The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{3}$. John has four throws and Rebecca has two throws. Find the ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once.

John hitting at least once

Let J = "John hits at least once"

$$\Pr(J) = 1 - \Pr(\text{John misses all 4}) = 1 - \left(1 - \frac{1}{5}\right)^4 = 1 - \left(\frac{4}{5}\right)^4 = 1 - \frac{256}{625} = \frac{369}{625}$$

Rebecca hitting at least once

Let R = "Rebecca hits at least once"

$$\Pr(R) = 1 - \Pr(\text{Rebecca misses both}) = 1 - \left(1 - \frac{1}{3}\right)^2 = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\text{Ratio} = \frac{\Pr(R)}{\Pr(J)} = \frac{\frac{5}{9}}{\frac{369}{625}} = \frac{5}{9} \cdot \frac{625}{369} = \frac{3125}{3321}$$

Final answer (exact ratio):

$$\boxed{\frac{3125}{3321}}$$

Space for Personal Notes

Question 29

A spinner is divided into 4 unequal-coloured sections: purple, orange, green, and pink. The spinner is biased, and the probabilities of landing on each colour are:

- $\Pr(\text{purple}) = \frac{1}{2}$
- $\Pr(\text{orange}) = \frac{1}{4}$
- $\Pr(\text{green}) = \frac{1}{8}$
- $\Pr(\text{pink}) = \frac{1}{8}$

Each player pays $\$\mathbf{3}$ to play the game. The player spins the spinner up to **2 times**, but the game ends immediately if the spinner lands on **purple**.

- If a player gets **green then pink**, or **pink then green**, they win $\$\mathbf{5}$.
- If the player lands on **orange both times**, they win $\$\mathbf{15}$.
- All other outcomes result in winning $\$\mathbf{0}$.

a. What is the probability that the first spin lands on pink or green?

$$P(\text{pink or green}) = \frac{1}{8} + \frac{1}{8} = \boxed{\frac{1}{4}}$$

b. What is the probability that a player plays both spins?

$$P(\text{not purple}) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

Let:

- A = Event that the first spin is orange
- B = Event that the second spin is green

c. Draw a Karnaugh table showing the probability of A and B. Use it to explain whether A and B are mutually exclusive or not.

2-Way_Table_for_Events_A_and_B

	Second Spin Green (B)	Second Spin Not Green (B')
First Spin Orange (A)	1/32	7/32
First Spin Not Orange (A')	3/32	21/32

No, they are **not mutually exclusive**, because they occur on different spins. Both can happen in the same trial (e.g., first orange, then green).

d. Calculate $\Pr(A \cup B)$, the probability that either the first spin is orange or the second spin is green.

Using:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since spins are independent:

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

$$P(A \cup B) = \frac{1}{4} + \frac{1}{8} - \frac{1}{32} = \frac{8}{32} + \frac{4}{32} - \frac{1}{32} = \boxed{\frac{11}{32}}$$

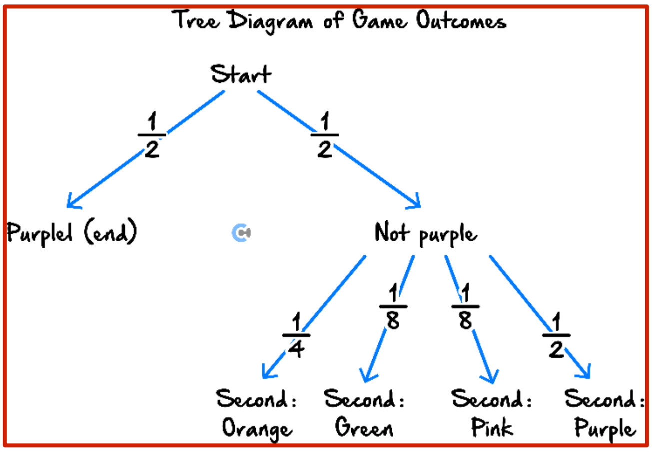
e. Are the events "first spin is green" and "second spin is green" independent?

Yes.

f. Are the events "first spin is purple" and "second spin occurs" mutually exclusive? Explain.

Yes. If first spin is purple, the game ends. So the second spin cannot occur. They are **mutually exclusive**.

g. Construct a tree diagram showing all possible outcomes of the game. Clearly label the branches with probabilities and outcomes.



h. What is the probability that a player wins \$5 given that they played both spins?

We already know:

- Total prob of playing both spins = $\frac{1}{2}$
- Prob of winning \$5 (green–pink or pink–green) = $2 \cdot \frac{1}{8} \cdot \frac{1}{8} = \frac{2}{64} = \frac{1}{32}$

So:

$$P(\text{win \$5} \mid 2 \text{ spins}) = \frac{\frac{1}{32}}{\frac{1}{2}} = \boxed{\frac{1}{16}}$$

i. List all outcomes that result in net winnings of \$2 or \$12.

\$2: green → pink or pink → green
\$12: orange → orange

j. Find the probability of losing \$3.

1. Winning \$5 → Net = 2

- Green then Pink: $\frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$
- Pink then Green: $\frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$
- Total: $\frac{2}{64} = \frac{1}{32} = 0.03125$

2. Winning \$15 → Net = 12

- Orange then Orange: $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16} = 0.0625$

Hence, all other outcomes: Net $= -3$
$1 - 0.03125 - 0.0625 = 0.90625$

Website: contoureducation.com.au | Phone: 1800 888 300 | Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾

Free 1-on-1 Support

Be Sure to Make the Most of These (Free) Services!

- Experienced Contour tutors (45 + raw scores, 99 + ATARs).
- For fully enrolled Contour students with up-to-date fees.
- After-school weekdays and all-day weekends.

1-on-1 Video Consults	Text-Based Support
➤ Book via bit.ly/contour-methods-consult-2025 (or QR code below). ➤ One active booking at a time (must attend before booking the next).	➤ Message +61 440 138 726 with questions. ➤ Save the contact as "Contour Methods".

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726