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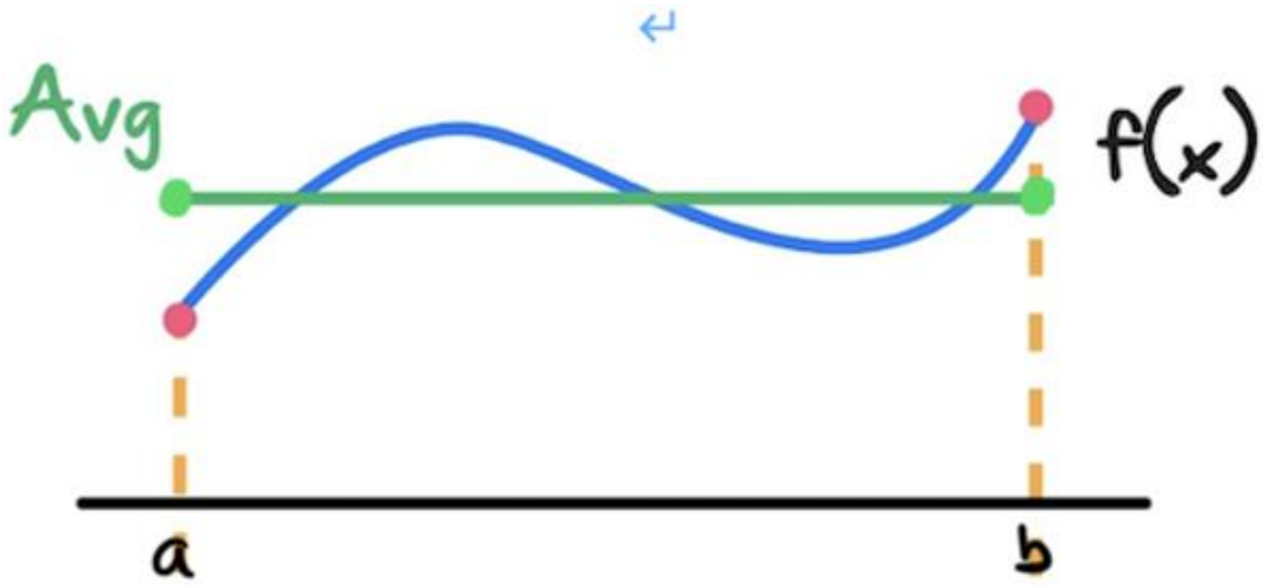
VCE Mathematical Methods ¾
Integration Exam Skills [4.4]
Workbook

Outline:

Section A: Recap

[4.3.1] - Find the Average Value of the Function

Average Value of a Function



- **Average value of the function**: Find the average y value of a function from $x = a$ and $x = b$.
- Average value of the function from $x = a$ to $x = b$ is given by:

Average Value = ________________

[4.3.2] - Apply Integration by Recognition

Integration by Recognition

- **Integration by Recognition:** Finding impossible antiderivatives using prior information.
- Questions follow the structure:

  a. Find the derivative of ...

  b. Hence, find the integral ...

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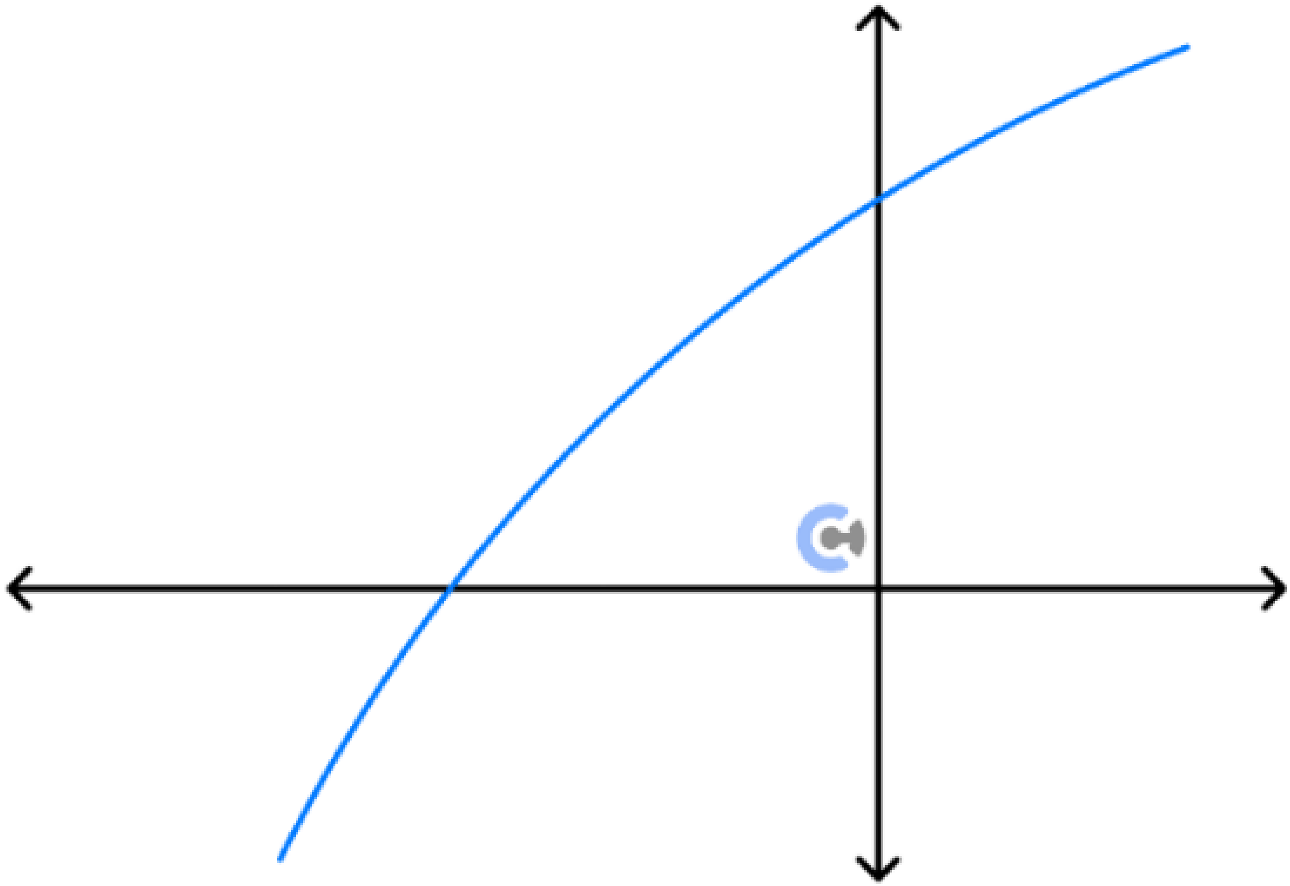
Definition

[4.3.3] – Advanced Areas

Horizontal Strips



- **Horizontal Strip:** Cut the area into thin horizontal strips.
- Useful when the function is hard to antiderive but the inverse is easy.

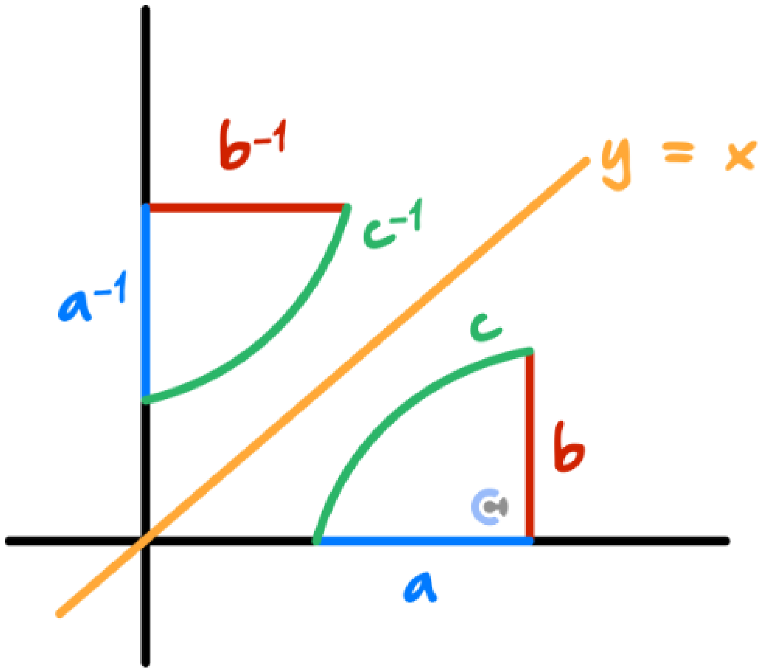
$$\text{Area of the thin strips} = x\,dy$$

- The area bounded by the function $y = f(x)$, the y-axis and the lines $y = a$ and $y = b$ are:

$$\boldsymbol{Area} =$$

- The terminals must be y values.

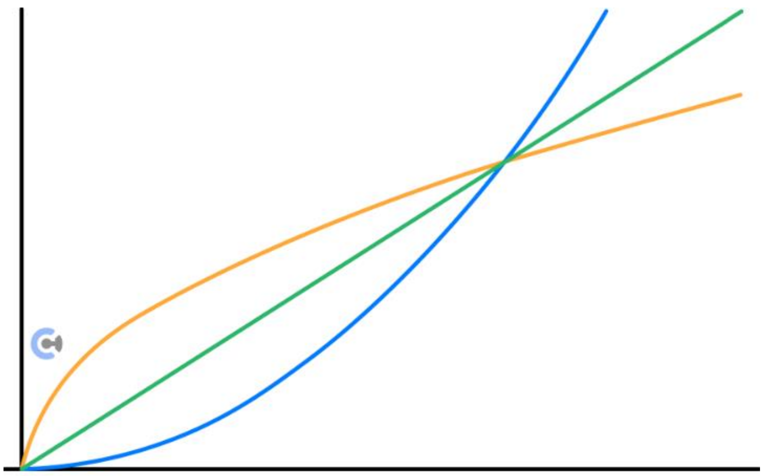
Using Inverse Boundaries



- **Idea:** Reflecting the area around the $y = x$ results in the same area.

Area bounded by a, b, c = Area bounded by ____________________

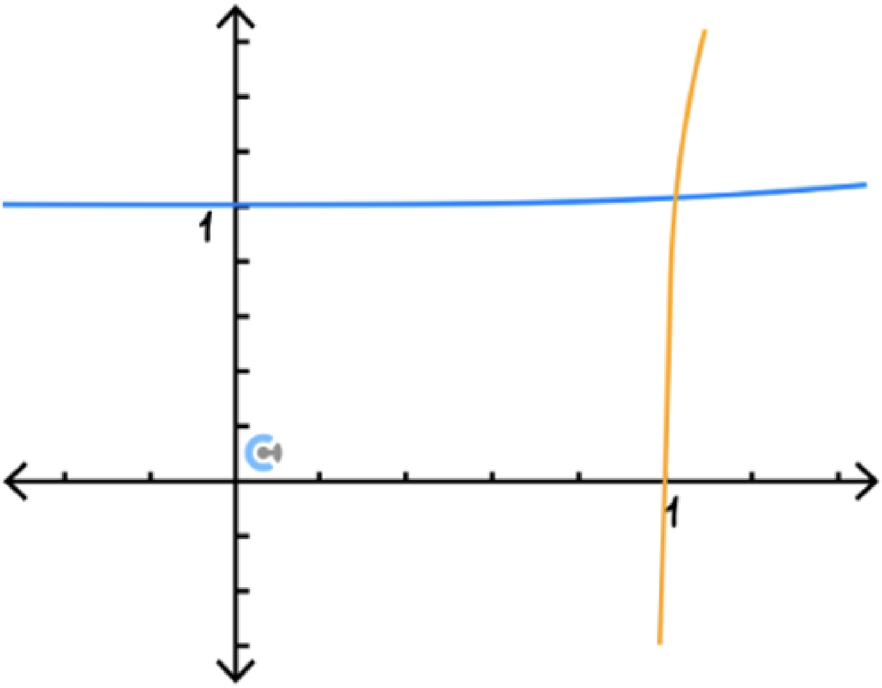
Area Between Inverses



- **The area between inverses:** Can be cut up by the line of symmetry: $y = x$.

$$\int_a^b f(x) - f^{-1}(x)\,dx =$$

Area and Family of Functions



$$\lim_{k \to l} Area = Area\ of\ a\ simple\ shape$$

- **Steps:**

1. Use Manipulate / Sliders.

2. Identify the “simple shape” that the area approaches to.

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Section B: Exam 1 (26 Marks)

Question 1 (4 marks)

a. Let $g: \left(\frac{3}{2}, \infty\right) \rightarrow R, g(x) = \frac{3}{2x-3}$.

Find the rule for an antiderivative of $g(x)$. (1 mark)

b. Evaluate $\int_0^1 \left(f(x)(2f(x) - 3)\right) dx$, where $\int_0^1 [f(x)]^2 dx = \frac{1}{5}$ and $\int_0^1 f(x) dx = \frac{1}{3}$. (3 marks)

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Question 2 (2 marks)

Find the average value of $y = x - e^x$ over the interval $x \in [-\ln(2), \ln(2)]$.

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Question 3 (4 marks)

Consider the function, $f(x) = -2x - x^2 + x^3$.

a. Find the equation of the tangent line of $f(x)$ at $x = 1$. (1 mark)

b. Hence, find the area bound by $f(x)$ and the tangent line found in **part a.** (3 marks)

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Question 4 (4 marks)

a. Find $\frac{d}{dx}(x\sin(2x))$. (1 mark)

b. Hence, evaluate $\int \left(\frac{1}{2}x\cos(2x)\right)$. (3 marks)

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Question 5 (3 marks)

Let $f(x) = \frac{1}{3-2x}$. Find the area bounded by the graph of $y = f(x)$, the y-axis and the line $y = 2$.

Question 6 (4 marks)

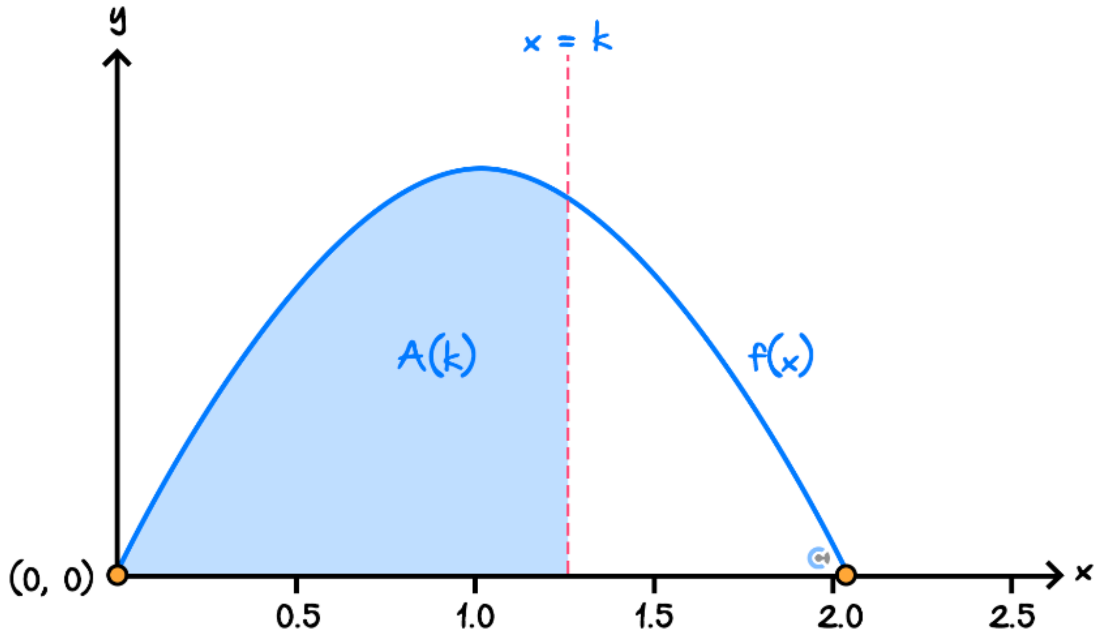
Consider a quadratic function of the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$.

The function f has an average value of $\frac{5}{3}$ on the interval $[0, a]$ and a turning point at $(1, 1)$.

Find the values of a,b, and c.

Question 7 (5 marks)

Part of the graph of $y = f(x)$ is shown below. The rule $A(k) = k\sin(k)$ gives the area bounded by the graph of f, the horizontal axis and the line $x = k$.



a. State the value of $A\left(\frac{\pi}{3}\right)$. (1 mark)

b. Evaluate $f\left(\frac{\pi}{3}\right)$. (2 marks)

c. Consider the average value of the function f over the interval $x \in [0, k]$, where $k \in [0, 2]$.

Find the value of k that results in the maximum average value. (2 marks)

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Section C: Tech-Active Exam Skills

Sub-Section: Quick Method for Integral Property Questions

Quick Method

- We can let $f(x)$ be ______________ number which is appropriate for the given integral.

$$f(x) = a, a \in R$$

- **Steps**

1. Let $f(x) = a$ and solve for a.
2. Save $f(x) = a$ that you have solved.
3. Find the integral the question is asking for.

Question 8 (1 mark) **Walkthrough.**

If $\int_1^2 f(2x)dx = 6$, then $\int_2^4 2f(x)dx$ is equal to:

A. 6

B. 12

C. 18

D. 24

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Question 9 (1 mark)

If $\int_0^2 f(x)dx = 10$, then $\int_0^1 2f(2x)dx$ is equal to:

A. 5

B. 10

C. 20

D. -5

Question 10 (1 mark) **Extension.**

If $\int_1^5 f(3x)dx = 8$, then $\int_6^2 2f(3x-3)dx$ is equal to:

A. 16

B. -16

C. 8

D. -8

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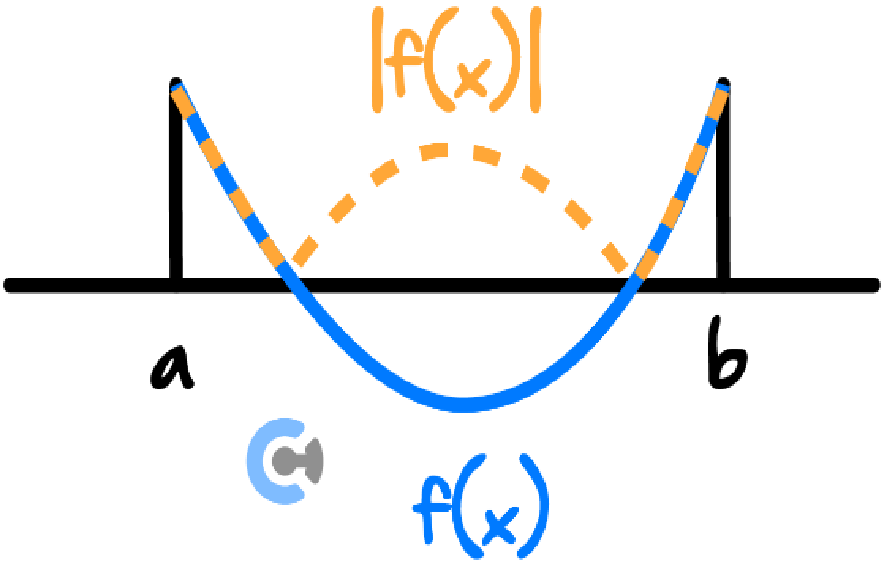
Sub-Section: Finding the Area Bounded Using One Integral

Using Modulus for Total Areas



$$\int_a^b |f(x)|dx$$

- **Finding area using one integral:** Requires modulus.

Calculator Commands: Total Bounded Area

- **Mathematica**
  - $|f(x)|$ can be written as

    `Abs[f[x]]`

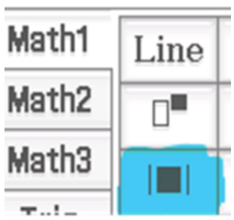
- **TI-Nspire**
  - Find the modulus sign

  - Under this button

- **Casio Classpad**
  - Find the modulus sign under Math1

Question 11 Tech-Active.

Find the **total** area bounded by the function $y = x^3 - 3x^2 - 4x + 12$ and the x-axis.

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Sub-Section: Pseudocode for Trapezoidal Approximation

REMINDER: VCAA formula for trapezoidal approximation.

Trapezium Rule Approximation	Area $\approx \frac{x_n - x_0}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)]$

Pseudocode For Trapezoidal Approximation

- Type 1: Adding an area of a trapezium in each loop.

Algorithm	Comments

<pre>define f(x)
  return (enter required function rule)
sum ← 0
a ← lowest x-value
b ← highest x-value
n ← number of trapeziums
h ← (b-a)/n
left ← a
right ← a + h
for i from 1 to n
  strip ← 0.5(f(left) + f(right)) x h
  sum ← sum + strip
  left ← left + h
  right ← right + h
end for
print sum</pre>
	➤ a and b represent where the area starts and ends.
	➤ n is the number of trapeziums chosen.
	➤ In the for loop, the area of each trapezium is calculated as you move from left to right and added to the sum.
	➤ The final value of the sum is printed.
	➤ The trapezium algorithm may be implemented by using a function structure. The function f must be defined first.
	$\boldsymbol{define}\ Trap(a, b, n)$
	$(Insert\ algorithm\ as\ shown)$
	$\boldsymbol{return}\ (sum)$
	➤ The advantage of this is that it may be implemented simply by changing the values of the variables.

- Type 2: Adding $f(x_0) + 2f(x_1) + \cdots 2f(x_{n-1}) + f(x_n)$ first and multiplying $\frac{h}{2}$ at the end.

Trapezium Rule Approximation	Area $\approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$

```
Inputs: f(x), the function to integrate
        a, the lower terminal of integration
        b, the upper terminal
        n, the number of trapeziums to use

Define: trapezium (f(x), a, b, n)
        h ← (b − a) ÷ n
        x ← a + h
        i ← 1
        While i < n Do
            sum ← sum + 2 × f(x)
            x ← x + h
            i ← i + 1
        EndWhile
        Area← (h ÷ 2) × sum
        Return area
```

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Question 12 (1 mark) **Walkthrough.**

The algorithm below, described in pseudocode, estimates the value of a definite integral using the trapezium rule.

Inputs: f(x), the function to integrate
a, the lower terminal of integration
b, the upper terminal of integration
n, the number of trapeziums to use

Define: trapezium (f(x), a, b, n)
$h \leftarrow (b - a) \div n$
$x \leftarrow a + h$
$i \leftarrow 1$
While $i < n$ **Do**
$sum \leftarrow sum + 2 \times f(x)$
$x \leftarrow x + h$
$i \leftarrow i + 1$
EndWhile
Area$\leftarrow (h \div 2) \times sum$
Return area

Consider the algorithm implemented with the following inputs:

$$\text{trapezium}\ (\log_e(x), 1, 7, 3)$$

The algorithm returns a value closest to:

A. 6.92

B. 5.93

C. 7.36

D. 7.62

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Question 13 (1 mark)

The algorithm below, described in pseudocode, estimates the value of a definite integral using the trapezium rule.

```
Inputs: f(x), the function to integrate
        a, the lower terminal of integration
        b, the upper terminal of integration
        n, the number of trapeziums to use

define: trapezium (f(x), a, b, n)
        sum ← 0
        h ← (b − a) ÷ n
        left ← a
        right ← a + h
        for i from 1 to n
            strip ← 0.5(f(left) + f(right)) × h
            sum ← sum + strip
            left ← left + h
            right ← right+h
        end for
        print sum
```

Consider the algorithm implemented with the following inputs:

$$\text{trapezium } (2x^2, 1, 7, 3)$$

After the second iteration of the for loop, the value of the *sum* is:

A. 236

B. 88

C. 44

D. 118

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Question 14 (1 mark) **Extension.**

The algorithm below, described in pseudocode, estimates the value of a definite integral using the trapezium rule.

```
Inputs: f(x), the function to integrate
        a, the lower terminal of integration
        b, the upper terminal of integration
        n, the number of trapeziums to use

Define: trapezium (f(x), a, b, n)
        h ← (b − a) ÷ n
        sum ← f(a) + f(b)
        x ← a + h
        i ← 1
        While i < n Do
            sum ← sum + 2 × f(x)
            x ← x + h
            i ← i + 1
        EndWhile
        area ← (h ÷ 2) × sum
        Return area
```

Consider the algorithm implemented with the following inputs:

$$\text{trapezium}\ (e^x, 1, 11, 5)$$

The value of the sum after the 2nd iteration is equal to:

A. $e + 2e^3 + e^{11}$

B. $e + 2e^3 + 2e^5 + e^{11}$

C. $e + 2e^3$

D. $e + 2e^3 + 2e^5$

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Section D: Exam 2 (36 Marks)

Question 15 (1 mark)

Let $f(x) = (2x + 1)^3$. The anti-derivative is given by:

A. $\frac{1}{4}(2x + 1)^3 + c$

B. $\frac{1}{8}(2x + 1)^4 + c$

C. $\frac{1}{6}(2x + 1)^4 + c$

D. $\frac{1}{4}(2x + 1)^4 + c$

Question 16 (1 mark)

Find the average value of $f(x) = \sin(x) - x$ over the interval $x = \frac{\pi}{2}$ to $x = \frac{3\pi}{2}$.

A. π

B. $-\pi$

C. $\frac{\pi}{2}$

D. $\frac{-\pi}{2}$

Question 17 (1 mark)

The area enclosed between $y = (x - 2)(x - 3)$, and $y = (x - 2)(3 - x)$ is:

A. $\frac{1}{12}$

B. $\frac{1}{6}$

C. $\frac{1}{3}$

D. $\frac{2}{3}$

Question 18 (1 mark)

If $\int_2^4 f(x)dx = 4$, then $\int_2^1 2f(2x)dx$ is equal to:

A. 2

B. -4

C. 8

D. 4

Question 19 (1 mark)

The area bounded by the graph of $y = -x^4 + 20x^2 - 64$ and the x-axis is:

A. $\frac{1024}{15}$

B. 256

C. $\frac{1024}{3}$

D. 128

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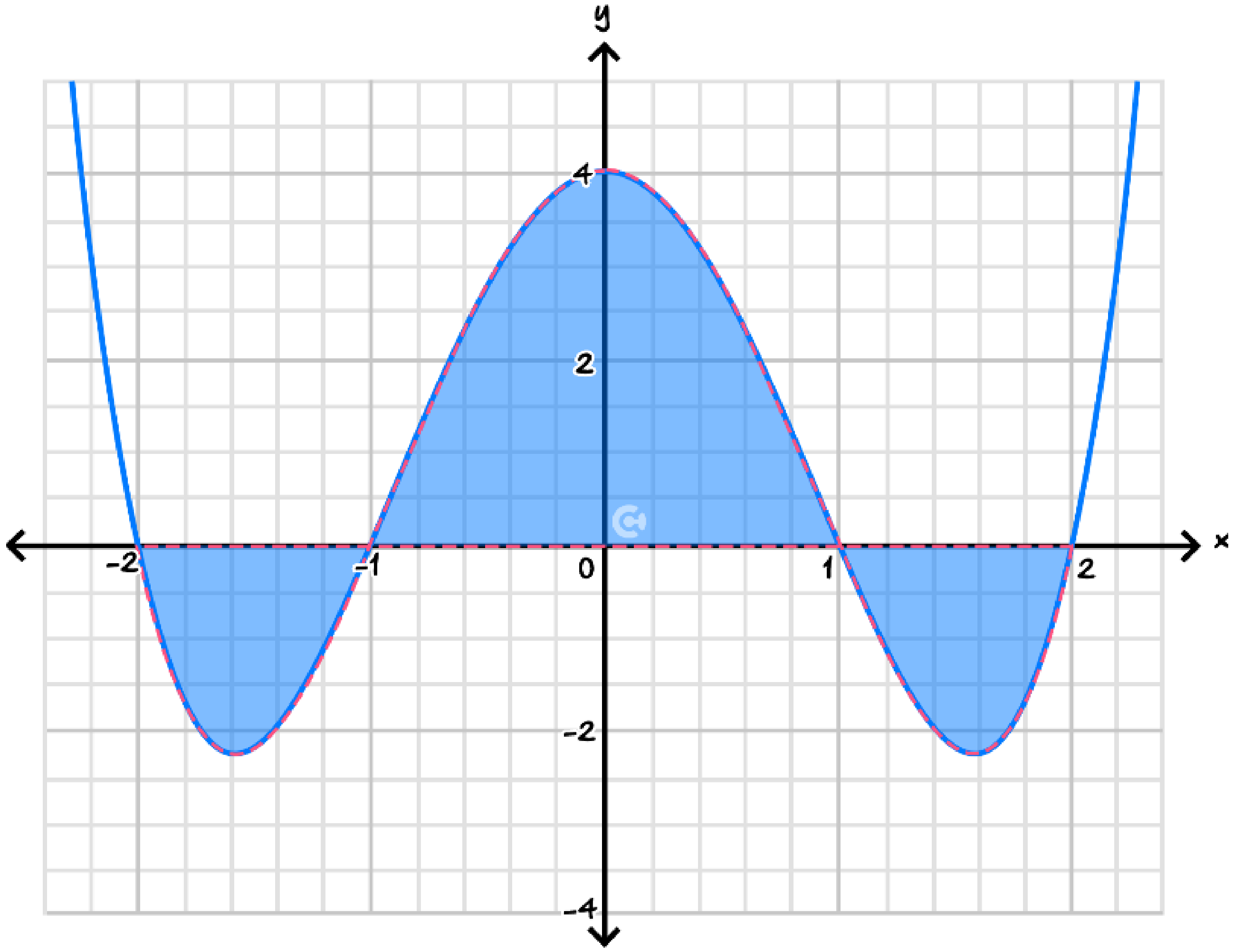
Question 20 (1 mark)

The following depicts the area bounded by the function $f(x) = (x-1)(x-2)(x+1)(x+2)$ and the x-axis.



Which of the following integrals represents the area highlighted above?

A. $\int_{-2}^{2} f(x)dx$

B. $2\int_{0}^{1} f(x)dx - \int_{1}^{2} f(x)dx$

C. $\int_{-1}^{1} f(x)dx + 2\int_{1}^{2} f(x)dx$

D. $\int_{-1}^{1} f(x)dx - 2\int_{1}^{2} f(x)dx$

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Question 21 (1 mark)

The algorithm below, described in pseudocode, estimates the value of a definite integral using the trapezium rule.

```
Inputs: f(x), the function to integrate
        a, the lower terminal of integration
        b, the upper terminal of integration
        n, the number of trapeziums to use

Define: trapezium (f(x), a, b, n)
        h ← (b − a) ÷ n
        x ← a + h
        i ← 1
        While i < n Do
            sum ← sum + 2 × f(x)
            x ← x + h
            i ← i + 1
        EndWhile
        Area ← (h ÷ 2) × sum
        Return area
```

Consider the algorithm implemented with the following inputs:

$$\text{trapezium }(e^{2x}, 0, 8, 4)$$

The value of the sum after the 2^{nd} iteration is equal to:

A. $e + 2e^4 + e^{16}$

B. $1 + 2e^4 + 2e^8 + e^{16}$

C. $1 + 2e^4 + 2e^8$

D. $1 + 2e^4 + 2e^8 + 2e^{12} + e^{16}$

Question 22 (1 mark)

If $\frac{d}{dx}(x\cos(x)) = \cos(x) - x\sin(x)$, then $\int x\sin(x)dx$ is equal to:

A. $-\sin(x) + x\cos(x)$

B. $\sin(x) - \int x\cos(x)dx$

C. $\int \cos(x)dx - x\cos(x)$

D. $\int \cos(x)dx + x\cos(x)$

Question 23 (1 mark)

Which of the following integrals gives the area bound between $f(x) = -x^3$ and its inverse?

A. $\int_{-1}^{1} f(x) - x \, dx$

B. $2\int_{-1}^{0} f(x) - f^{-1}(x) \, dx$

C. $2\int_{0}^{1} f(x) - f^{-1}(x) \, dx$

D. $2\int_{0}^{1} f(x) + f^{-1}(x) \, dx$

Question 24 (1 mark)

It is known that $g(x)$ is an odd function, whilst $f(x)$ is an even function. Then, the integral $\int_{-a}^{a} g(x) - f(x)dx$ simplifies to:

A. $\int_{-a}^{a} f(x)dx$

B. $\int_{a}^{-a} f(x)dx$

C. $\int_{-a}^{a} g(x)dx$

D. $\int_{a}^{-a} g(x)dx$

Question 25 (1 mark)

Consider the function $h(x) = \frac{1}{-k(x+2)} + 1$, where $k > 0$. Let $A(k)$ be the area between $h(x)$ and the axes. The value of $\lim_{k\to\infty} A(k)$ is given by:

A. k^2

B. $2k$

C. 2

D. $10k$

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Question 26 (1 mark)

An estimate of the area under the curve $y = \sqrt{x}$ using the trapezoidal method, from the interval $x = 0$ to $x = 4$, with a step size of 1 is equal to:

A. $\sqrt{2} + \sqrt{3}$

B. $1 + \sqrt{2} + \sqrt{3}$

C. $2 + \sqrt{2} + \sqrt{3}$

D. $4 + 2\sqrt{2} + 2\sqrt{3}$

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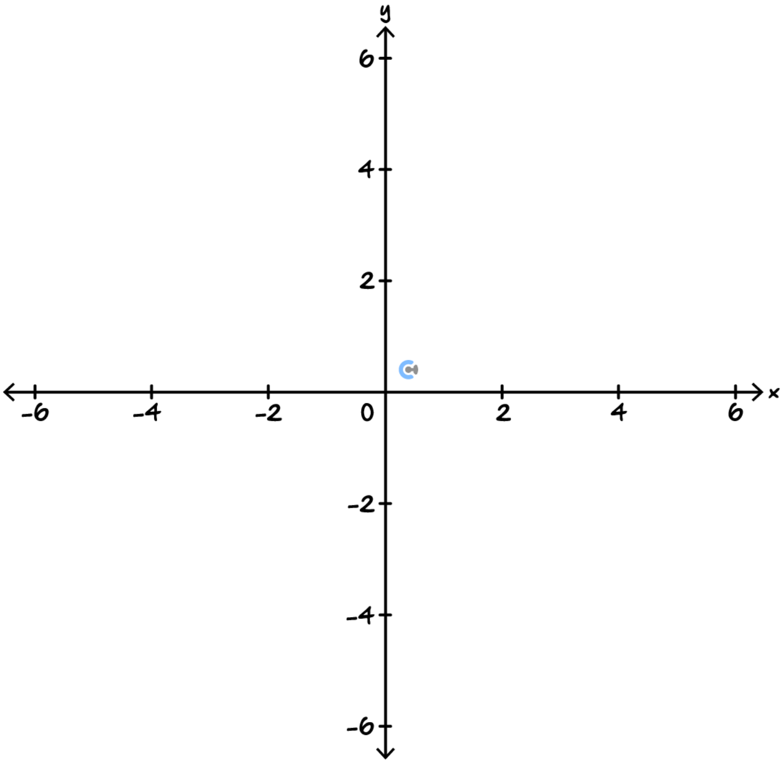
Question 27 (14 marks)

Let $f(x) = \frac{4x}{2x+1}$.



a. Express $f(x)$ in the form $\frac{a}{bx+c} + d$. (1 mark)

b. Find the equation of the asymptotes of $f(x)$. (1 mark)

c. Find the equation of the normal line that is normal to two different points on $f(x)$. (3 marks)

d. Hence, find the minimum distance from the portion of the graph that lies on the right-hand side and the left-hand side of the vertical asymptote. (2 marks)

e. Now, consider the function:

$$f: [a, b] \to R, f_k(x) = \frac{2kx}{kx+1}, k > 0.5$$

i. a and b are the intersection points between $f_k(x)$ and x. Provide your answer in terms of k. (2 marks)

ii. Let $A(k)$ be the area between $f_k(x)$ and x. State $A(k)$. (3 marks)

iii. Find the range of $A(k)$. (2 marks)

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Question 28 (10 marks)

Let $f : [0,3] \to \mathbb{R}, f(x) = 3x - x^2$.

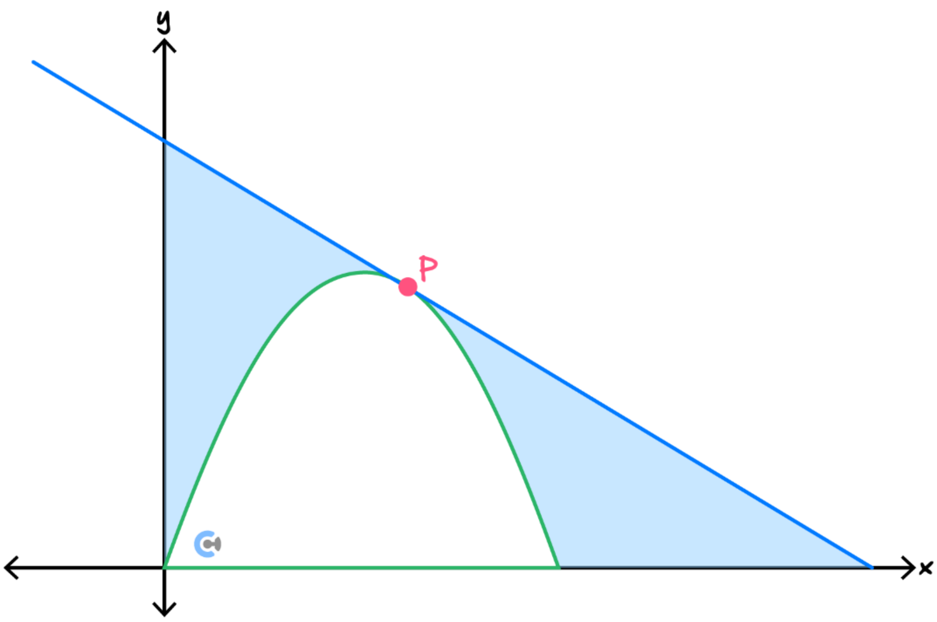
a. A rectangle is to be fitted in the region bounded by $y = f(x)$ and the x-axis.

Let $x = a$ where $\frac{3}{2} < a < 3$, be the x-coordinate of one of the vertices of the rectangle on the curve.

i. Find the area of the rectangle in terms of a. (2 marks)

ii. State the area of the largest rectangle and the value of a when it occurs. (2 marks)

b. Let $P(b, 3b - b^2)$ be a point on the curve $y = f(x)$, where $\frac{3}{2} < b < 3$.



i. Find the equation of the tangent to the curve $y = f(x)$ at the point P. (1 mark)

ii. Find the x and y-intercepts of the tangent in terms of b. (1 mark)

iii. Find the shaded area, on the diagram above, in terms of b. (2 marks)

iv. Find the value of b for which the shaded area is minimum and state this minimum area. (2 marks)

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